A NEW APPROACH TO THE L^p-THEORY OF $-\Delta + b\cdot\nabla$, AND ITS APPLICATIONS TO FELLER PROCESSES WITH GENERAL DRIFTS

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Abstract. We develop a detailed regularity theory of $-\Delta + b\cdot\nabla$ in $L^p(\mathbb{R}^d)$, for a wide class of vector fields. The L^p-theory allows us to construct associated strong Feller process in $C_\infty(\mathbb{R}^d)$. Our starting object is an operator-valued function, which, we prove, determines the resolvent of an operator realization of $-\Delta + b\cdot\nabla$, the generator of a holomorphic C_0-semigroup on $L^p(\mathbb{R}^d)$. Then the very form of the operator-valued function yields crucial information about smoothness of the domain of the generator.

1. Let $\mathcal{L}^d$ be the Lebesgue measure on $\mathbb{R}^d$, $L^p = L^p(\mathbb{R}^d,\mathcal{L}^d)$ and $W^{1,p} = W^{1,p}(\mathbb{R}^d,\mathcal{L}^d)$ the standard (complex) Lebesgue and Sobolev spaces, $C^{0,\gamma} = C^{0,\gamma}(\mathbb{R}^d)$ the space of Hölder continuous functions $(0<\gamma<1)$, $C_b = C_b(\mathbb{R}^d)$ the space of bounded continuous functions endowed with the sup-norm, $C_\infty \subset C_b$ the closed subspace of functions vanishing at infinity, $\mathcal{W}^{\alpha,p}$, $\alpha>0$, the Bessel space endowed with norm $\|u\|_{p,\alpha} := \|g\|_p$, $u = (1-\Delta)^{-\frac{\alpha}{2}}g$, $g\in L^p$, and $\mathcal{W}^{-\alpha,p'}$, $p' = p/(p-1)$, the anti-dual of $\mathcal{W}^{\alpha,p}$. $\mathcal{W}^{\alpha,p}_{\rm loc}$ denotes the class of (distributions) u such that $(1-\Delta)^{\frac{\alpha}{2}}u \in L^p_{\rm loc}$. We denote by $\mathcal{B}(X,Y)$ the space of bounded linear operators between complex Banach spaces $X\to Y$, endowed with operator norm $\|\cdot\|_{X\to Y}$; $\mathcal{B}(X) := \mathcal{B}(X,X)$. Set $\|\cdot\|_{p\to q} := \|\cdot\|_{L^p\to L^q}$.

For each $p\geqslant 1$, by $\langle u,v\rangle$ we denote the $(L^p,L^{p'})$ pairing, so that

$$\langle u,v\rangle = \langle u\bar{v}\rangle := \int_{\mathbb{R}^d} u\bar{v}\,d\mathcal{L}^d \qquad (u\in L^p, v\in L^{p'}).$$

2. Let $d\geqslant 3$. Consider the following classes of vector fields:

(1) We say that a $b:\mathbb{R}^d\to\mathbb{C}^d$ belongs to the Kato class $\mathbf{K}^{d+1}_\delta$, and write $b\in\mathbf{K}^{d+1}_\delta$, if b is $\mathcal{L}^d$-measurable, and there exists $\lambda=\lambda_\delta>0$ such that

$$\|b(\lambda-\Delta)^{-\frac{1}{2}}\|_{1\to 1}\leqslant\delta.$$

(2) We say that a $b:\mathbb{R}^d\to\mathbb{C}^d$ belongs to $\mathbf{F}_\delta$, the class of form-bounded vector fields, and write $b\in\mathbf{F}_\delta$, if b is $\mathcal{L}^d$-measurable, and there exists $\lambda=\lambda_\delta>0$ such that

$$\|b(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2}\leqslant\sqrt{\delta}.$$

(3) We say that a $b:\mathbb{R}^d\to\mathbb{C}^d$ belongs to $\mathbf{F}^{\frac{1}{2}}_\delta$, the class of *weakly* form-bounded vector fields, and write $b\in\mathbf{F}^{\frac{1}{2}}_\delta$, if b is $\mathcal{L}^d$-measurable, and there exists $\lambda=\lambda_\delta>0$ such that

$$\||b|^{\frac{1}{2}}(\lambda-\Delta)^{-\frac{1}{4}}\|_{2\to 2}\leqslant\sqrt{\delta}.$$

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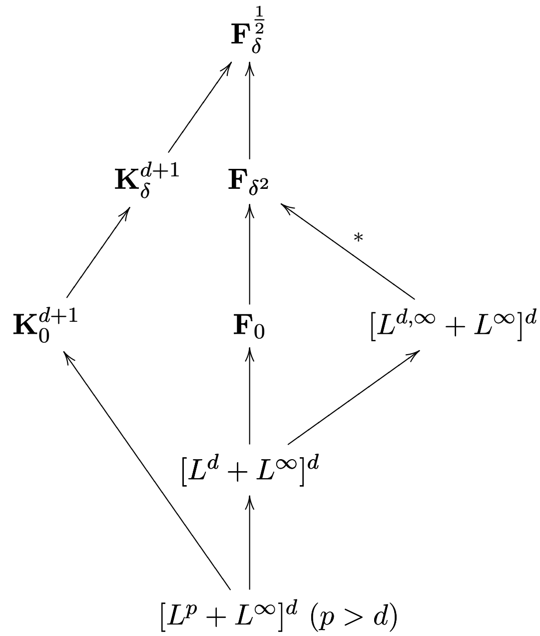

General classes of vector fields $b:\mathbb{R}^d\to\mathbb{R}^d$ studied in the literature in connection with operator $-\Delta+b\cdot\nabla$. Here $\rightarrow$ stands for strict inclusion, and $\overset{*}{\rightarrow}$ reads "if $b=b_1+b_2\in[L^{d,\infty}+L^\infty]^d$, then $b\in\mathbf{F}_{\delta^2}$ with $\delta>0$ determined by the value of the $L^{d,\infty}$-norm of $|b_1|$, see Remark 2 below for details, $\mathbf{K}_0^{d+1}:=\bigcap_{\delta>0}\mathbf{K}_\delta^{d+1}$, $\mathbf{F}_0:=\bigcap_{\delta>0}\mathbf{F}_\delta$.

Simple examples show:

$$\mathbf{F}_{\delta_1}-\mathbf{K}_\delta^{d+1}\neq\varnothing,\quad\text{and}\quad\mathbf{K}_{\delta_1}^{d+1}-\mathbf{F}_\delta\neq\varnothing\quad\text{for any }\delta,\delta_1>0,$$

for instance,

1) by the Hardy inequality, $b(x):=\sqrt{\delta_1}\frac{d-2}{2}x|x|^{-2}\in\mathbf{F}_{\delta_1}-\mathbf{K}_\delta^{d+1}$ for any $\delta,\delta_1>0$,

2) $b(x):=e\mathbf{1}_{|x_1|<1}|x_1|^{s-1}$, where $\frac{1}{2}<s<1$, $e=(1,\dots,1)\in\mathbb{R}^d$, $x=(x_1,\dots,x_d)$, is in $\mathbf{K}_0^{d+1}-\mathbf{F}_\delta$, for any $\delta>0$. (An example of a $b\in\mathbf{K}_\delta^{d+1}-\mathbf{K}_0^{d+1}$ can be obtained e.g. by modifying [AS, p. 250, Example 1].)

The classes $\mathbf{F}_\delta$, $\mathbf{K}_\delta^{d+1}$ cover singularities of b of critical order[1], at isolated points or along hypersurfaces, respectively. The class L^d doesn't contain vector fields having critical order singularities.

