FELLER GENERATORS WITH MEASURE-VALUED DRIFTS

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Abstract. We construct a L^p-strong Feller process associated with the formal differential operator $-\Delta + \sigma \cdot \nabla$ on $\mathbb{R}^d$, $d \geqslant 3$, with drift σ in a wide class of measures (e.g. the sum of a measure having density in weak L^d space and a Kato class measure), by exploiting a quantitative dependence of the smoothness of the domain of an operator realization of $-\Delta + \sigma \cdot \nabla$ generating a holomorphic C_0-semigroup on $L^p(\mathbb{R}^d)$, $p > d-1$, on the value of the relative bound of σ.

1. Let $\mathcal{L}^d$ be the Lebesgue measure on $\mathbb{R}^d$, $L^p = L^p(\mathbb{R}^d, \mathcal{L}^d)$, $L^{p,\infty} = L^{p,\infty}(\mathbb{R}^d, \mathcal{L}^d)$ and $W^{1,p} = W^{1,p}(\mathbb{R}^d, \mathcal{L}^d)$ $(p \geqslant 1)$ the standard Lebesgue, weak Lebesgue and Sobolev spaces, $C^{0,\gamma} = C^{0,\gamma}(\mathbb{R}^d)$ the space of γ-Hölder continuous functions $(0 < \gamma < 1)$, $C_b = C_b(\mathbb{R}^d)$ the space of bounded continuous functions, endowed with the sup-norm, $C_\infty \subset C_b$ the closed subspace of functions vanishing at infinity, $\mathcal{W}^{s,p}$, $s > 0$, the Bessel potential space endowed with norm $\|u\|_{p,s} := \|g\|_p$, $u = (1-\Delta)^{-\frac{s}{2}} g$, $g \in L^p$, $\mathcal{W}^{-s,p'}$, $p' := p/(p-1)$, the anti-dual of $\mathcal{W}^{s,p}$, and $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ the L. Schwartz space of test functions. Given a $v = (v_i)_{i=1}^d \in \mathbb{C}^d$, set $|v|_1 := \sum_{i=1}^d |v_i|$. We denote by $\mathcal{B}(X,Y)$ the space of bounded linear operators between complex Banach spaces $X \to Y$, endowed with operator norm $\|\cdot\|_{X \to Y}$; $\mathcal{B}(X) := \mathcal{B}(X,X)$. Set $\|\cdot\|_{p \to q} := \|\cdot\|_{L^p \to L^q}$. Depending on the context, $\overset{w}{\to}$ will denote either the weak convergence of measures, or the weak convergence in a given Banach space. $\overset{s}{\to}$ denotes the strong convergence (or the strong convergence of bounded linear operators) in a given Banach space.

By $\langle u, v \rangle$ we denote the inner product in L^2,

$$\langle u, v \rangle = \langle u \bar{v} \rangle := \int_{\mathbb{R}^d} u \bar{v} \mathcal{L}^d \qquad (u, v \in L^2).$$

2. Let $d \geqslant 3$. The problem of constructing an operator realization on C_∞ of the formal differential operator $-\Delta + \sigma \cdot \nabla$, with σ a singular vector field $\mathbb{R}^d \to \mathbb{R}^d$ (or a $\mathbb{R}^d$–valued measure), that generates a contraction positivity preserving C_0-semigroup there (Feller semigroup), has been thoroughly studied in the literature (motivated, in particular, by applications to the theory of stochastic processes: by the classical result, such a semigroup determines the transition (sub-) probability function of a Hunt process). In the context of this problem, we consider the following classes of vector fields and vector-valued measures on $\mathbb{R}^d$.

1. A $\mathbb{R}^d$-valued Borel measure $\sigma = (\sigma_i)_{i=1}^d$ on $\mathbb{R}^d$ is said to belong to $\bar{\mathbf{F}}_\delta^{\frac{1}{2}}$, $\delta > 0$, the class of *weakly form-bounded measures*, if there exists $\lambda = \lambda_\delta > 0$ such that

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (\lambda - \Delta)^{-\frac{1}{4}}(x,y) f(y) dy \right)^2 |\sigma|_1(dx) \leqslant \delta \|f\|_2^2, \quad f \in \mathcal{S},$$

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where $(\lambda-\Delta)^{-\frac{1}{4}}(x,y)$ is the Bessel potential kernel, $|\sigma|_1 := \sum_{i=1}^d |\sigma_i|$, $|\sigma_i|$ is the variation of σ_i.

2. A $\mathbb{R}^d$-valued Borel measure σ on $\mathbb{R}^d$ is said to belong to the Kato class $\bar{\mathbf{K}}_\delta^{d+1}$, $\delta>0$, if there exists $\lambda=\lambda_\delta>0$ such that

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}(\lambda-\Delta)^{-\frac{1}{2}}(x,y)|\sigma|_1(dy)\leqslant\delta.$$

3. A vector field $b:\mathbb{R}^d\to\mathbb{R}^d$ belongs to $\mathbf{F}_\delta$, $\delta>0$, the class of form-bounded vector fields, if b is $\mathcal{L}^d$-measurable and there exists $\lambda=\lambda_\delta>0$ such that

$$\||b|_1(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2}\leqslant\sqrt{\delta}.$$

4. $\mathbf{F}_\delta^{\frac{1}{2}} := \bar{\mathbf{F}}_\delta^{\frac{1}{2}}\cap\{b\mathcal{L}^d$ with a $\mathcal{L}^d$-measurable $b:\mathbb{R}^d\to\mathbb{R}^d\}$,

5. $\mathbf{K}_\delta^{d+1} := \bar{\mathbf{K}}_\delta^{d+1}\cap\{b\mathcal{L}^d$ with a $\mathcal{L}^d$-measurable $b:\mathbb{R}^d\to\mathbb{R}^d\}$

6. $\mathbf{K}_0^{d+1} := \bigcap_{\delta>0}\mathbf{K}_\delta^{d+1}$, $\bar{\mathbf{K}}_0^{d+1} := \bigcap_{\delta>0}\bar{\mathbf{K}}_\delta^{d+1}$, and $\mathbf{F}_0 := \bigcap_{\delta>0}\mathbf{F}_\delta$.

Simple examples show:

$$\mathbf{K}_0^{d+1}-\mathbf{F}_\delta\neq\varnothing,\quad\text{and}\quad\mathbf{F}_{\delta_1}-\mathbf{K}_\delta^{d+1}\neq\varnothing\quad\text{for any }\delta,\delta_1>0,$$

for instance,

1) $b\mathcal{L}^d$, where $b(x):=\sqrt{\delta}\frac{d-2}{2}x|x|^{-2}$, is in $\mathbf{F}_\delta-\mathbf{K}_{\delta_1}^{d+1}$ for any $\delta,\delta_1>0$ (by the Hardy inequality).

2) Let $b(x):=e\mathbf{1}_{|x_1|<1}|x_1|^{s-1}$ for some $e\in\mathbb{R}^d$, $|e|=1$, where $0<s<1$, $x=(x_1,\dots,x_d)$, and $\mathbf{1}_{|x_1|<1}$ is the indicator function of $\{x\in\mathbb{R}^d:|x_1|<1\}$. Then $b\mathcal{L}^d\in\mathbf{K}_0^{d+1}-\mathbf{F}_\delta$ for any $\delta>0$.

The examples above show that there exist $b\in\mathbf{F}_\delta$ (resp. $\mathbf{K}_\delta^{d+1}$) such that $\varepsilon b\notin\mathbf{F}_0$ (resp. $\mathbf{K}_0^{d+1}$) for any $\varepsilon>0$. The classes $\mathbf{F}_\delta$, $\mathbf{K}_\delta^{d+1}$ cover singularities of b of critical order, i.e. 'sensitive' to multiplication by a constant (replacing a $b\in\mathbf{F}_\delta$ with $cb\,(\in\mathbf{F}_{c^2\delta})$, $c>1$, destroys e.g. the uniqueness of the solution of Cauchy problem for $-\Delta+b\cdot\nabla$, cf. [KS, Example 5]). The classes $\mathbf{K}_0^{d+1}$, $\bar{\mathbf{K}}_0^{d+1}$, $\mathbf{F}_0$ (and, thus, $L^d(\mathbb{R}^d,\mathbb{R}^d)\subsetneq\mathbf{F}_0$ – the inclusion follows by the Sobolev embedding theorem, cf. the diagram below) don't contain vector fields having critical order singularities.