Remark 1. The classes $\mathbf{F}_\delta$ and $\mathbf{K}_\delta^{d+1}$ have been intensely studied in the literature: after 1996, the Kato class $\mathbf{K}_\delta^{d+1}$, with $\delta>0$ sufficiently small (yet allowed to be non-zero), has been recognized as 'the right' class for the Gaussian upper and lower bounds on the fundamental solution of $-\Delta+b\cdot\nabla$, see [Se3], which, in turn, allow to construct an associated Feller process (in C_b). The class $\mathbf{F}_\delta$, $\delta<4$, is responsible for dissipativity of $\Delta-b\cdot\nabla$ in L^p, $p\geqslant\frac{2}{2-\sqrt{\delta}}$, needed to run the iterative procedure of [KS] (taking $p\to\infty$, assuming additionally $\delta<\min\{4/(d-2)^2,1\}$), which produces an associated Feller process. We emphasize that, in general, the Gaussian bounds are not valid if $|b|\in L^d$, while $b\in\mathbf{K}_0^{d+1}$, in general, destroys L^p-dissipativity.

[1]In particular, the uniqueness of solution of Cauchy problem for $-\Delta+b\cdot\nabla$ can fail if $b\in\mathbf{F}_\delta$ is replaced with cb $(\in\mathbf{F}_{c^2\delta})$, for a sufficiently large constant c, cf. [KS, Example 5].

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The class $\mathbf{F}_\delta^{\frac{1}{2}}$ combines critical point and critical hypersurface singularities:

$$\mathbf{K}_\delta^{d+1} \subsetneq \mathbf{F}_\delta^{\frac{1}{2}}, \qquad \mathbf{F}_{\delta_1} \subsetneq \mathbf{F}_\delta^{\frac{1}{2}} \quad \text{for } \delta = \sqrt{\delta_1},$$
$$\Big(b \in \mathbf{F}_{\delta_1} \text{ and } \mathsf{f} \in \mathbf{K}_{\delta_2}^{d+1}\Big) \Longrightarrow \Big(b + \mathsf{f} \in \mathbf{F}_\delta^{\frac{1}{2}},\ \sqrt{\delta} = \sqrt[4]{\delta_1} + \sqrt{\delta_2}\Big) \tag{1}$$

(for the proof, if needed, see Appendix B).

Remark 2. The inclusion $|b| \in L^d \Rightarrow b \in \mathbf{F}_0$ (cf. the diagram above) follows by the Sobolev embedding theorem. For $|b| \in L^{d,\infty}$, we can verify, using [KPS, Prop. 2.5, 2.6, Cor. 2.9]:

$$b \in \mathbf{F}_\delta^{\frac{1}{2}}, \quad \text{with } \sqrt{\delta} = \||b|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}}\|_{2\to 2} \leqslant \|(|b|^*)^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}}\|_{2\to 2}$$
$$\leqslant \Big(\|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}}\Big)^{\frac{1}{2}} \||x|^{-\frac{1}{2}}(\lambda-\Delta)^{-\frac{1}{4}}\|_{2\to 2} = \Big(\|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}}\Big)^{\frac{1}{2}} 2^{-\frac{1}{2}} \frac{\Gamma\big(\frac{d-1}{4}\big)}{\Gamma\big(\frac{d+1}{4}\big)},$$

where $\Omega_d = \pi^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)$, and $|b|^*$ is the symmetric decreasing rearrangement of $|b|$. Similarly,

$$b \in \mathbf{F}_{\delta_1}, \quad \text{with } \sqrt{\delta_1} = \||b|(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2}$$
$$\leqslant \|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}}\||x|^{-1}(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2}$$
$$\leqslant \|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}} 2^{-1}\frac{\Gamma\big(\frac{d-2}{4}\big)}{\Gamma\big(\frac{d+2}{4}\big)} = \|b\|_{d,\infty}\Omega_d^{-\frac{1}{d}}\frac{2}{d-2}.$$

In particular, using [KPS, Cor. 2.9],

$$x|x|^{-2} \in \mathbf{F}_\delta^{\frac{1}{2}}, \quad \sqrt{\delta} = 2^{-\frac{1}{2}}\frac{\Gamma\big(\frac{d-1}{4}\big)}{\Gamma\big(\frac{d+1}{4}\big)},$$

$$x|x|^{-2} \in \mathbf{F}_{\delta_1}, \quad \sqrt{\delta_1} = \frac{2}{d-2},$$

and so $\delta < \sqrt{\delta_1}$ (cf. (1)).

3. Denote

$$m_d := \pi^{\frac{1}{2}}(2e)^{-\frac{1}{2}}d^{\frac{d}{2}}(d-1)^{\frac{1-d}{2}}, \qquad c_p := pp'/4.$$

The following two theorems are the main results of our paper.

Theorem 1 (L^p-theory)**.** *Let $d \geqslant 3$ and $b : \mathbb{R}^d \to \mathbb{C}^d$. Assume that $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, $m_d\delta < 1$. Then, for every*

$$p \in \mathcal{I} := \left(\frac{2}{1+\sqrt{1-m_d\delta}}, \frac{2}{1-\sqrt{1-m_d\delta}}\right),$$

there exists a C_0-semigroup $e^{-t\Lambda_p(b)}$ in L^p such that

(*i*) *The resolvent set $\rho(-\Lambda_p(b))$ contains the half-plane $\mathcal{O} := \{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta \geqslant \kappa_d\lambda_\delta\}$, $\kappa_d := \frac{d}{d-1}$, and the resolvent admits the representation:*

$$(\zeta + \Lambda_p(b))^{-1} = \Theta_p(\zeta, b), \quad \zeta \in \mathcal{O},$$

where

$$\Theta_p(\zeta, b) := (\zeta - \Delta)^{-1} - Q_p(1+T_p)^{-1}G_p, \tag{2}$$

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the operators $Q_p, G_p, T_p \in \mathcal{B}(L^p)$,

$$\|G_p\|_{p\to p} \leqslant C_1|\zeta|^{-\frac{1}{2p'}},\ \|Q_p\|_p \leqslant C_2|\zeta|^{-\frac{1}{2}-\frac{1}{2p}},\ \|T_p\|_p \leqslant m_d c_p \delta < 1,$$

$$G_p \equiv G_p(\zeta, b) := b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1}, \quad b^{\frac{1}{p}} := |b|^{\frac{1}{p}-1}b,$$

Q_p, T_p *are the extensions by continuity of densely defined (on* $\mathcal{E} := \bigcup_{\epsilon>0} e^{-\epsilon|b|}L^p$*) operators*

$$Q_p|_{\mathcal{E}} \equiv Q_p(\zeta, b)|_{\mathcal{E}} := (\zeta - \Delta)^{-1}|b|^{\frac{1}{p'}}, \qquad T_p|_{\mathcal{E}} \equiv T_p(\zeta, b)|_{\mathcal{E}} := b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1}|b|^{\frac{1}{p'}}.$$

(ii) *It follows from* (i) *that* $e^{-t\Lambda_p(b)}$ *is holomorphic: there is a constant* C_p *such that*

$$\|(\zeta + \Lambda_p(b))^{-1}\|_{p\to p} \leqslant C_p|\zeta|^{-1}, \quad \zeta \in \mathcal{O}.$$

(iii) *For each* $1 \leqslant r < p < q$ *and* $\zeta \in \mathcal{O}$, *define*

$$G_p(r) \equiv G_p(r, \zeta, b) := b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-\frac{1}{2}-\frac{1}{2r}}, \quad G_p(r) \in \mathcal{B}(L^p),$$
$$Q_p(q) \equiv Q_p(q, \zeta, b) := (\zeta - \Delta)^{-\frac{1}{2q'}}|b|^{\frac{1}{p'}} \text{ on } \mathcal{E}.$$

The extension of $Q_p(q)$ *by continuity we denote again by* $Q_p(q)$. *Then, for each* $\zeta \in \mathcal{O}$,

$$\Theta_p(\zeta, b) = (\zeta - \Delta)^{-1} - (\zeta - \Delta)^{-\frac{1}{2}-\frac{1}{2q}}Q_p(q)(1 + T_p)^{-1}G_p(r)(\zeta - \Delta)^{-\frac{1}{2r'}};$$
$$\Theta_p(\zeta, b) \text{ extends by continuity to an operator in } \mathcal{B}\big(\mathcal{W}^{-\frac{1}{r'},p},\ \mathcal{W}^{1+\frac{1}{q},p}\big).$$

(iv) *By* (i) *and* (iii), $D\big(\Lambda_p(b)\big) \subset \mathcal{W}^{1+\frac{1}{q},p}$ $(q > p)$. *In particular, if* $m_d\delta < 4\frac{d-2}{(d-1)^2}$, *there exists* $p \in \mathcal{I}$, $p > d-1$, *so* $D\big(\Lambda_p(b)\big) \subset C^{0,\gamma}$, $\gamma < 1 - \frac{d-1}{p}$.