We have:

$$\bar{\mathbf{K}}_\delta^{d+1}\subsetneq\bar{\mathbf{F}}_\delta^{\frac{1}{2}},\tag{1}$$

$$\mathbf{K}_\delta^{d+1}\subsetneq\mathbf{F}_\delta^{\frac{1}{2}},\quad\mathbf{F}_\delta\subsetneq\mathbf{F}_{\delta_1}^{\frac{1}{2}}\quad\text{for }\delta_1=\sqrt{\delta},\tag{2}$$

$$b\mathcal{L}^d\in\mathbf{F}_{\delta_1}^{\frac{1}{2}}\text{ and }\nu\in\bar{\mathbf{K}}_{\delta_2}^{d+1}\quad\Longrightarrow\quad b\mathcal{L}^d+\nu\in\bar{\mathbf{F}}_\delta^{\frac{1}{2}},\ \sqrt{\delta}=\sqrt{\delta_1}+\sqrt{\delta_2}\tag{3}$$

The inclusion (1) is Proposition 1 below. The first inclusion in (2) follows e.g. by interpolation between $\|(\lambda-\Delta)^{-\frac{1}{2}}|b|_1\|_\infty\leqslant\delta$ and (by duality) $\||b|_1(\lambda-\Delta)^{-\frac{1}{2}}\|_{1\to 1}\leqslant\delta$, the second inclusion in (2) follows by the Heinz inequality; for details, if needed, see [K, Appendix B].

[BC] constructed an operator realization on C_b of $-\Delta+\sigma\cdot\nabla$, $\sigma\in\bar{\mathbf{K}}_0^{d+1}$, generating a strong Feller semigroup there, thus obtaining e.g. a Brownian motion drifting upward when filtering through certain fractal-like sets. Below we construct an operator realization on C_∞ of $-\Delta+\sigma\cdot\nabla$ generating a L^p-strong Feller semigroup, with drift σ of the form

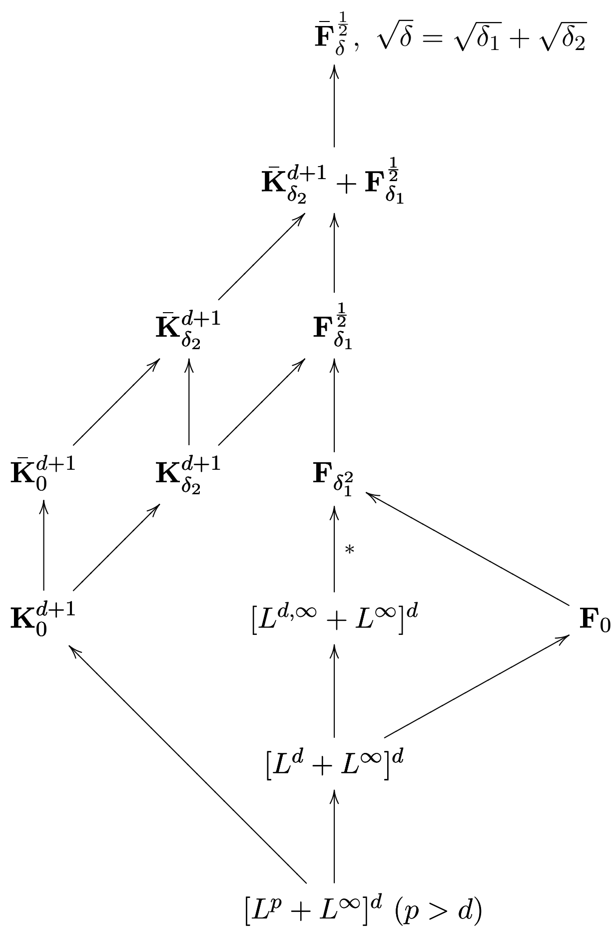
$$\sigma = b\mathcal{L}^d + \nu, \quad b\mathcal{L}^d \in \mathbf{F}_{\delta_1}^{\frac{1}{2}}, \quad \nu \in \bar{\mathbf{K}}_{\delta_2}^{d+1}, \tag{4}$$

$$\left(\Longrightarrow \quad \sigma \in \bar{\mathbf{F}}_{\delta}^{\frac{1}{2}} \quad \text{ with } \sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2} \quad \text{by (3)} \right)$$

provided $m_d \delta < \frac{2d-5}{(d-2)^2}$, where

$$m_d := \inf_{\kappa > 0} \sup_{\substack{x \neq y, \\ \mathrm{Re}\,\zeta > 0}} \frac{|\nabla(\zeta - \Delta)^{-1}(x,y)|}{\left(\kappa^{-1}\mathrm{Re}\,\zeta - \Delta\right)^{-\frac{1}{2}}(x,y)} \tag{5}$$

(m_d is bounded from above by $\pi^{\frac{1}{2}}(2e)^{-\frac{1}{2}} d^{\frac{d}{2}} (d-1)^{\frac{1-d}{2}} < \infty$, see [K, (A.1)]). See Theorem 2 below.



The general classes of drifts σ studied in the literature in connection with the operator $-\Delta + \sigma \cdot \nabla$.
Here $\delta, \delta_1, \delta_2 > 0$. We identify $b : \mathbb{R}^d \to \mathbb{R}^d$ with $b\mathcal{L}^d$.
$\to$ stands for strict inclusion, and $\overset{*}{\to}$ reads "if $b = b_1 + b_2 \in [L^{d,\infty} + L^\infty]^d$, then $b \in \mathbf{F}_{\delta_1^2}$ with $\delta_1 > 0$ determined by the value of the $L^{d,\infty}$-norm of $|b_1|$ (by the Strichartz inequality with sharp constants [KPS, Prop. 2.5, 2.6, Cor. 2.9]).

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EXAMPLE. 1. An example of a $b\mathcal{L}^d \in \mathbf{K}_\delta^{d+1} - \mathbf{K}_0^{d+1}$, $\delta > 0$, can be obtained as follows (modifying [AS, p. 250, Example 1]). Fix $e \in \mathbb{R}^d$, $|e| = 1$. Let $z_n := (2^{-n}, 0, \dots, 0) \in \mathbb{R}^d$, $n \geqslant 1$. Set

$$b(x) := eF(x), \quad F(x) := \sum_{n=1}^{\infty} 8^n \mathbf{1}_{B(z_n, 8^{-n})}(x), \quad x \in \mathbb{R}^d,$$

where $B(z_n, 8^{-n})$ is the open ball of radius 8^{-n} centered at z_n. Arguing as in [AS, p. 250, Example 1], we obtain that $b\mathcal{L}^d \in \mathbf{K}_\delta^{d+1} - \mathbf{K}_0^{d+1}$ for appropriate $\delta > 0$.

2. Recall that a Borel-measurable set $\Gamma \subset \mathbb{R}^d$ is called a κ-set, $0 < \kappa \leqslant d$, if for all $x \in \Gamma$, all $0 < \rho < 1$,

$$c_1 \rho^\kappa \leqslant \mathcal{H}^\kappa(\Gamma \cap B(x, \rho)) \leqslant c_2 \rho^\kappa,$$

for some constants $0 < c_1, c_2 < \infty$, where $\mathcal{H}^\kappa$ is the κ-dimensional Hausdorff measure in $\mathbb{R}^d$ (e.g. $\Gamma = A \times \mathbb{R}$, where A is the Sierpinski gasket in $\mathbb{R}^2$, is a $(1 + \log 3/\log 2)$-set).

Then, for a fixed $e \in \mathbb{R}^d$, $|e| = 1$, if $\Gamma \subset \mathbb{R}^d$ is a κ-set, $\kappa > d - 1$, the measure

$$\sigma := e\mathbf{1}_\Gamma \mathcal{H}^\kappa|_\Gamma \in \bar{\mathbf{K}}_0^{d+1},$$

see [BC, Prop. 2.1].

An example of $\sigma \in \bar{\mathbf{K}}_\delta^{d+1} - \bar{\mathbf{K}}_0^{d+1}$ can be obtained e.g. by modifying the example in 1, e.g. for $d = 3$ as $\sigma := eF\mathbf{1}_\Gamma \mathcal{H}^\kappa|_\Gamma$, where $\Gamma := A \times \mathbb{R}$, $\kappa = 1 + \log 3/\log 2$, $z_n \in \Gamma$ are chosen at the distance of at least 2^{-n} from each other, and the coefficients 8^{-n} in F are replaced with $8^{-(\kappa-d+1)n}$.

REMARKS. After 1996, the Kato class of vector fields $\mathbf{K}_\delta^{d+1}$, with $\delta > 0$ sufficiently small (yet allowed to be non-zero), has been recognized as the general class 'responsible' for the Gaussian upper and lower bounds on the fundamental solution of $-\Delta + b \cdot \nabla$ [S, Z] which, in turn, allow to construct an associated Feller process (in C_b). The class $\mathbf{F}_\delta$, $\delta < 4$, provides dissipativity of $\Delta - b \cdot \nabla$ in L^p, $p \geqslant 2/(2 - \sqrt{\delta})$, needed to run the iterative procedure of [KS] (taking $p \to \infty$, assuming additionally $\delta < \min\{1, (2/(d-2))^2\}$), which produces an associated Feller semigroup in C_∞. We emphasize that, in general, the Gaussian bounds are not valid if $|b| \in L^d$, while $b\mathcal{L}^d \in \mathbf{K}_0^{d+1}$, in general, destroys L^p-dissipativity.