(v) *Let* $u \in D(\Lambda_p(b))$. *Then*

$$\langle \Lambda_p(b)u, v\rangle = \langle u, -\Delta v\rangle + \langle b \cdot \nabla u, v\rangle, \qquad v \in C_c^\infty(\mathbb{R}^d);$$
$$u \in \mathcal{W}^{2,1}_{\text{loc}}.$$

(vi) $e^{-t\Lambda_p(b_n)} \xrightarrow{s} e^{-t\Lambda_p(b)}$ *in* L^p, $\quad t > 0$,

where $b_n := b$ *if* $|b| \leqslant n$, $b_n := n|b|^{-1}b$ *if* $|b| > n$, *and* $\Lambda_p(b_n) := -\Delta + b_n \cdot \nabla$, $D(\Lambda_p(b_n)) = \mathcal{W}^{2,p}$.

(vii) *If* b *is real-valued, then* $e^{-t\Lambda_p(b)}$ *is positivity preserving.*

$(viii)$ *By Theorem 3(b) below,* $\|e^{-t\Lambda_p(b)}\|_{p\to r} \leqslant c_{p,r}t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})}$, $0 < t \leqslant 1$, $p < r$

Remark 3. Theorem 1 provides a complete description of $\Lambda_p(b)$, an operator realization of $-\Delta + b \cdot \nabla$, $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, generating a holomorphic C_0-semigroup on L^p.

Let

$$\eta(x) := \begin{cases} c\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geqslant 1, \end{cases}$$

where c is adjusted to $\int_{\mathbb{R}^d} \eta(x)dx = 1$. Define the standard mollifier

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^d}\eta\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0, \ \ x \in \mathbb{R}^d.$$

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Theorem 2 (C_∞-theory)**.** *Let $d \geqslant 3$. Assume that*

$$b:\mathbb{R}^d \to \mathbb{R}^d, \quad b \in \mathbf{F}_\delta^{\frac{1}{2}}, \quad m_d\delta < 4\frac{d-2}{(d-1)^2}.$$

Then for every $\tilde{\delta} > \delta$ satisfying $m_d\tilde{\delta} < 4\frac{d-2}{(d-1)^2}$ there exists $\{\varepsilon_n\}$, $\varepsilon_n \downarrow 0$, such that

$$\tilde{b}_n := \eta_{\varepsilon_n} * b_n \in C^\infty(\mathbb{R}^d,\mathbb{R}^d) \cap \mathbf{F}_{\tilde{\delta}}^{\frac{1}{2}}, \quad n=1,2,\dots,$$

and

(*i*)

$$e^{-t\Lambda_{C_\infty}(b)} := s\text{-}C_\infty\text{-}\lim_n e^{-t\Lambda_{C_\infty}(\tilde{b}_n)}, \quad t>0,$$

determines a positivity preserving contraction C_0-semigroup on C_∞, where b_n's were defined in Theorem 1, $\Lambda_{C_\infty}(\tilde{b}_n) := -\Delta + \tilde{b}_n \cdot \nabla$, $D(\Lambda_{C_\infty}(\tilde{b}_n)) = (1-\Delta)^{-1}C_\infty$.

(*ii*) *(Strong Feller property)* $(\mu + \Lambda_{C_\infty}(b))^{-1}[L^p \cap C_\infty] \subset C^{0,\alpha}$, $\mu > 0$, $p \in \big(d-1, \frac{2}{1-\sqrt{1-m_d\delta}}\big)$, $\alpha < 1 - \frac{d-1}{p}$.

(*iii*) *The integral kernel $e^{-t\Lambda_{C_\infty}(b)}(x,y)$ $(x,y \in \mathbb{R}^d)$ of $e^{-t\Lambda_{C_\infty}(b)}$ determines the (sub-Markov) transition probability function of a strong Feller process.*

Remark 4. 1. In the proof of Theorem 2, we define

$$(\mu + \Lambda_{C_\infty}(b))^{-1}|_{\mathcal{S}} := s\text{-}C_\infty\text{-}\lim_n \big((\mu + \Lambda_p(\tilde{b}_n))^{-1}|_{\mathcal{S}}, \quad \mu \geqslant \kappa_d\lambda, \quad p \in \big(d-1, \frac{2}{1-\sqrt{1-m_d\delta}}\big),$$

appealing to Theorem 1(*iv*), which allows us to move the proof of convergence in C_∞ to L^p, $p > d-1$, a space having much weaker topology (locally). Earlier proofs for a smaller class $\mathbf{K}_0^{d+1}$ verified convergence in C_∞ (in fact, in C_b) directly.

2. The problem of constructing a Feller process associated with $-\Delta + b\cdot\nabla$, for an unbounded $b:\mathbb{R}^d \to \mathbb{R}^d$ ("a diffusion with drift b"), has been thoroughly studied in the literature, see [KR] and references therein, motivated by applications, as well as by the search for the *maximal* general class of vector fields b such that the associated process exists. To the author's knowledge, Theorem 2 is the first result on diffusion processes with drifts combining different kinds of singularities, e.g. $\big||x|-1\big|^{-\beta}$, $\beta < 1$, and $|x|^{-1}$ (originally, the main motivation for this work).

4. On the existing results prior to our work. First, it had been known for a long time, see [KS], that, for $b:\mathbb{R}^d \to \mathbb{R}^d$, $d \geqslant 3$, and $b \in \mathbf{F}_\delta$,

(i) *(The basic fact) $D(\Lambda_p(b)) \subset W^{1,jp}$ for every $p \in (d-2, 2/\sqrt{\delta})$, $j = \frac{d}{d-2}$,* provided that $0 < \delta < \min\{1, (\frac{2}{d-2})^2\}$.

(ii) *If, in addition to the assumptions in (i), $|b| \in L^2 + L^\infty$, then*

$$s\text{-}C_\infty\text{-}\lim_n e^{-t\Lambda_{C_\infty}(b_n)}$$

exists uniformly in each finite interval of $t \geqslant 0$, and hence determines a strongly Feller semigroup on C_∞.

Remark 5. The additional (to $|b| \in L^2_{\rm loc}$) assumption $|b| \in L^2 + L^\infty$ in (ii) was removed in [Ki] (albeit at expense of imposing a more restrictive assumption on the maximal admissible value of $\delta > 0$).