In [K], we constructed an associated with $-\Delta + b \cdot \nabla$ Feller semigroup in C_∞ for $b\mathcal{L}^d \in \mathbf{F}_\delta^{\frac{1}{2}}$, $m_d \delta < 1$. The starting object for the method is an operator-valued function in L^p, $p \in \mathcal{I} := \left(\frac{2}{1+\sqrt{1-m_d\delta}}, \frac{2}{1-\sqrt{1-m_d\delta}}\right)$,

$$\Theta_p(\zeta, b\mathcal{L}^d) := (\zeta - \Delta)^{-1} - (\zeta - \Delta)^{-\frac{1}{2} - \frac{1}{2q}} Q_p(q)(1 + T_p)^{-1} G_p(r)(\zeta - \Delta)^{-\frac{1}{2r'}}, \tag{6}$$
$$1 \leqslant r < p < q, \quad \operatorname{Re} \zeta \geqslant \lambda d/(d-1),$$

where $Q_p(q), T_p, G_p(r) \in \mathcal{B}(L^p)$, $\|T_p\|_{p \to p} \leqslant m_d \frac{pp'}{4} \delta < 1$,

$$G_p(r) := b^{\frac{1}{p}} \cdot \nabla (\zeta - \Delta)^{-\frac{1}{2} - \frac{1}{2r}}, \quad b^{\frac{1}{p}} := b|b|^{\frac{1}{p} - 1},$$

$Q_p(q)$ and T_p are extensions by continuity of densely defined operators

$$Q_p(q) := (\zeta - \Delta)^{-\frac{1}{2q'}} |b|^{\frac{1}{p'}}, \quad T_p := b^{\frac{1}{p}} \cdot \nabla (\zeta - \Delta)^{-1} |b|^{\frac{1}{p'}}.$$

We prove that

$$\Theta_p(\zeta, b\mathcal{L}^d) = (\zeta + \Lambda_p(b))^{-1}, \quad \mathrm{Re}\,\zeta \geqslant \lambda d/(d-1),$$

where $\Lambda_p(b)$ is the generator of a holomorphic C_0-semigroup $e^{-t\Lambda_p(b)}$ on L^p. The proof uses ideas from [BS], and appeals to the L^p-inequalities between the operator $(\lambda - \Delta)^{\frac{1}{2}}$ and the "potential" $|b|$. Then, as follows from the definition of $\Theta_p(\zeta, b\mathcal{L}^d)$, $D(\Lambda_p(b)) \subset \mathcal{W}^{1+\frac{1}{q},p}$, $q > p$. In particular, if $m_d\delta < 4\frac{d-2}{(d-1)^2}$, then there exists $p \in \mathcal{I}$, $p > d-1$, and by the Sobolev embedding theorem $D(\Lambda_p(b)) \subset C^{0,\gamma}$, $\gamma < 1 - \frac{d-1}{p}$. The latter allows us to construct

$$(\mu + \Lambda_{C_\infty}(b))^{-1}|_{\mathcal{S}} := \Theta_p(\mu, b\mathcal{L}^d)|_{\mathcal{S}}, \quad \mu \geqslant \frac{d}{d-1}\lambda, \quad p > d-1,$$

where $\Lambda_{C_\infty}(b)$ is the required generator of a Feller semigroup on C_∞.

For $\sigma \in \bar{\mathbf{F}}_\delta^{\frac{1}{2}}$, $\sigma \not\ll \mathcal{L}^d$ (a subject of this work) $\Theta_p(\zeta, \sigma)$, as in (6), is not well defined. We modify the method in [K]. This modification highlights the fundamental role of the L^2-theory in the C_∞-theory of $-\Delta + \sigma \cdot \nabla$, in particular, the role of the alternative representation of (6) in L^2,

$$\Theta_2(\zeta, \sigma) := (\zeta - \Delta)^{-\frac{3}{4}}(1 + B)^{-1}(\zeta - \Delta)^{-\frac{1}{4}},$$

$$B := (\zeta - \Delta)^{-\frac{1}{4}}\sigma \cdot \nabla(\zeta - \Delta)^{-\frac{3}{4}} \quad \text{(well defined)},$$

used in [S2, Theorem 5.1].

Also, in contrast to the setup of [K], a σ as above doesn't admit a monotone approximation by regular vector fields v_k (i.e. by $v_k\mathcal{L}^d$), which complicates the proof of the convergence $\Theta_2(\zeta, v_k\mathcal{L}^d) \overset{s}{\to} \Theta_2(\zeta, \sigma)$ in L^2, needed to carry out the method. We resolve this using an important variant of the Kato-Ponce inequality by [GO] (Proposition 6 below); there, we also employ a modification of an argument from [SV, proof of Theorem 2.1].

The method depends on the fact that the two operators constituting $-\Delta + \sigma \cdot \nabla$, i.e. $-\Delta$ and ∇, commute; in particular, the method admits a straightforward generalization to fractional powers of the Laplacian (for fundamental results on potential theory of the latter, see [BJ]).

Remark. Our main results (Theorems 1 and 2 below) are valid for $\sigma \in \bar{\mathbf{F}}_\delta^{\frac{1}{2}}$ such that there exist C^∞-smooth approximating vector fields v_k such that

$$v_k\mathcal{L}^d \in \mathbf{F}^{\frac{1}{2}}_{\delta + 2^{-k}}, \quad v_k\mathcal{L}^d \overset{w}{\to} \sigma.$$

We construct these v_k's for σ as in (4) (Proposition 1 below), but do not consider the problem of constructing such an approximation for an arbitrary weakly form-bounded measure σ.

Remark. The symmetry assumption on the generator allows to include drifts of the form: the countable sum of certain (possibly accumulating) hypersurface measures, see [ST].

3. We proceed to precise formulations of our results.

In the next theorem we allow $\mathbb{C}^d$-valued measures (the modification of the definitions 1, 2 is straightforward).

Theorem 1 (L^p-theory)**.** *Let $d \geqslant 3$. Assume that σ is a $\mathbb{C}^d$-valued Borel measure in $\bar{\mathbf{F}}_\delta^{\frac{1}{2}}$ such that $\sigma = b\mathcal{L}^d + \nu$, where $b : \mathbb{R}^d \to \mathbb{C}^d$ is $\mathcal{L}^d$-measurable,*

$$b\mathcal{L}^d \in \mathbf{F}_{\delta_1}^{\frac{1}{2}}, \qquad \nu \in \bar{\mathbf{K}}_{\delta_2}^{d+1}, \qquad \sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2}.$$

There exist vector fields $v_k \in C_0^\infty(\mathbb{R}^d, \mathbb{C}^d)$ such that $v_k\mathcal{L}^d \in \mathbf{F}_{\delta+2^{-k}}^{\frac{1}{2}}$, $v_k\mathcal{L}^d \overset{w}{\to} \sigma$.

If $m_d\delta < 1$, then for every

$$p \in \mathcal{J} := \left(1 + \frac{1}{1+\sqrt{1-m_d\delta}}, 1 + \frac{1}{1-\sqrt{1-m_d\delta}}\right)$$

we have:

(i) There exists a holomorphic C_0-semigroup $e^{-t\Lambda_p(\sigma)}$ in L^p such that, possibly after replacing $v_k\mathcal{L}^d$'s with a sequence of their convex combinations (also weakly converging to measure σ), we have

$$e^{-t\Lambda_p(v_k\mathcal{L}^d)} \overset{s}{\longrightarrow} e^{-t\Lambda_p(\sigma)} \text{ in } L^p,$$

as $k \to \infty$, where

$$\Lambda_p(v_k\mathcal{L}^d) := -\Delta + v_k \cdot \nabla, \quad D(\Lambda_p(v_k\mathcal{L}^d)) = \mathcal{W}^{2,p}.$$

(ii) The resolvent set $\rho(-\Lambda_p(\sigma))$ contains a half-plane $\mathcal{O} \subset \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > 0\}$, and the resolvent $(\zeta + \Lambda_p(\sigma))^{-1}$, $\zeta \in \mathcal{O}$, admits extension by continuity to a bounded linear operator in $\mathcal{B}\left(\mathcal{W}^{-\frac{1}{r'},p}, \mathcal{W}^{1+\frac{1}{q},p}\right)$, where $1 \leqslant r < \min\{2,p\}$, $\max\{2,p\} < q$.

(iii) The domain of the generator $D\big(\Lambda_p(\sigma)\big) \subset \mathcal{W}^{1+\frac{1}{q},p}$ for every $q > \max\{p,2\}$.

Theorem 1 allows us to prove

Theorem 2 (C_∞-theory)**.** *Let $d \geqslant 3$. Assume that σ is a $\mathbb{R}^d$-valued Borel measure in $\bar{\mathbf{F}}_\delta^{\frac{1}{2}}$ such that $\sigma = b\mathcal{L}^d + \nu$, where $b : \mathbb{R}^d \to \mathbb{R}^d$ is $\mathcal{L}^d$-measurable,*

$$b\mathcal{L}^d \in \mathbf{F}_{\delta_1}^{\frac{1}{2}}, \qquad \nu \in \bar{\mathbf{K}}_{\delta_2}^{d+1}, \qquad \sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2}.$$

There exist vector fields $v_k \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ such that $v_k\mathcal{L}^d \in \mathbf{F}_{\delta+2^{-k}}^{\frac{1}{2}}$, $v_k\mathcal{L}^d \overset{w}{\to} \sigma$.