Theorem 3 (Yu.A. Semenov). *Let $b : \mathbb{R}^d \to \mathbb{R}^d$, $d \geqslant 3$.*

a) [Se] *If $b \in \mathbf{K}_\delta^{d+1}$, $m_d\delta < 1$, then, for each $p \in [1,\infty)$, s-L^p-$\lim_n e^{-t\Lambda_p(b_n)}$ exists uniformly on each finite interval of $t \geqslant 0$, and hence determines a C_0-semigroup $e^{-t\Lambda_p(b)}$.*

$e^{-t\Lambda_p(b)}$ is a quasi-bounded positivity preserving L^∞-contraction C_0- semigroup;

$$\|e^{-t\Lambda_r(b)}\|_{r\to q} \leqslant c_{d,\delta}\, t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})} \text{ for all } 0 < t \leqslant 1,\ 1 \leqslant r < q \leqslant \infty;$$

The resolvent set $\rho(-\Lambda_p(b))$ contains the half-plane $\mathcal{O}$,

$$(\zeta + \Lambda_p(b))^{-1} = \Theta_p(\zeta, b),\ \ \zeta \in \mathcal{O},$$

$$\Theta_p(\zeta,b) := (\zeta-\Delta)^{-1} - (\zeta-\Delta)^{-\frac{1}{2}} S_p(1+T_p)^{-1}G_p,$$

$$S_p := (\zeta-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}, \quad G_p := b^{\frac{1}{p}} \cdot \nabla(\zeta-\Delta)^{-1}, \quad T_p := b^{\frac{1}{p}} \cdot \nabla(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}};$$

$$\Theta_p(\zeta,b) \in \mathcal{B}(L^p, \mathcal{W}^{1,p});$$

$$D(\Lambda_p(b)) \subset \mathcal{W}^{1,\,p}. \text{ In particular, for } p > d,\ D(\Lambda_p(b)) \subset C^{0,\,\alpha},\ \alpha = 1 - \frac{d}{p};$$

$$\langle \Lambda_p(b) f, g\rangle = \langle \nabla f, \nabla g\rangle + \langle b\cdot\nabla f, g\rangle, \qquad f \in D(\Lambda_p(b)),\ g \in C_c^\infty(\mathbb{R}^d).$$

b) [Se2, Theorem 5.1] *If $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, $\delta < 1$, then, for each $p \in [2,\infty)$, s-L^p-$\lim_n e^{-t\Lambda_p(b_n)}$ exists uniformly on each finite interval of $t \geqslant 0$, and hence determines a C_0-semigroup $e^{-t\Lambda_p(b)}$.*

$e^{-t\Lambda_p(b)}$ is a quasi-bounded positivity preserving L^∞-contraction C_0- semigroup.

$\|e^{-t\Lambda_r(b)}\|_{r\to q} \leqslant c_{d,\delta}\, t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})}$ *for all* $0 < t \leqslant 1,\ 2 \leqslant r < q \leqslant \infty$.

$D(\Lambda_2(b)) \subset W^{\frac{3}{2},2}$.

$\langle \Lambda_2(b) f, g\rangle = \langle f, -\Delta g\rangle + \langle b\cdot\nabla f, g\rangle, \qquad f \in D(\Lambda_2(b)),\ g \in C_c^\infty(\mathbb{R}^d)$.

Remark 6. The additional (to $|b| \in L^1_{\rm loc}$) assumption $|b| \in L^1 + L^\infty$ in [Se2, Theorem 5.1] is not essential for the proof, and can be eliminated.

For the sake of completeness, we now outline the proof of Theorem 3, with permission of its author.

Proof. **a**) Indeed, for all ζ with Re $\zeta > 0$,

$$|\nabla(\zeta-\Delta)^{-1}(x,y)| \leqslant m_d(\kappa_d^{-1}{\rm Re}\,\zeta - \Delta)^{-\frac{1}{2}}(x,y) \text{ pointwise on } \mathbb{R}^d\times\mathbb{R}^d$$

(see (A.1) in the Appendix). Therefore, for $b \in \mathbf{K}_\delta^{d+1}$,

$$\|b\cdot\nabla(\zeta-\Delta)^{-1}\|_{1\to 1} \leqslant m_d\delta, \qquad {\rm Re}\ \zeta \geqslant \kappa_d\lambda,$$

and so by the Miyadera perturbation theorem, the operator $-\Lambda_1(b) := \Delta - b\cdot\nabla$ of domain $D(\Lambda_1(b)) = \mathcal{W}^{2,1}$ is the generator of a quasi-bounded C_0 semigroup on L^1 *whenever* $m_d\delta < 1$.

Clearly $b_n \in \mathbf{K}_\delta^{d+1}$, $\|b_n\cdot\nabla(\zeta-\Delta)^{-1}\|_{1\to 1} \leqslant m_d\delta$, and, for $m_d\delta < 1$ and every $f \in D(\Lambda_1(b))$, $\Lambda_1(b_n)f \overset{s}{\to} \Lambda_1(b)f$ by the Dominated Convergence Theorem. (See, if needed, (A.0).) The latter easily implies the strong resolvent and the semigroup convergence of $\Lambda_1(b_n)$ to $\Lambda_1(b)$.

Then, for each $n = 1,2,\dots$, the semigroups $e^{-t\Lambda_1(b_n)}, t > 0$, are positivity preserving L^∞-contractions, and so is $e^{-t\Lambda_1(b)}$. The bounds

$$\|e^{-t\Lambda_1(b)}\|_{1\to 1} \leqslant Me^{t\omega},\ \omega = \kappa_d\lambda, \text{ and } \|e^{-t\Lambda_1(b)}f\|_\infty \leqslant \|f\|_\infty,\ f \in L^1\cap L^\infty,$$

yield via the Riesz interpolation theorem

$$\|e^{-t\Lambda_1(b)}f\|_p \leqslant M^{1/p}e^{t\omega/p}\|f\|_p, \quad f \in L^1 \cap L^\infty.$$

Therefore, we obtain a family $\{e^{-t\Lambda_p(b)}\}_{1\leqslant p<\infty}$ of consistent C_0-semigroups by setting $e^{-t\Lambda_p(b)} :=$ the extension by continuity in L^p of $e^{-t\Lambda_1(b)} \mid L^1 \cap L^\infty$.

Next, for each $p \in [1,\infty)$ and all $f \in \mathcal{E} := \bigcup_{\epsilon>0} e^{-\epsilon|b|}L^p$, the inequality

$$\||b|^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}f\|_p \leqslant \delta\|f\|_p$$

as well as inequality

$$\|(|b|+\sqrt{\lambda})^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2}}(|b|+\sqrt{\lambda})^{\frac{1}{p'}}f\|_p \leqslant (1+\delta)\|f\|_p$$

follow from the very definition of $\mathbf{K}_\delta^{d+1}$ (e.g. by interpolating between $\|(|b|+\sqrt{\lambda})(\lambda-\Delta)^{-\frac{1}{2}}\|_{1\to 1} \leqslant 1+\delta$ and (by duality) $\|(\lambda-\Delta)^{-\frac{1}{2}}(|b|+\sqrt{\lambda})\|_\infty \leqslant 1+\delta$). The latter implies that

$$\||b|^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2}}\|_{p\to p} \leqslant (1+\delta)\lambda^{-\frac{1}{2p'}},$$

and the first inequality implies that, for every $\zeta \in \mathcal{O}$, $p \in [1,\infty)$ and all $f \in \mathcal{E}$,

$$\|b^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-1}\,|b|^{\frac{1}{p'}}f\|_p \leqslant m_d\||b|^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}|f|\|_p \leqslant m_d\delta\|f\|_p.$$

Now, it is seen that for every $p \in [1,\infty)$ and $\zeta \in \mathcal{O}$ the operator G_p is bounded:

$$\|G_p\|_{p\to p} \leqslant m_d\|b^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2}}\|_{p\to p} \leqslant m_d(1+\delta)\lambda^{-\frac{1}{2p'}}.$$

S_p and T_p are densely defined (on $\mathcal{E}$) and, for all $f \in \mathcal{E}$,

$$\|S_pf\|_p \leqslant (1+\delta)^{-1}\lambda^{-\frac{1}{2p}}\|f\|_p \text{ and } \|T_pf\|_p \leqslant m_d\delta\|f\|_p.$$

Their extensions by continuity we denote again by S_p, T_p.