If $m_d\delta < \frac{2d-5}{(d-2)^2}$, then:

(i) There exists a positivity preserving contraction C_0-semigroup $e^{-t\Lambda_{C_\infty}(\sigma)}$ on C_∞ such that, possibly after replacing $v_k\mathcal{L}^d$'s with a sequence of their convex combinations (also weakly converging to measure σ) we have

$$e^{-t\Lambda_{C_\infty}(v_k\mathcal{L}^d)} \overset{s}{\longrightarrow} e^{-t\Lambda_{C_\infty}(\sigma)} \text{ in } C_\infty, \quad t \geqslant 0,$$

as $k \to \infty$, where

$$\Lambda_{C_\infty}(v_k\mathcal{L}^d) := -\Delta + v_k \cdot \nabla, \quad D(\Lambda_{C_\infty}(v_k\mathcal{L}^d)) = (1-\Delta)^{-1}C_\infty.$$

(ii) [L^p-strong Feller property] $(\mu + \Lambda_{C_\infty}(\sigma))^{-1}|_{\mathcal{S}}$, $\mu > 0$, can be extended by continuity to a bounded linear operator in $\mathcal{B}(L^p, C^{0,\gamma})$, $\gamma < 1 - \frac{d-1}{p}$, for every $d-1 < p < 1 + \frac{1}{1-\sqrt{1-m_d\delta}}$.

(iii) The integral kernel $e^{-t\Lambda_{C_\infty}(\sigma)}(x,y)$ $(x, y \in \mathbb{R}^d)$ of $e^{-t\Lambda_{C_\infty}(\sigma)}$ determines the (sub-Markov) transition probability function of a Feller process.

Remark. If $\sigma \ll \mathcal{L}^d$, then the interval $\mathcal{J}$ in Theorem 1 can be expanded; accordingly, the assumption on δ in Theorem 2 can be relaxed, cf. [K, Theorems 1, 2]. There, we work directly in L^p, while in the proof of Theorem 1 we have to first prove our convergence results in L^2, and then transfer them to L^p (Proposition 8), which leads to more restrictive constraints on p and, respectively, δ.

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Proofs of Theorem 1 and Theorem 2

In the proofs of both Theorem 1 and Theorem 2 we use the following

Proposition 1. *Let $\sigma = b\mathcal{L}^d + \nu$, where $b : \mathbb{R}^d \to \mathbb{C}^d$ is $\mathcal{L}^d$-measurable,*

$$b\mathcal{L}^d \in \mathbf{F}_{\delta_1}^{\frac{1}{2}}, \qquad \nu \in \bar{\mathbf{K}}_{\delta_2}^{d+1}, \qquad \sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2}.$$

Then $\sigma \in \bar{\mathbf{F}}_{\delta}^{\frac{1}{2}}$, and there exist vector fields $v_k \in C_0^\infty(\mathbb{R}^d, \mathbb{C}^d)$ such that

$$v_k\mathcal{L}^d \in \mathbf{F}_{\delta+2^{-k}}^{\frac{1}{2}}, \quad \sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2},$$

$$v_k\mathcal{L}^d \overset{w}{\to} \sigma \text{ as } k \to \infty.$$

If σ is $\mathbb{R}^d$-valued, then v_k's are $\mathbb{R}^d$-valued.

Proof. First, let us construct the vector fields v_k. We fix functions $\rho_k \in C_0^\infty$, $0 \leqslant \rho_k \leqslant 1$, $\rho \equiv 1$ in $\{x \in \mathbb{R}^d : |x| \leqslant k\}$, $\rho \equiv 0$ in $\{x \in \mathbb{R}^d : |x| \geqslant k+1\}$, and a sequence $\varepsilon_k \downarrow 0$. We define

$$\nu_k := \rho_k e^{\varepsilon_k \Delta}\nu, \quad b_k := \rho_k e^{\varepsilon_k \Delta}\mathbf{1}_k b,$$

where $\mathbf{1}_k := \mathbf{1}_{\{x \in \mathbb{R}^d : |x| \leqslant k, |b(x)| \leqslant k\}}$, and

$$v_k\mathcal{L}^d := b_k\mathcal{L}^d + \nu_k\mathcal{L}^d.$$

Clearly, $v_k \in C_0^\infty(\mathbb{R}^d, \mathbb{C}^d)$ and $v_k\mathcal{L}^d \overset{w}{\longrightarrow} \sigma$ as $k \to \infty$.

Let us prove that $v_k\mathcal{L}^d \in \mathbf{F}_{\delta+2^{-k}}^{\frac{1}{2}}$. First, let us show that $\nu_k\mathcal{L}^d \in \mathbf{K}_{\delta_2}^{d+1}$. We have a.e. on $\mathbb{R}^d$:

$$(\lambda - \Delta)^{-\frac{1}{2}}|\nu_k|_1 \leqslant (\lambda - \Delta)^{-\frac{1}{2}}|e^{\varepsilon_k\Delta}\nu|_1 \leqslant (\lambda - \Delta)^{-\frac{1}{2}}e^{\varepsilon_k\Delta}|\nu|_1 = e^{\varepsilon_k\Delta}(\lambda - \Delta)^{-\frac{1}{2}}|\nu|_1.$$

Since $\|e^{\varepsilon_k\Delta}(\lambda - \Delta)^{-\frac{1}{2}}|\nu|_1\|_\infty \leqslant \|(\lambda - \Delta)^{-\frac{1}{2}}|\nu|_1\|_\infty$ and, in turn, $\|(\lambda - \Delta)^{-\frac{1}{2}}|\nu|_1\|_\infty \leqslant \delta_2$ ($\Leftrightarrow \nu \in \bar{\mathbf{K}}_{\delta_2}^{d+1}$), we have $\nu_k\mathcal{L}^d \in \mathbf{K}_{\delta_2}^{d+1}$. Now, since $\mathbf{K}_{\delta_2}^{d+1} \subset \mathbf{F}_{\delta_2}^{\frac{1}{2}}$ (cf. (2)), we have $\nu_k\mathcal{L}^d \in \mathbf{F}_{\delta_2}^{\frac{1}{2}}$. Next, since $\mathbf{1}_k|b| \leqslant |b|$, we have $b_k\mathcal{L}^d \in \mathbf{F}_{\delta_1+8^{-k}}^{\frac{1}{2}}$ (possibly, after selecting smaller $\varepsilon_k \downarrow 0$). Thus, $v_k\mathcal{L}^d \in \mathbf{F}_{\delta+2^{-k}}^{\frac{1}{2}}$, as needed.

The latter, and the convergence $v_k\mathcal{L}^d \overset{w}{\to} \sigma$, imply that $\sigma \in \bar{\mathbf{F}}_{\delta}^{\frac{1}{2}}$. □

Proof of Theorem 1. Due to the strict inequality $m_d\delta<1$, we may assume that the infimum m_d (cf. (5)) is attained, i.e. there is $\kappa_d>0$ such that

$$|\nabla(\zeta-\Delta)^{-1}(x,y)|\leqslant m_d\Big(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta\Big)^{-\frac{1}{2}}(x,y),\qquad x,y\in\mathbb{R}^d,\,x\neq y,\,\mathrm{Re}\,\zeta>0,$$

Set

$$\mathcal{O}:=\{\zeta\in\mathbb{C}:\mathrm{Re}\,\zeta\geqslant\kappa_d\lambda_\delta\}.$$

We fix σ as in the formulation of the theorem. In view of the strict strict inequality $m_d\delta<1$, we may assume that the approximating vector fields $v_k:\mathbb{R}^d\to\mathbb{C}^d$ in Proposition 1 are in $\mathbf{F}_\delta^{\frac{1}{2}}$.

The method of proof. Let us fix $p\in\mathcal{J}$ and r,q satisfying $1\leqslant r<\min\{2,p\},\max\{2,p\}<q$. Our starting object is an operator-valued function

$$\Theta_p(\zeta,\sigma):=(\zeta-\Delta)^{-\frac{1}{2}-\frac{1}{2q}}\Omega_p(\zeta,\sigma,q,r)(\zeta-\Delta)^{-\frac{1}{2r'}}\in\mathcal{B}(L^p),\quad\zeta\in\mathcal{O},$$

which is 'a candidate' for the resolvent of the desired operator realization $\Lambda_p(\sigma)$ of $-\Delta+\sigma\cdot\nabla$ on L^p. Here

$$\Omega_p(\zeta,\sigma,q,r):=\Big(\Omega_2(\zeta,\sigma,q,r)\Big|_{L^p\cap L^2}\Big)^{\mathrm{clos}}_{L^p}\in\mathcal{B}(L^p)\tag{7}$$

($(\cdot)^{\mathrm{clos}}_{L^p}$ denotes the extension of an operator by continuity to L^p), where, on L^2,

$$\Omega_2(\zeta,\sigma,q,r):=(\zeta-\Delta)^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}(1+Z_2(\zeta,\sigma))^{-1}(\zeta-\Delta)^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{r'}\right)}\in\mathcal{B}(L^2),$$

$$\begin{aligned}Z_2(\zeta,\sigma)h(x)&:=(\zeta-\Delta)^{-\frac{1}{4}}\sigma\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}h(x)\\&=\int_{\mathbb{R}^d}(\zeta-\Delta)^{-\frac{1}{4}}(x,y)\bigg(\int_{\mathbb{R}^d}\nabla(\zeta-\Delta)^{-\frac{3}{4}}(y,z)h(z)dz\bigg)\cdot\sigma(y)dy,\quad x\in\mathbb{R}^d,\quad h\in\mathcal{S},\end{aligned}$$

and $\|Z_2\|_{2\to2}<1$, so $\Omega_2(\zeta,\sigma,q,r)\in\mathcal{B}(L^2)$, see Proposition 2 below. We prove that $\Omega_p(\zeta,\sigma,q,r)\in\mathcal{B}(L^p)$ in Proposition 7 below.

We show that $\Theta_p(\zeta,\sigma)$ is the resolvent of $\Lambda_p(\sigma)$ (assertion (i) of Theorem 1) by verifying conditions of the Trotter approximation theorem:

1) $\Theta_p(\zeta,v_k\mathcal{L}^d)=(\zeta+\Lambda_p(v_k\mathcal{L}^d))^{-1}$, $\zeta\in\mathcal{O}$, where $\Lambda_p(v_k\mathcal{L}^d):=-\Delta+v_k\cdot\nabla$, $D(\Lambda_p(v_k\mathcal{L}^d))=\mathcal{W}^{2,p}$.

2) $\sup_{n\geqslant1}\|\Theta_p(\zeta,v_k\mathcal{L}^d)\|_{p\to p}\leqslant C_p|\zeta|^{-1}$, $\zeta\in\mathcal{O}$.

3) $\mu\Theta_p(\zeta,v_k\mathcal{L}^d)\overset{s}{\to}1$ in L^p as $\mu\uparrow\infty$ uniformly in k.

4) $\Theta_p(\zeta,v_k\mathcal{L}^d)\overset{s}{\to}\Theta_p(\zeta,\sigma)$ in L^p for some $\zeta\in\mathcal{O}$ as $k\to\infty$ (possibly after replacing $v_k\mathcal{L}^d$'s with a sequence of their convex combinations, also weakly converging to measure σ), see Propositions 3 - 8 below for details.

We note that a priori in 1) the set of ζ's for which $\Theta_p(\zeta,v_k\mathcal{L}^d)=(\zeta+\Lambda_p(v_k\mathcal{L}^d))^{-1}$ could depend on k; the fact that it does not is the content of Proposition 4.

The proofs of 2), 3), contained in Proposition 3 and Proposition 5, are based on an explicit representation of $\Omega_p(\zeta,v_k\mathcal{L}^d,q,r)$, $k\geqslant1$, that doesn't exist if σ has a non-zero singular part.

Next, 4) follows from $\Theta_2(\zeta, v_k\mathcal{L}^d) \overset{s}{\to} \Theta_2(\zeta,\sigma)$, combined with $\sup_n \|\Theta_p(\zeta, v_k\mathcal{L}^d)\|_{2(p-1)\to 2(p-1)} < \infty$ ($\Leftarrow$ 2)) and Hölder's inequality, see Proposition 8. Our proof of $\Theta_2(\zeta, v_k\mathcal{L}^d) \overset{s}{\to} \Theta_2(\zeta,\sigma)$ (Proposition 6) uses the Kato-Ponce inequality by [GO].

Finally, we note that the very definition of the operator-valued function $\Theta_p(\zeta,\sigma)$ implies that it admits extension to an operator-valued function in $\mathcal{B}\big(\mathcal{W}^{-\frac{1}{r'},p}, \mathcal{W}^{1+\frac{1}{q},p}\big) \Rightarrow$ assertion (ii). Assertion (iii) is immediate from (ii).

We proceed to formulating and proving Propositions 2-8.

Proposition 2. *We have, for every $\zeta \in \mathcal{O}$,*

(1) $\|Z_2(\zeta, v_k\mathcal{L}^d)\|_{2\to 2} \leqslant \delta$ *for all* k.

(2) $\|Z_2(\zeta,\sigma)f\|_2 \leqslant \delta\|f\|_2$, *for all* $f \in \mathcal{S}$, *all* k.

Proof. (1) Define $H := |v_k|^{\frac{1}{2}}(\zeta-\Delta)^{-\frac{1}{4}}$, $S := v_k^{\frac{1}{2}} \cdot \nabla(\zeta-\Delta)^{-\frac{3}{4}}$, where $v_k^{\frac{1}{2}} := |v_k|^{-\frac{1}{2}}v_k$. Then $Z_2(\zeta, v_k\mathcal{L}^d) = H^*S$, and we have

$$\|Z_2(\zeta, v_k\mathcal{L}^d)\|_{2\to 2} \leqslant \|H\|_{2\to 2}\|S\|_{2\to 2} \leqslant \|H\|_{2\to 2}^2 \|\nabla(\zeta-\Delta)^{-\frac{1}{2}}\|_{2\to 2} \leqslant \delta,$$

where we have used $\|\nabla(\zeta-\Delta)^{-\frac{1}{2}}\|_{2\to 2} = 1$, and

$$\begin{aligned}
&\|H\|_{2\to 2} \quad \text{(here we are using } |v_k| \leqslant |v_k|_1\text{)}\\
&\leqslant \||v_k|_1^{\frac{1}{2}}(\zeta-\Delta)^{-\frac{1}{4}}\|_{2\to 2}\\
&\text{(here we are using } |(\zeta-\Delta)^{-1}(x,y)| \leqslant |(\operatorname{Re}\zeta-\Delta)^{-1}(x,y)|,\ x,y \in \mathbb{R}^d,\ x \neq y\text{)}\\
&\leqslant \sqrt{\delta} \quad \text{(since } v_k\mathcal{L}^d \in \mathbf{F}_\delta^{\frac{1}{2}}\text{)}.
\end{aligned}$$

(2) We have, for every f, $g \in \mathcal{S}$,

$$\begin{aligned}
\big\langle g, Z_2(\zeta,\sigma)f\big\rangle =& \big\langle (\zeta-\Delta)^{-\frac{1}{4}}g, \sigma\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}f\big\rangle\\
&\text{(here we are using } v_k\mathcal{L}^d \overset{w}{\to} \sigma\text{)}\\
&= \lim_k \big\langle (\zeta-\Delta)^{-\frac{1}{4}}g, v_k\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}f\big\rangle\\
&\text{(here we are using assertion (1))}\\
&\leqslant \delta\|g\|_2\|f\|_2,
\end{aligned}$$

i.e. $\|Z_2(\zeta,\sigma)f\|_2 \leqslant \delta\|f\|_2$, as needed. □

The extension of $Z_2(\zeta,\sigma)|_{\mathcal{S}}$ by continuity to a bounded linear operator in $\mathcal{B}(L^2)$ will be denoted again by $Z_2(\zeta,\sigma)$. Since $\|Z_2(\zeta, v_k\mathcal{L}^d)\|_{2\to 2}, \|Z_2(\zeta,\sigma)\|_{2\to 2} \leqslant \delta$ (< 1), we have $\Omega_2(\zeta, v_k\mathcal{L}^d, q, r)$, $\Omega_2(\zeta,\sigma,q,r) \in \mathcal{B}(L^2)$.

Set

$$\mathcal{I} := \left(\frac{2}{1+\sqrt{1-m_d\delta}}, \frac{2}{1-\sqrt{1-m_d\delta}}\right).$$

In the following propositions, given a $p \in \mathcal{I}$, we assume that r, q satisfy $1 \leqslant r < \min\{2,p\}$, $\max\{2,p\} < q$.

The following proposition plays a principal role:

Proposition 3. *Let $p \in \mathcal{I}$. There exist constants C_p, $C_{p,q,r} < \infty$ such that, for every $\zeta \in \mathcal{O}$,*

(1) $\|\Omega_p(\zeta, v_k\mathcal{L}^d, q, r)\|_{p\to p} \leqslant C_{p,q,r}$ *for all k,*

(2) $\|\Omega_p(\zeta, v_k\mathcal{L}^d, \infty, 1)\|_{p\to p} \leqslant C_p|\zeta|^{-\frac{1}{2}}$ *for all k.*