Next, we define an operator function $\Theta_p(\zeta,b)$ in L^p by

$$\Theta_p(\zeta,b) := (\zeta-\Delta)^{-1} - (\zeta-\Delta)^{-\frac{1}{2}}S_p\,(1+T_p)^{-1}\,G_p \qquad \zeta \in \mathcal{O}.$$

Obviously,

$$\Theta_p(\zeta,b) \in \mathcal{B}(L^p) \text{ and } \Theta_p(\zeta,b) \in \mathcal{B}(L^p, W^{1,p}).$$

It is also seen that

$(\zeta+\Lambda_1(b))^{-1} = \Theta_1(\zeta,b)$, $(\zeta+\Lambda_p(b))^{-1} \mid L^1\cap L^p = \Theta_p(\zeta,b) \mid L^1 \cap L^p$, and so

$$(\zeta+\Lambda_p(b))^{-1} = \Theta_p(\zeta,b), \qquad \zeta \in \mathcal{O}.$$

The latter implies that $D(\Lambda_p(b)) \subset W^{1,p}$, for all $p \in [1,\infty)$. The main assertion is proved.

b) Let $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, $\delta < 1$. Define $H = |b|^{\frac{1}{2}}(\zeta-\Delta)^{-\frac{1}{4}}$, $S = b^{\frac{1}{2}}\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}$ and

$$\begin{aligned}\Theta_2(\zeta,b) :=&(\zeta-\Delta)^{-\frac{3}{4}}(1+H^*S)^{-1}(\zeta-\Delta)^{-\frac{1}{4}} \qquad (*)\\ =&(\zeta-\Delta)^{-1} - (\zeta-\Delta)^{-\frac{3}{4}}H^*(1+SH^*)^{-1}S(\zeta-\Delta)^{-\frac{1}{4}}, \ \ \mathrm{Re}\,\zeta \geqslant \lambda.\end{aligned}$$

We represent $S = \hat{H}\nabla(\zeta-\Delta)^{-\frac{1}{2}}$, where the operator $\hat{H}$ defined by $\hat{H}h := b^{\frac{1}{2}}\cdot(\zeta-\Delta)^{-\frac{1}{4}}h$, $h : \mathbb{R}^d \to \mathbb{R}^d$, with $(\zeta-\Delta)^{-\frac{1}{4}}$ acting on h component-wise, clearly satisfies $\|\hat{H}h\|_2 \leqslant \||b|^{\frac{1}{2}}(\mathrm{Re}\,\zeta -$

$\Delta)^{-\frac{1}{4}}|h|\|_2 \leqslant \sqrt{\delta}\|h\|_2$, $\operatorname{Re}\zeta \geqslant \lambda$. Therefore,

$$\begin{aligned}\|H^*S\|_{2\to 2} &\leqslant \|H\|_{2\to 2}\|S\|_{2\to 2}\\ &\leqslant \|H\|_{2\to 2}\|\hat{H}\|_{2\to 2}\|\nabla(\zeta-\Delta)^{-\frac{1}{2}}\|_{2\to 2} \leqslant \delta,\end{aligned}$$

and

$$\|\Theta_2(\zeta,b)\|_{2\to 2} \leqslant (1-\delta)^{-1}|\zeta|^{-1}.$$

Note that $D(\Lambda_2(b_n)) = W^{2,2}$ and, for all Re $\zeta \geqslant \lambda$, by the first representation of $\Theta_2(\zeta, b_n)$,

$$\Theta_2(\zeta,b_n)^{-1}|W^{2,2} = (\zeta+\Lambda_2(b_n))|W^{2,2}, \quad \Theta_2(\zeta,b_n) = (\zeta+\Lambda_2(b_n))^{-1},$$

$$\zeta\Theta_2(\zeta,b_n) \overset{s}{\to} 1 \text{ as } \zeta\uparrow\infty \text{ by the second representation of } \Theta_2(\zeta,b_n).$$

Therefore, $\Theta_2(\zeta, b_n)$ is the resolvent of $-\Lambda_2(b_n)$.

Since $\|\Theta_2(\zeta,b_n)\|_{2\to 2} \leqslant (1-\delta)^{-1}|\zeta|^{-1}$, the semigroups $e^{-t\Lambda_2(b_n)}$ are holomorphic and equi-bounded.

Finally, it is seen that $\Theta_2(\zeta,b_n) \overset{s}{\to} \Theta_2(\zeta,b)$ in L^2 on Re $\zeta \geqslant \lambda$, and $\mu\ \Theta_2(\mu,b_n) \overset{s}{\to} 1$ in L^2 as $\mu\uparrow\infty$ uniformly in n. Therefore, by the Trotter approximation theorem s-L^2-$\lim_n e^{-t\Lambda_2(b_n)}$ exists and determines a C_0-semigroup in L^2. It is also clear that this semigroup is holomorphic and L^∞-contractive. □

5. Comments. 1. The fact that $b:\mathbb{R}^d\to\mathbb{R}^d$ belongs to $\mathbf{K}_\delta^{d+1}$ or $\mathbf{F}_\delta$ allows us to construct operator realizations of the formal differential operator $-\Delta+b\cdot\nabla$ as (minus) generators of strongly continuous semigroups in L^p for some or all $p\in[1,\infty)$, C_∞ and/or C_b, by means of general tools of the standard perturbation theory (e.g. theorems of Miyadera [Vo] or Phillips [Ph], respectively).

2. Concerning the class $\mathbf{F}_\delta^{\frac{1}{2}}$ one can not appeal to the standard perturbation theory (in contrast to $\mathbf{K}_\delta^{d+1}$ and $\mathbf{F}_\delta$) in order to properly characterize the domain of the generator $\Lambda_p(b)$. Indeed, the arguments in [Se2, p. 413-416] (repeated above in the proof of Theorem 3b) say nothing about $\mathcal{W}^{\alpha,p}$-smoothness of $D(\Lambda_p(b))$ for $p\neq 2$. The natural analogue of $(*)$ in L^p is valid only for a smaller class of vector fields: $|b|\in L^{d,\infty}$.

3. For $|b|\in L^{d,\infty}$, the assertion of Theorem 1(iv) can be strengthened:

$$|b|\in L^{d,\infty} \quad\Rightarrow\quad D(\Lambda_p(b))\subset\mathcal{W}^{1+\frac{1}{p},s}, \quad s<dp. \tag{3}$$

Indeed, arguing as in Remark 2 (i.e. appealing to [KPS, Prop. 2.5, 2.6, Cor. 2.9]), we can estimate, using (A.1), for every $f\in\mathcal{E}$,

$$\|b^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-\frac{1}{2}-\frac{1}{2p}}f\|_s \leqslant c_1\|f\|_s, \qquad c_1 := m_d(\Omega_d^{-\frac{1}{d}}\|b\|_{d,\infty})^{\frac{1}{p}}c(p,d),$$

$$\|(\zeta-\Delta)^{-\frac{1}{2p'}}|b|^{\frac{1}{p'}}f\|_s \leqslant c_2\|f\|_s, \quad c_2 := (\Omega_d^{-\frac{1}{d}}\|b\|_{d,\infty})^{\frac{1}{p'}}c(p',d),$$

where $c(p,d) := 2^{-\frac{1}{p}}\dfrac{\Gamma\left(\frac{d}{2p'}\right)}{\Gamma\left(\frac{d}{2p}\right)}\dfrac{\Gamma\left(\frac{d-1}{2p}\right)}{\Gamma\left(\frac{1}{2p}+\frac{d}{2p'}\right)}$, so

$$\|b^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}}f\|_s \leqslant c_3\|f\|_s, \quad c_3 := m_d\Omega_d^{-\frac{1}{d}}\|b\|_{d,\infty}c(p,d)c(p',d).$$

Now, we can estimate in Theorem 1(iii): $\|Q_p(p)\|_{s\to s}$, $\|G_p(p)\|_{s\to s}$, $\|T_p\|_{s\to s}<\infty$, to conclude that $\|\Theta_p(\zeta,b)\|_{s\to s}<\infty$, $1<s<dp$. In view of Theorem 1(i), the last estimate implies the required.