Proof. Denote $v_k^{\frac{1}{p}} := |v_k|^{\frac{1}{p}-1}v_k$. Set

$$\tilde{\Omega}_p(\zeta, v_k\mathcal{L}^d, q, r) := (\zeta-\Delta)^{\frac{1}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} - Q_p(q)(1+T_p)^{-1}G_p(r), \quad \zeta \in \mathcal{O}, \tag{8}$$

where

$$Q_p(q) := (\zeta-\Delta)^{-\frac{1}{2q'}}|v_k|^{\frac{1}{p'}}, \quad T_p := v_k^{\frac{1}{p}} \cdot \nabla(\zeta-\Delta)^{-1}|v_k|^{\frac{1}{p'}}, \quad G_p(r) := v_k^{\frac{1}{p}} \cdot \nabla(\zeta-\Delta)^{-\frac{1}{2}-\frac{1}{2r}},$$

are uniformly in k bounded in $\mathcal{B}(L^p)$ (see the proof of [K, Prop. 1(i)]); in particular, $\|T_p\|_{p\to p} \leqslant \frac{pp'}{4}m_d\delta$, where $\frac{pp'}{4}m_d\delta < 1$ since $p \in \mathcal{I}$. Therefore, $C_{p,q,r} := \sup_k \|\tilde{\Omega}_p(\zeta, v_k\mathcal{L}^d, q, r)\|_{p\to p} < \infty$. Now, $\tilde{\Omega}_p(\zeta, v_k\mathcal{L}^d, q, r)|_{L^2\cap L^p} = \Omega_2(\zeta, v_k\mathcal{L}^d, q, r)|_{L^2\cap L^p}$ (by expanding $(1+T_p)^{-1}$, $(1+Z_2)^{-1}$ in the K. Neumann series in L^p and in L^2, respectively). Therefore, $\tilde{\Omega}_p(\zeta, v_k\mathcal{L}^d, q, r) = \Omega_p(\zeta, v_k\mathcal{L}^d, q, r) \Rightarrow$ assertion (1). The proof of assertion (2) follows closely the proof of [K, Prop. 1(ii)]. □

Clearly, $\Theta_p(\zeta, v_k\mathcal{L}^d)$ does not depend on q, r. Taking $q = \infty$, $r = 1$, by Proposition 3 we obtain:

$$\|\Theta_p(\zeta, v_k\mathcal{L}^d)\|_{p\to p} \leqslant C_p|\zeta|^{-1}, \quad \zeta \in \mathcal{O}. \tag{9}$$

Proposition 4. *Let $p \in \mathcal{I}$. For every $k \geqslant 1$ $\mathcal{O} \subset \rho(-\Lambda_p(v_k\mathcal{L}^d))$, the resolvent set of $-\Lambda_p(v_k\mathcal{L}^d)$, and*

$$\Theta_p(\zeta, v_k\mathcal{L}^d) = (\zeta + \Lambda_p(v_k\mathcal{L}^d))^{-1}, \quad \zeta \in \mathcal{O},$$

where $\Lambda_p(v_k\mathcal{L}^d) := -\Delta + v_k \cdot \nabla$, $D(\Lambda_{C_\infty}(v_k\mathcal{L}^d)) = \mathcal{W}^{2,p}$.

Proof. The proof repeats the proof of [K, Prop. 4]. □

Proposition 5. *For $p \in \mathcal{I}$, $\mu\Theta_p(\mu, v_k\mathcal{L}^d) \overset{s}{\to} 1$ in L^p as $\mu \uparrow \infty$ uniformly in k.*

Proof. The proof repeats the proof of [K, Prop. 3]. □

Proposition 6. *There exists a sequence $\{\hat{v}_n\} \subset \operatorname{conv}\{v_k\}$ $(\subset C_0^\infty(\mathbb{R}^d, \mathbb{C}^d))$ such that*

$$\hat{v}_n\mathcal{L}^d \overset{w}{\longrightarrow} \sigma \text{ as } n \to \infty, \tag{10}$$

and

$$\Omega_2(\zeta, \hat{v}_n\mathcal{L}^d, q, r) \overset{s}{\to} \Omega_2(\zeta, \sigma, q, r) \text{ in } L^2, \quad \zeta \in \mathcal{O}. \tag{11}$$

Proof. To prove (11), it suffices to establish the convergence $Z_2(\zeta, \hat{v}_n\mathcal{L}^d) \overset{s}{\to} Z_2(\zeta, \sigma)$ in L^2.

Let $\eta_r \in C_0^\infty$, $0 \leqslant \eta_r \leqslant 1$, $\eta_r \equiv 1$ on $\{x \in \mathbb{R}^d : |x| \leqslant r\}$ and $\eta_r \equiv 0$ on $\{x \in \mathbb{R}^d : |x| \geqslant r+1\}$.

Claim 1. *We have, for every $\mu \geqslant \lambda_\delta$,*

(j) $\|(\mu-\Delta)^{-\frac{1}{4}}|v_k|_1(\mu-\Delta)^{-\frac{1}{4}}\|_{2\to 2} \leqslant \delta$, *for all k.*

(jj) $\|(\mu-\Delta)^{-\frac{1}{4}}|\sigma|_1(\mu-\Delta)^{-\frac{1}{4}}f\|_2 \leqslant \delta\|f\|_2$, *for all $f \in \mathcal{S}$.*

Proof. Define $H := |v_k|_1^{\frac{1}{2}}(\mu-\Delta)^{-\frac{1}{4}}$. We have

$$\|(\mu-\Delta)^{-\frac{1}{4}}|v_k|_1(\mu-\Delta)^{-\frac{1}{4}}\|_{2\to 2} = \|H^*H\|_{2\to 2} = \|H\|^2_{2\to 2} \leqslant \delta,$$

where $\|H\|^2_{2\to 2} \leqslant \delta$ since $v_k\mathcal{L}^d \in \mathbf{F}_\delta^{\frac{1}{2}}$, i.e. we have proved (j). An argument similar to the one in the proof of Proposition 2, but using assertion (j), yields (jj). □

Claim 2. There exists a sequence $\{\hat{v}_n\} \subset \mathrm{conv}\{v_k\}$ such that (10) holds, and for every $r \geqslant 1$

$$(\zeta-\Delta)^{-\frac{1}{4}}\eta_r(\hat{v}_n-\sigma)\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}} \overset{s}{\to} 0 \textit{ in } L^2, \quad \operatorname{Re}\zeta \geqslant \lambda_\delta.$$

(here and below we use the shorthand $\hat{v}_n - \sigma := \hat{v}_n\mathcal{L}^d - \sigma$).

Proof of Claim 2. In view of Claim 1(j), (jj), it suffices to establish this convergence over $\mathcal{S}$.

Fix some $\mu \geqslant \lambda_\delta$. Set $c(x) := e^{-x^2}$. Clearly, $c \in \mathcal{S}$, $|(\mu-\Delta)^{-\frac{1}{4}}c| > 0$ on $\mathbb{R}^d$.

Step 1. Let $r = 1$. Let us show that there exists a sequence $\{v^1_{\ell_1}\} \subset \mathrm{conv}\{v_k\}$ such that

$$(\mu-\Delta)^{-\frac{1}{4}}\eta_1(v^1_{\ell_1}-\sigma)\cdot\nabla(\mu-\Delta)^{-\frac{3}{4}} \overset{s}{\to} 0 \text{ in } L^2 \text{ as } \ell_1 \to \infty. \tag{12}$$

First, we show that

$$(\mu-\Delta)^{-\frac{1}{4}}\eta_1(v_k-\sigma)(\mu-\Delta)^{-\frac{1}{4}}c \overset{w}{\to} 0 \text{ in } L^2. \tag{13}$$

Indeed, by Claim 1(j), (jj), $\|(\mu-\Delta)^{-\frac{1}{4}}\eta_1(v_k-\sigma)(\mu-\Delta)^{-\frac{1}{4}}c\|_2 \leqslant 2\delta\|c\|_2$ for all k. Hence, there exists a subsequence of $\{v_k\}$ (without loss of generality, it is $\{v_k\}$ itself) such that $(\mu-\Delta)^{-\frac{1}{4}}\eta_1(v_k-\sigma)(\mu-\Delta)^{-\frac{1}{4}}c \overset{w}{\to} h$ for some $h \in L^2$. Therefore, given any $f \in \mathcal{S}$, we have $\langle f, (\mu-\Delta)^{-\frac{1}{4}}\eta_1(v_k-\sigma)(\mu-\Delta)^{-\frac{1}{4}}c\rangle \to \langle f, h\rangle$. Along with that, since $v_k\mathcal{L}^d \overset{w}{\to} \sigma$, we also have

$$\langle f, (\mu-\Delta)^{-\frac{1}{4}}\eta_1(v_k-\sigma)(\mu-\Delta)^{-\frac{1}{4}}c\rangle = \langle(\mu-\Delta)^{-\frac{1}{4}}f, \eta_1(v_k-\sigma)(\mu-\Delta)^{-\frac{1}{4}}c\rangle \to 0,$$

i.e. $\langle f, h\rangle = 0$. Since $f \in \mathcal{S}$ was arbitrary, we have $h = 0$, which yields (13).

Now, in view of (13), by Mazur's Theorem, there exists a sequence $\{v^1_{\ell_1}\} \subset \mathrm{conv}\{v_k\}$ such that

$$(\mu-\Delta)^{-\frac{1}{4}}\eta_1(v^1_{\ell_1}-\sigma)(\mu-\Delta)^{-\frac{1}{4}}c \overset{s}{\to} 0 \text{ in } L^2. \tag{14}$$

We may assume without loss of generality that each $v^1_{\ell_1} \in \mathrm{conv}\{v_n\}_{n\geqslant\ell_1}$.

Next, set $\ell := \ell_1$, $\varphi_\ell := \eta_1(v^1_\ell - \sigma)$, $\Phi := (\mu-\Delta)^{-\frac{1}{4}}c$, fix some $u \in \mathcal{S}$. We estimate (cf. [SV, proof of Theorem 2.1]):

$$
\begin{aligned}
&\|(\mu-\Delta)^{-\frac{1}{4}}\varphi_\ell \cdot \nabla(\mu-\Delta)^{-\frac{3}{4}}u\|_2^2 \\
&= \Big\langle \varphi_\ell \cdot \nabla(\mu-\Delta)^{-\frac{3}{4}}u, (\mu-\Delta)^{-\frac{1}{2}}\varphi_\ell \cdot \nabla(\mu-\Delta)^{-\frac{3}{4}}u \Big\rangle \\
&\left(\text{since } \varphi_\ell \equiv 0 \text{ on } \{|x| \geqslant 2\}, \text{ in the left multiple } \varphi_\ell = \varphi_\ell \Phi \frac{\eta_2}{\Phi}\right) \\
&= \Big\langle \varphi_\ell \Phi \frac{\eta_2}{\Phi} \cdot \nabla(\mu-\Delta)^{-\frac{3}{4}}u, (\mu-\Delta)^{-\frac{1}{2}}\varphi_\ell \cdot \nabla(\mu-\Delta)^{-\frac{3}{4}}u \Big\rangle \\
&= \Big\langle \varphi_\ell \Phi, \frac{\eta_2}{\Phi}\nabla(\mu-\Delta)^{-\frac{3}{4}}u \Big[(\mu-\Delta)^{-\frac{1}{2}}\varphi_\ell \cdot \nabla(\mu-\Delta)^{-\frac{3}{4}}u\Big] \Big\rangle \\
&\qquad \text{(here we are using in the left multiple that } \varphi_\ell = (\mu-\Delta)^{\frac{1}{4}}(\mu-\Delta)^{-\frac{1}{4}}\varphi_\ell) \\
&= \Big\langle (\mu-\Delta)^{-\frac{1}{4}}\varphi_\ell \Phi, (\mu-\Delta)^{\frac{1}{4}}(fg_\ell) \Big\rangle
\end{aligned}
$$

where we set $f := \frac{\eta_2}{\Phi}\nabla(\mu-\Delta)^{-\frac{3}{4}}u \in C_0^\infty(\mathbb{R}^d, \mathbb{C}^d)$, $g_\ell := (\mu-\Delta)^{-\frac{1}{2}}\varphi_\ell \cdot \nabla(\mu-\Delta)^{-\frac{3}{4}}u \in (\mu-\Delta)^{-\frac{1}{4}}L^2$ (in view of Claim 1(j), (jj)). Thus, in view of the above estimates,

$$
\|(\mu-\Delta)^{-\frac{1}{4}}\varphi_\ell \cdot \nabla(\mu-\Delta)^{-\frac{3}{4}}u\|_2^2 \leqslant \|(\mu-\Delta)^{-\frac{1}{4}}\varphi_\ell\Phi\|_2 \|(\mu-\Delta)^{\frac{1}{4}}(fg_\ell)\|_2.
$$

By the Kato-Ponce inequality of [GO, Theorem 1],

$$
\|(\mu-\Delta)^{\frac{1}{4}}(fg_\ell)\|_2 \leqslant C\bigg(\|f\|_\infty \|(\mu-\Delta)^{\frac{1}{4}}g_\ell\|_2 + \|(\mu-\Delta)^{\frac{1}{4}}f\|_\infty \|g_\ell\|_2\bigg),
$$

for some $C = C(d) < \infty$. Clearly, $\|f\|_\infty$, $\|(\mu-\Delta)^{\frac{1}{4}}f\|_\infty < \infty$, and $\|(\mu-\Delta)^{\frac{1}{4}}g_\ell\|_2$, $\|g_\ell\|_2$ are uniformly (in ℓ) bounded from above according to Claim 1(j), (jj). Thus, in view of (14), we obtain (12) (recalling that $\ell_1 = \ell$, and $\varphi_{\ell_1} = \eta_1(v^1_{\ell_1} - \sigma)$).

Step 2. Now, we can repeat the argument of Step 1, but starting with sequence $\{v^1_{\ell_1}\}$ in place of $\{v_l\}$, thus obtaining a sequence $\{v^2_{\ell_2}\} \subset \operatorname{conv}\{v^1_{\ell_1}\}$ such that

$$
(\mu-\Delta)^{-\frac{1}{4}}\eta_2(v^2_{\ell_2} - \sigma) \cdot \nabla(\mu-\Delta)^{-\frac{3}{4}} \overset{s}{\to} 0 \text{ in } L^2 \text{ as } \ell_2 \to \infty.
$$

We may assume without loss of generality that each $v^2_{\ell_2} \in \operatorname{conv}\{v^1_{\ell_1}\}_{\ell_1 \geqslant \ell_2}$. Therefore, we also have

$$
(\mu-\Delta)^{-\frac{1}{4}}\eta_1(v^2_{\ell_2} - \sigma) \cdot \nabla(\mu-\Delta)^{-\frac{3}{4}} \overset{s}{\to} 0 \text{ in } L^2 \text{ as } \ell_2 \to \infty.
$$

Repeating this procedure $n-2$ times, we obtain a sequence $\{v^n_{\ell_n}\} \subset \operatorname{conv}\{v^{n-1}_{\ell_{n-1}}\}$ ($\subset \operatorname{conv}\{v_k\}$) such that

$$
(\mu-\Delta)^{-\frac{1}{4}}\eta_i(v^n_{\ell_n} - \sigma) \cdot \nabla(\mu-\Delta)^{-\frac{3}{4}} \overset{s}{\to} 0 \text{ in } L^2 \text{ as } \ell_n \to \infty, \quad 1 \leqslant i \leqslant n.
$$

Step 3. We set $\hat{v}_n := v^n_{\ell_n}$, $n \geqslant 1$, so for every $r \geqslant 1$

$$
(\mu-\Delta)^{-\frac{1}{4}}\eta_r(\hat{v}_n - \sigma) \cdot \nabla(\mu-\Delta)^{-\frac{3}{4}} \overset{s}{\to} 0 \text{ in } L^2. \tag{15}
$$

Since $v^n_{\ell_n} \in \operatorname{conv}\{v^{n-1}_{\ell_{n-1}}\}_{\ell_{n-1} \geqslant \ell_n}$, $v^{n-1}_{\ell_{n-1}} \in \operatorname{conv}\{v^{n-2}_{\ell_{n-2}}\}_{\ell_{n-2} \geqslant \ell_{n-1}}$, etc, we obtain that $v^n_{\ell_n} \in \operatorname{conv}\{v_k\}_{k \geqslant \ell_n}$, i.e. we also have (10). Finally, (15), combined with the resolvent identity, yield

$$
(\zeta-\Delta)^{-\frac{1}{4}}\eta_r(\hat{v}_n - \sigma) \cdot \nabla(\zeta-\Delta)^{-\frac{3}{4}} \overset{s}{\to} 0 \text{ in } L^2, \quad \operatorname{Re}\zeta \geqslant \lambda_\delta.
$$

i.e. we have proved Claim 2. □

We are in a position to complete the proof of Proposition 6. Let us show that, for every $\zeta \in \mathcal{O}$,

$$Z_2(\zeta, \hat{v}_n \mathcal{L}^d)g - Z_2(\zeta,\sigma)g = (\zeta-\Delta)^{-\frac{1}{4}}(\hat{v}_n - \sigma)\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}g \overset{s}{\to} 0 \text{ in } L^2, \quad g \in \mathcal{S}.$$

Let us fix some $g \in \mathcal{S}$. We have

$$\begin{aligned}(\zeta-\Delta)^{-\frac{1}{4}}(\hat{v}_n - \sigma)\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}g &= (\zeta-\Delta)^{-\frac{1}{4}}(\hat{v}_n - \eta_r\hat{v}_n)\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}g \\ &+ (\zeta-\Delta)^{-\frac{1}{4}}(\eta_r\hat{v}_n - \eta_r\sigma)\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}g \\ &+ (\zeta-\Delta)^{-\frac{1}{4}}(\eta_r\sigma - \sigma)\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}g =: I_{1,r,n} + I_{2,r,n} + I_{3,r}.\end{aligned}$$

Claim 3. *Given any $\varepsilon > 0$, there exists r such that $\|I_{3,r}\|_2$, $\|I_{1,r,n}\|_2 < \varepsilon$, for all n, $\zeta \in \mathcal{O}$.*

Proof of Claim 3. It suffices to prove $\|I_{1,r,n}\|_2 < \varepsilon$ for all n. We will need the following elementary estimate: $|\nabla(\zeta-\Delta)^{-\frac{3}{4}}(x,y)| \leqslant M_d(\kappa_d^{-1}\mathrm{Re}\,\zeta - \Delta)^{-\frac{1}{4}}(x,y)$, $x,y \in \mathbb{R}^d$, $x \neq y$, for some $M_d < \infty$ (cf. [K, Appendix A]). We have

$$\begin{aligned}\|I_{1,r,n}\|_2 &= \|(\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}(1-\eta_r)\hat{v}_n\cdot\nabla(\mathrm{Re}\,\zeta-\Delta)^{-\frac{3}{4}}g\|_2 \\ &\leqslant c_d M_d\|(\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}(1-\eta_r)|\hat{v}_n|(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g\|_2 \\ &\leqslant c_d M_d\big\|(\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}|\hat{v}_n|^{\frac{1}{2}}\big\|_{2\to2}\big\|(1-\eta_r)|\hat{v}_n|^{\frac{1}{2}}(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g\big\|_2\end{aligned}$$