4. Theorem 3 can be obtained as a side product of the proof of Theorem 1. Indeed, the constraints on p and δ in Theorem 1 come solely from the estimate on $\|T_p\|_{p\to p}$. Now, if $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, $\delta < 1$, then (representing $S = \hat{H}\nabla(\zeta-\Delta)^{-\frac{1}{2}}$)

$$\|T_2\|_{2\to 2} \leqslant \|\hat{H}\|_{2\to 2}\|H^*\|_{2\to 2}\|\nabla(\zeta-\Delta)^{-\frac{1}{2}}\|_{2\to 2} \leqslant \delta < 1.$$

And if $b \in \mathbf{K}_\delta^{d+1}$, $m_d\delta < 1$, then $\|T_p\|_{p\to p} < 1$ for *all* $p \in [1,\infty)$, so that the interval $\mathcal{I} \ni p$ transforms into $[1,\infty)$, and a possible causal dependence of the properties of $D(\Lambda_p(b))$ on δ gets lost. The latter indicates the smallness of $\mathbf{K}_\delta^{d+1}$ as a subclass of $\mathbf{F}_\delta^{\frac{1}{2}}$.

5. Both proofs of Theorem 1 and Theorem 3 are based on similar operator-valued functions, although the arguments involved differ considerably.

6. Note that for $b \in \mathbf{K}_\delta^{d+1}$, $m_d\delta < 1$, $D(\Lambda_1(b)) = \mathcal{W}^{2,1}$; for $b \in \mathbf{F}_\delta$, $\delta < 1$, $D(\Lambda_2(b)) = W^{2,2}$, while for $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, $\delta < 1$, $D(\Lambda_2(b)) \subset \mathcal{W}^{2,1}_{\text{loc}}$.

7. Let $b : \mathbb{R}^d \to \mathbb{R}^d$, $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, $m_d\delta < 1$. Theorem 1(i),(vi) and the argument in the proof of Theorem 3a (using the Riesz interpolation theorem) yield a consistent family of positivity preserving quasi-bounded C_0-semigroups $e^{-t\Lambda_p(b)}$ on L^p, for all $p \in (\frac{2}{1+\sqrt{1-m_d\delta}},\infty)$.

8. The author considers the assertion (iv) of Theorem 1 (the $\mathcal{W}^{1+\frac{1}{q},p}$-smoothness) as the main result of the paper. Theorem 1, compared to [KS] and Theorem 3a, covers the larger class of vector fields, and at the same time establishes stronger smoothness properties of $D(\Lambda_p(b))$: $D(\Lambda_p(b)) \subset \mathcal{W}^{1+\frac{1}{q},p}$, $p \in \mathcal{I}$ $(q > p)$, while in [KS] $D(\Lambda_p(b)) \subset W^{1,jp}$, $jp \in (d, 2j/\sqrt{\delta})$, and in Theorem 3a $D(\Lambda_p(b)) \subset \mathcal{W}^{1,p}$, $p \in [1,\infty)$.

9. The C_∞-theory of operator $-\Delta + b\cdot\nabla$, $b \in \mathbf{F}_\delta^{\frac{1}{2}}$ (Theorem 2) follows almost automatically from the L^p-theory (Theorem 1) (with $p > d-1$), in contrast to [KS], where the C_∞-theory is obtained from the L^p-theory by running a specifically tailored iterative procedure (see also [Ki]).

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1. Proof of Theorem 1

The method of the proof. We start with an operator-valued function

$$\Theta_p(\zeta,b) := (\zeta-\Delta)^{-1} - Q_p(1+T_p)^{-1}G_p, \quad \zeta \in \mathcal{O},$$

defined in L^p for each p from the interval

$$\mathcal{I} := \left]\frac{2}{1+\sqrt{1-m_d\delta}}, \frac{2}{1-\sqrt{1-m_d\delta}}\right[, \quad m_d\delta < 1,$$

and step by step prove that, for $n = 1, 2, \dots$,

$$\|\Theta_p(\zeta,b_n)\|_{p\to p},\ \|\Theta_p(\zeta,b)\|_{p\to p} \leqslant c|\zeta|^{-1};$$

$$\Theta_p(\zeta,b_n) \text{ is a pseudo-resolvent;}$$

$$\Theta_p(\zeta,b_n) \text{ coincides with the resolvent } R(\zeta,-\Lambda_p(b_n)) = (\zeta+\Lambda_p(b_n))^{-1} \text{ on } \mathcal{O};$$

$$\Theta_p(\zeta, b_n) \overset{s}{\to} \Theta_p(\zeta, b) \text{ in } L^p \text{ as } n \uparrow \infty;$$

$$\mu\, \Theta_p(\mu, b_n) \overset{s}{\to} 1 \text{ as } \mu \uparrow \infty \text{ in } L^p \text{ uniformly in } n.$$

All this combined leads to the conclusion: for each $p \in \mathcal{I}$ there is a holomorphic semigroup $e^{-t\Lambda_p(b)}$ in L^p such that the resolvent $R(\zeta, -\Lambda_p(b))$ on $\zeta \in \mathcal{O}$ has the representation $\Theta_p(\zeta, b)$;

$\Theta_p(\zeta, b)$ can be written as $(\zeta - \Delta)^{-1} + ABC$, where $C \in \mathcal{B}(\mathcal{W}^{-\frac{1}{r'}, p}, L^p)$, $B \in \mathcal{B}(L^p)$, $A \in \mathcal{B}(L^p, \mathcal{W}^{1+\frac{1}{q}, p})$, $r < p < q$, $r' = r/(r-1)$.

Propositions 1-4 below constitute the core of the proof of Theorem 1.

Proposition 1. *Let* $p \in \mathcal{I}$.

(i) For every $1 \leqslant r < p < q \leqslant \infty$ *and* $\zeta \in \mathcal{O}$ $(= \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta \geqslant \kappa_d \lambda\},\ \lambda = \lambda_\delta)$ *define operators on* L^p

$$Q_p(q) = (\zeta - \Delta)^{-\frac{1}{2q'}} |b|^{\frac{1}{p'}}, \quad G_p(r) = b^{\frac{1}{p}} \cdot \nabla (\zeta - \Delta)^{-\frac{1}{2} - \frac{1}{2r}}, \quad T_p = b^{\frac{1}{p}} \cdot \nabla (\zeta - \Delta)^{-1} |b|^{\frac{1}{p'}}.$$

Then $G_p(r)$ *is bounded:* $\|G_p(r)\|_{p \to p} \leqslant K_{1,r}$. $Q_p(q)$ *and* T_p *are densely defined (on* $\mathcal{E}$*), and for all* $f \in \mathcal{E}$,

$$\|Q_p(q) f\|_p \leqslant K_{2,q} \|f\|_p,$$

$$\|T_p f\|_p \leqslant m_d c_p \delta \|f\|_p, \quad m_d c_p \delta < 1, \quad c_p = pp'/4. \tag{4}$$

Their extensions by continuity we denote again by $Q_p(q)$, T_p.

(ii) Set $G_p = b^{\frac{1}{p}} \cdot \nabla (\zeta - \Delta)^{-1}$, $Q_p = (\zeta - \Delta)^{-1} |b|^{\frac{1}{p'}}$, $P_p = |b|^{\frac{1}{p}} (\zeta - \Delta)^{-1}$. *The operator* Q_p *is densely defined on* $\mathcal{E}$. *There exist constants* C_i, $i = 1, 2, 3$, *such that*

$$\|G_p\|_{p \to p} \leqslant C_1 |\zeta|^{-\frac{1}{2p'}}, \quad \|P_p\|_{p \to p} \leqslant C_3 |\zeta|^{-\frac{1}{2} - \frac{1}{2p'}}, \quad \|Q_p f\|_p \leqslant C_2 |\zeta|^{-\frac{1}{2} - \frac{1}{2p}} \|f\|_p \quad (f \in \mathcal{E}), \quad \zeta \in \mathcal{O}. \tag{5}$$

The extension of Q_p *by continuity we denote again by* Q_p.

Remark 7. The proof of Proposition 1 uses ideas from [BS], [LS], and appeals to the L^p-inequalities between the operator $(\lambda - \Delta)^{\frac{1}{2}}$ and the "potential" $|b|$.

Proof. (i) Let $r \in (1, \infty)$. Then

(**a**) $\mu \geqslant \lambda \ \Rightarrow\ \||b|^{\frac{1}{r}} (\mu - \Delta)^{-\frac{1}{2}}\|_{r \to r} \leqslant C_{r,\delta} \mu^{-\frac{1}{2r'}}$, $C_{r,\delta} = (c_r \delta)^{\frac{1}{r}}$, $c_r = rr'/4$.

Indeed, define in L^2 $A = (\mu - \Delta)^{\frac{1}{2}}$, $D(A) = W^{1,2}$. Then $-A$ is a symmetric Markov generator. Therefore (see e.g. [LS, Theorem 2.1]), for any $r \in (1, \infty)$,

$$0 \leqslant u \in D(A_r) \Rightarrow v := u^{\frac{r}{2}} \in D(A^{\frac{1}{2}}) \text{ and } c_r^{-1} \|A^{\frac{1}{2}} v\|_2^2 \leqslant \langle A_r u, u^{r-1} \rangle.$$

Now let u be the solution of $A_r u = |f|$, $f \in L^r$. Note that $\|u\|_r \leqslant \mu^{-\frac{1}{2}} \|f\|_r$ (using $\|(\mu - \Delta)^{-1}\|_{r \to r} \leqslant \mu^{-1}$ in (A.5) with $\alpha = \frac{1}{2}$).

Since $b \in \mathbf{F}_\delta^{\frac{1}{2}}$, we have

$$(c_r \delta)^{-1} \||b|^{\frac{1}{2}} v\|_2^2 \leqslant \langle A_r u, u^{r-1} \rangle,$$

and so $\||b|^{\frac{1}{r}} u\|_r^r \leqslant c_r \delta \|f\|_r \|u\|_r^{r-1}$, $\||b|^{\frac{1}{r}} A_r^{-1} |f|\|_r^r \leqslant c_r \delta \mu^{-\frac{r-1}{2}} \|f\|_r^r$. (**a**) is proved.

(**b**) $\mu \geqslant \lambda \ \Rightarrow\ \||b|^{\frac{1}{r}} (\mu - \Delta)^{-\frac{1}{2}} |b|^{\frac{1}{r'}} f\|_r \leqslant c_r \delta \|f\|_r$, $f \in \mathcal{E}$.

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Indeed, let u be the solution of $Au = |b|^{\frac{1}{r'}}|f|$, $f \in \mathcal{E}$. Then, arguing as in (**a**), we have

$$\||b|^{\frac{1}{r}}u\|_r^r \leqslant c_r\delta\|f\|_r\||b|^{\frac{1}{r}}u\|_r^{r-1},$$

or $\||b|^{\frac{1}{r}}u\|_r \leqslant c_r\delta\|f\|_r$. (**b**) is proved.

(**c**) $\mu \geqslant \lambda \;\Rightarrow\; \|(\mu-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{r'}}f\|_r \leqslant C_{r',\delta}\,\mu^{-\frac{1}{2r}}\|f\|_r$, $f \in \mathcal{E}$.

Indeed, (**c**) follows from (**a**) by duality.

Let us prove (4). Let $\zeta \in \mathcal{O}$. Using (A.1) + (**b**) with $r = p \in \mathcal{I}$, $\mu = \lambda$, we obtain:

$$\|T_p f\|_p \leqslant m_d\|b^{\frac{1}{p}}(\kappa_d^{-1}\mathrm{Re}\,\zeta - \Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}|f|\|_p \leqslant m_d c_p\delta\|f\|_p, \quad f \in \mathcal{E}.$$

$m_d c_p \delta < 1$ since $p \in \mathcal{I}$.

Next, we estimate $\|Q_p(q)\|_{p\to p}$, $\|G_p(r)\|_{p\to p}$. Let $\mathrm{Re}\,\zeta \geqslant \lambda$, $p<q$. We obtain:

$$\begin{aligned}\|Q_p(q)f\|_p &\leqslant \|(\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{2q'}}|b|^{\frac{1}{p'}}|f|\|_p \\ &\leqslant \|(\lambda-\Delta)^{-\frac{1}{2q'}}|b|^{\frac{1}{p'}}|f|\|_p \\ &\quad \text{(here we are using (A.5) with } \alpha = 1/2q') \\ &\leqslant k_{q'}\int_0^\infty t^{-\frac{1}{2q'}}\|(t+\lambda-\Delta)^{-1}|b|^{\frac{1}{p'}}|f|\|_p dt \qquad \left(k_{q'} := \frac{\sin\frac{\pi}{2q'}}{\pi}\right) \\ &\leqslant k_{q'}\int_0^\infty t^{-\frac{1}{2q'}}(t+\lambda)^{-\frac{1}{2}}\|(t+\lambda-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}|f|\|_p dt \\ &\quad \text{(here we are using (c) with } r = p \in \mathcal{I},\ \mu = t+\lambda) \\ &\leqslant k_{q'}C_{p',\delta}\int_0^\infty t^{-\frac{1}{2q'}}(t+\lambda)^{-\frac{1}{2}-\frac{1}{2p}}dt\,\|f\|_p = K_{2,q}\|f\|_p, \quad f \in \mathcal{E},\end{aligned}$$

where, clearly, $K_{2,q} < \infty$ due to $q > p$.

Let $\zeta \in \mathcal{O}$, $1 \leqslant r < p$. Using (A.2), we obtain:

$$\begin{aligned}\|G_p(r)f\|_p &\leqslant m_{r,d}\||b|^{\frac{1}{p}}(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{2r}}|f|\|_p \\ &\leqslant m_{r,d}\||b|^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2r}}|f|\|_p \\ &\quad \text{(here we are using (A.5) with } \alpha = 1/2r) \\ &\leqslant m_{r,d}k_r\int_0^\infty t^{-\frac{1}{2r}}\||b|^{\frac{1}{p}}(t+\lambda-\Delta)^{-1}|f|\|_p dt \\ &\leqslant m_{r,d}k_r\int_0^\infty t^{-\frac{1}{2r}}\||b|^{\frac{1}{p}}(t+\lambda-\Delta)^{-\frac{1}{2}}\|_{p\to p}\|(t+\lambda-\Delta)^{-\frac{1}{2}}|f|\|_p dt \\ &\quad \text{(here we are using (a) with } r = p \in \mathcal{I},\ \mu = t+\lambda) \\ &\leqslant m_{r,d}k_r C_{p,\delta}\int_0^\infty t^{-\frac{1}{2r}}(t+\lambda)^{-\frac{1}{2p'}-\frac{1}{2}}dt\,\|f\|_p = K_{1,r}\|f\|_p, \quad f \in \mathcal{E},\end{aligned}$$

where, clearly, $K_{1,r} < \infty$ due to $r < p$. The proof of (i) is completed.

(ii) Let $\mathrm{Re}\,\zeta \geqslant \lambda$. We have

$$\begin{aligned}\|Q_p(2\zeta,b)f\|_p &\leqslant \|(2\zeta-\Delta)^{-\frac{1}{2}}\|_{p\to p}\|(2\zeta-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}f\|_p \\ &\quad \text{(here we are applying (A.4) twice + (c) with } r = p \in \mathcal{I},\ \mu = |\zeta|) \\ &\leqslant 2^{\frac{d}{4}+\frac{1}{4}}2^{-\frac{1}{2}}|\zeta|^{-\frac{1}{2}}C_{p',\delta}2^{\frac{d}{4}+\frac{1}{4}}|\zeta|^{-\frac{1}{2p}}\|f\|_p, \quad f \in \mathcal{E}.\end{aligned}$$

Now, using the identity $(\zeta-\Delta)^{-1}=\big(1+\zeta(\zeta-\Delta)^{-1}\big)(2\zeta-\Delta)^{-1}$, we obtain:

$$\begin{aligned}\|Q_p(\zeta,b)f\|_p &\leqslant \|1+\zeta(\zeta-\Delta)^{-1}\|_{p\to p}\|Q_p(2\zeta,b)f\|_p\\ &\leqslant 2^{\frac12}|\zeta|^{-\frac12}C_{p',\delta}2^{\frac d2+\frac12}|\zeta|^{-\frac{1}{2p}}\|f\|_p\\ &= C_2|\zeta|^{-\frac12-\frac{1}{2p}}\|f\|_p, \quad f\in\mathcal{E}.\end{aligned}$$