We have $\big\|(\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}|\hat{v}_n|^{\frac{1}{2}}\big\|_{2\to2} \leqslant \delta$ since (by construction) $\hat{v}_n\mathcal{L}^d \in \mathbf{F}_\delta^{\frac{1}{2}}$. In turn,

$$\begin{aligned}&(1-\eta_r)|\hat{v}_n|^{\frac{1}{2}}(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g \\ &\qquad = |\hat{v}_n|^{\frac{1}{2}}(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}\,(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{\frac{1}{4}}(1-\eta_r)(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g,\end{aligned}$$

so

$$\big\|(1-\eta_r)|\hat{v}_n|^{\frac{1}{2}}(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g\big\|_2 \leqslant \delta\|(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{\frac{1}{4}}(1-\eta_r)(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g\|_2,$$

where $\delta\|(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{\frac{1}{4}}(1-\eta_r)(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g\|_2 \to 0$ as $r\to\infty$. The proof of Claim 3 is completed. □

Claim 2, which yields the convergence $\|I_{2,r,n}\|_2 \to 0$ as $n \to \infty$ for every r, and Claim 3, imply that

$$Z_2(\zeta,\hat{v}_n\mathcal{L}^d)g - Z_2(\zeta,\sigma)g \overset{s}{\to} 0 \text{ in } L^2, \quad g\in\mathcal{S}, \quad \zeta\in\mathcal{O},$$

which, in view of Claim 1(j), (jj), yields $Z_2(\zeta,\hat{v}_n\mathcal{L}^d) - Z_2(\zeta,\sigma) \overset{s}{\to} 0$ in L^2 ($\Rightarrow$(11)). By Claim 2, we also have (10). This completes the proof of Proposition 6. □

Proposition 7. *Let $p \in \mathcal{I}$. There exist constants C_p, $C_{p,q,r} < \infty$ such that, for every $\zeta \in \mathcal{O}$,*

(1) $\|\Omega_p(\zeta,\sigma,q,r)\|_{p\to p} \leqslant C_{p,q,r}$ *for all k,*

(2) $\|\Omega_p(\zeta,\sigma,\infty,1)\|_{p\to p} \leqslant C_p|\zeta|^{-\frac{1}{2}}$, *for all k.*

Proof. Immediate from Proposition 3, Proposition 6 and the definition (7). □

Now, we assume that $p \in \mathcal{J}$ ($\subset \mathcal{I}$).

Proposition 8. *Let $\{\hat{v}_n\}$ be the sequence in Proposition 6. For any $p \in \mathcal{J}$,*

$$\Omega_p(\zeta, \hat{v}_n\mathcal{L}^d, q, r) \overset{s}{\to} \Omega_p(\zeta, \sigma, q, r) \text{ in } L^p, \quad \zeta \in \mathcal{O}.$$

Proof. Set $\Omega_p \equiv \Omega_p(\zeta, \sigma, q, r)$, $\Omega_p^n \equiv \Omega_p(\zeta, \hat{v}_n\mathcal{L}^d, q, r)$. Since $p \in \mathcal{J}$, we have $2(p-1) \in \mathcal{I}$. Since Ω_p, $\Omega_p^n \in \mathcal{B}(L^p)$, it suffices to prove the required convergence over $\mathcal{S}$. We have ($f \in \mathcal{S}$):

$$\|\Omega_p f - \Omega_p^n f\|_p^p \leqslant \|\Omega_p f - \Omega_p^n f\|_{2(p-1)}^{p-1} \|\Omega_p f - \Omega_p^n f\|_2. \tag{16}$$

Let us estimate the right-hand side in (16):

1) $\Omega_p f - \Omega_p^n f$ $\big(= \Omega_{2(p-1)} f - \Omega_{2(p-1)}^n f\big)$ is uniformly bounded in $L^{2(p-1)}$ in view of Proposition 3 and Proposition 7,

2) $\Omega_p f - \Omega_p^n f = \Omega_2 f - \Omega_2^n f \overset{s}{\to} 0$ in L^2 as $k \to \infty$ by Proposition 6.

Therefore, by (16), $\Omega_p^n f \overset{s}{\to} \Omega_p f$ in L^p, as needed. ☐

This completes the proof of assertion (i), and, thus, the proof of Theorem 1.

Proof of Theorem 2. (i) The approximating vector fields v_k ($\in C_0(\mathbb{R}^d, \mathbb{R}^d)$) were constructed in Proposition 1. The proof essentially repeats the proof of [K, Theorem 2]. Namely, we verify conditions of the Trotter approximation theorem for $\Lambda_{C_\infty}(v_k\mathcal{L}^d) := -\Delta + v_k \cdot \nabla$, $D(\Lambda_{C_\infty}(v_k\mathcal{L}^d)) = (1-\Delta)^{-1}C_\infty$:

1°) $\sup_n \|(\mu + \Lambda_{C_\infty}(v_k\mathcal{L}^d))^{-1}\|_{\infty\to\infty} \leqslant \mu^{-1}$, $\mu \geqslant \kappa_d\lambda$.

2°) $\mu(\mu + \Lambda_{C_\infty}(v_k\mathcal{L}^d))^{-1} \to 1$ in C_∞ as $\mu \uparrow \infty$ uniformly in n.

3°) There exists s-C_∞-$\lim_n(\mu + \Lambda_{C_\infty}(v_k\mathcal{L}^d))^{-1}$ for some $\mu \geqslant \kappa_d\lambda$.

1°) is immediate. Let us verify 2°) and 3°). Fix some $p \in \mathcal{J}$, $p > d-1$ (such p exists since $m_d\delta < \frac{2d-5}{(d-2)^2}$). Let

$$\Theta_p(\mu, \sigma) := (\mu - \Delta)^{-\frac{1}{2}-\frac{1}{2q}} \Omega_p(\mu, \sigma, q, 1) \in \mathcal{B}(L^p), \quad \mu \geqslant \kappa_d\lambda, \tag{17}$$

where $\max\{2, p\} < q$, see the proof of Theorem 1 for notation. We will be using the properties of the operator-valued function $\Omega_p(\mu, \sigma, q, 1)$ established there. Without loss of generality, we may assume that $\{v_k\}$ is the sequence constructed in Proposition 8, that is, $v_k\mathcal{L}^d \overset{w}{\to} \sigma$, and $\Omega_p(\mu, v_k\mathcal{L}^d, q, 1) \overset{s}{\to} \Omega_p(\mu, \sigma, q, 1)$ in L^p as $k \to \infty$.

Given any $\gamma < 1 - \frac{d-1}{p}$, we can select q sufficiently close to p so that by the Sobolev embedding theorem,

$$(\mu - \Delta)^{-\frac{1}{2}-\frac{1}{2q}}[L^p] \subset C^{0,\gamma} \cap L^p, \quad \text{and} \quad (\mu - \Delta)^{-\frac{1}{2}-\frac{1}{2q}} \in \mathcal{B}(L^p, C_\infty).$$

Then Proposition 8 yields $\Theta_p(\mu, \hat{v}_n\mathcal{L}^d)f \overset{s}{\to} \Theta_p(\mu, \sigma)f$ in C_∞, $f \in \mathcal{S}$, as $n \to \infty$. The latter, combined with the next proposition and 1°), verifies condition 3°):

Proposition 9. *For every $k \geqslant 1$, $\Theta_p(\mu, v_k\mathcal{L}^d)\mathcal{S} \subset \mathcal{S}$, and*

$$(\mu + \Lambda_{C_\infty}(v_k\mathcal{L}^d))^{-1}|_{\mathcal{S}} = \Theta_p(\mu, v_k\mathcal{L}^d)|_{\mathcal{S}}, \quad \mu \geqslant \kappa_d\lambda.$$

Proof. The proof repeats the proof of [K, Prop. 6]. ☐

Proposition 10. *$\mu\Theta_p(\mu, v_k) \overset{s}{\to} 1$ in C_∞ as $\mu \uparrow \infty$ uniformly in k.*

Proof. The proof repeats the proof of [K, Prop. 8]. □

The last two propositions yield 2°). This completes the proof of assertion (i).

(ii) follows from $\Theta_p(\mu,\sigma)|_{\mathcal{S}} = (\mu + \Lambda_{C_\infty}(C_\infty))^{-1}|_{\mathcal{S}}$ (by construction), representation (17), and the Sobolev embedding theorem.

(iii) It follows from (i) that $e^{-t\Lambda_{C_\infty}(\sigma)}$ is positivity preserving. The latter, 1°) and the Riesz-Markov-Kakutani representation theorem imply (iii).

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