Let Re $\zeta\geqslant\lambda$. We have:

$$\begin{aligned}\|P_p(2\zeta,b)\|_{p\to p} &\leqslant \||b|^{\frac1p}(2\zeta-\Delta)^{-\frac12}\|_{p\to p}\|(2\zeta-\Delta)^{-\frac12}\|_{p\to p}\\ &\quad\text{(here we are applying (A.4) twice)}\\ &\leqslant 2^{\frac d2+\frac12}\||b|^{\frac1p}(|\zeta|-\Delta)^{-\frac12}\|_{p\to p}|\zeta|^{-\frac12}\\ &\quad\text{(here we are using } (\mathbf{a}) \text{ with } r=p\in\mathcal{I},\ \mu=|\zeta|)\\ &\leqslant C_{p,\delta}2^{\frac d2+\frac12}|\zeta|^{-\frac12-\frac{1}{2p'}}.\end{aligned}$$

Now, using the identity $(\zeta-\Delta)^{-1}=(2\zeta-\Delta)^{-1}\big(1+\zeta(\zeta-\Delta)^{-1}\big)$, we obtain:

$$\begin{aligned}\|P_p(\zeta,b)\|_{p\to p} &\leqslant 2C_{p,\delta}2^{\frac d2+\frac12}|\zeta|^{-\frac12-\frac{1}{2p'}}\\ &= C_3|\zeta|^{-\frac12-\frac{1}{2p'}}.\end{aligned}$$

Let $\zeta\in\mathcal{O}$. Using (A.3) + ($\mathbf{a}$) with $r=p\in\mathcal{I}$, $\mu=|\zeta|$, we obtain:

$$\|G_p(2\kappa_d\zeta,b)\|_{p\to p}\leqslant m_dC_{p,\delta}2^{\frac d4}|\zeta|^{-\frac{1}{2p'}}.$$

Now, using the identity $(\zeta-\Delta)^{-1}=(2\kappa_d\zeta-\Delta)^{-1}\big(1+(2\kappa_d-1)\zeta(\zeta-\Delta)^{-1}\big)$, we obtain:

$$\begin{aligned}\|G_p(\zeta,b)\|_{p\to p} &\leqslant 2\kappa_d m_dC_{p,\delta}2^{\frac d4}|\zeta|^{-\frac{1}{2p'}}\\ &= C_1|\zeta|^{-\frac{1}{2p'}}.\end{aligned}$$

The proof of (ii) is completed. □

Remark 8. Since $|b_n|\leqslant|b|$ a.e., Proposition 1 is valid for b_n, $n=1,2,\dots$, with the same constants.

Proposition 2. *For every $p\in\mathcal{I}$, and $n=1,2,\dots$, the operator-valued function $\Theta_p(\zeta,b_n)$ is a pseudo-resolvent on $\mathcal{O}$, i.e.*

$$\Theta_p(\zeta,b_n)-\Theta_p(\eta,b_n)=(\eta-\zeta)\Theta_p(\zeta,b_n)\Theta_p(\eta,b_n),\quad \zeta,\eta\in\mathcal{O}.$$

Proof. Define $S^k_\zeta:=(-1)^k(\zeta-\Delta)^{-1}b_n\cdot\nabla(\zeta-\Delta)^{-1}\dots b_n\cdot\nabla(\zeta-\Delta)^{-1}$, $k:=\#\ b_n$'s. Obviously,

$$\begin{aligned}\Theta_p(\zeta,b_n) &:= (\zeta-\Delta)^{-1}-Q\left(1+T\right)^{-1}G\\ &= (\zeta-\Delta)^{-1}-Q\sum_{k=0}^\infty(-1)^kT^k\,G=\sum_{k=0}^\infty S^k_\zeta \quad \text{(absolutely convergent in } L^p),\end{aligned}$$

$$\Theta_p(\zeta,b_n)\Theta_p(\eta,b_n)=\sum_{\ell=0}^\infty\sum_{i=0}^\ell S^i_\zeta S^{\ell-i}_\eta,\quad \zeta,\eta\in\mathcal{O}. \tag{6}$$

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Define

$$I^k_{j,m}(\zeta,\eta) := (\zeta-\Delta)^{-1} b_n\cdot\nabla(\zeta-\Delta)^{-1}\dots b_n\cdot\nabla(\zeta-\Delta)^{-1}$$
$$b_n\cdot\nabla(\eta-\Delta)^{-1} b_n\cdot\nabla(\eta-\Delta)^{-1}\dots b_n\cdot\nabla(\eta-\Delta)^{-1},$$
$$j := \#\zeta\text{'s},\quad m := \#\eta\text{'s},\quad k := \#b_n\text{'s}.$$

Substituting the identity $(\zeta-\Delta)^{-1}(\eta-\Delta)^{-1} = (\eta-\zeta)^{-1}\big((\zeta-\Delta)^{-1}-(\eta-\Delta)^{-1}\big)$ inside the product

$$S^k_\zeta S^j_\eta$$
$$= (-1)^{k+j}(\zeta-\Delta)^{-1} b_n\cdot\nabla(\zeta-\Delta)^{-1}\dots\, b_n\cdot\nabla\, \underbrace{(\zeta-\Delta)^{-1}(\eta-\Delta)^{-1}}_{(\eta-\zeta)^{-1}((\zeta-\Delta)^{-1}-(\eta-\Delta)^{-1})}\, b_n\cdot\nabla(\eta-\Delta)^{-1}\dots\, b_n\cdot\nabla(\eta-\Delta)^{-1},$$

we obtain $S^k_\zeta S^j_\eta = (\eta-\zeta)^{-1}(-1)^{k+j}\big[I^{k+j}_{k+1,j} - I^{k+j}_{k,j+1}\big]$. Therefore,

$$\sum_{i=0}^{\ell} S^i_\zeta S^{\ell-i}_\eta = (\eta-\zeta)^{-1}(-1)^\ell\Big[I^\ell_{1,\ell} - I^\ell_{0,\ell+1} + I^\ell_{2,\ell-1} - I^\ell_{1,\ell} + \dots + I^\ell_{\ell+1,0} - I^\ell_{\ell,1}\Big]$$
$$= (\eta-\zeta)^{-1}(-1)^\ell\big(I^\ell_{\ell+1,0} - I^\ell_{0,\ell+1}\big).$$

Substituting the last identity in the right-hand side of (6), we obtain

$$\Theta_p(\zeta,b_n)\Theta(\eta,b_n) = (\eta-\zeta)^{-1}\sum_{\ell=0}^{\infty}(-1)^\ell\big(I^\ell_{\ell+1,0} - I^\ell_{0,\ell+1}\big) = (\eta-\zeta)^{-1}\big(\Theta_p(\zeta,b_n)h - \Theta_p(\eta,b_n)\big).$$

□

Proposition 3. *For every $p\in\mathcal{I}$, and $n=1,2,\dots,$*

(i) $\|\Theta_p(\zeta,b_n)\|_{p\to p} \leqslant C_p|\zeta|^{-1}$, $\zeta\in\mathcal{O}$, *for a constant C_p independent of n.*

(ii) $\mu\Theta_p(\mu,b_n) \overset{s}{\to} 1$ *in L^p as $\mu\uparrow\infty$ (uniformly in n),*

Proof. Proof of (i). Put $Q_p\equiv Q_p(\zeta,b_n)$, $T_p\equiv T_p(\zeta,b_n)$, $G_p\equiv G_p(\zeta,b_n)$. By the definition of $\Theta_p(\zeta,b_n)$, see (2), for every $\zeta\in\mathcal{O}$,

$$\|\Theta_p(\zeta,b_n)\|_{p\to p} \leqslant \|(\zeta-\Delta)^{-1}\|_{p\to p} + \|Q_p\|_{p\to p}\|(1+T_p)^{-1}\|_{p\to p}\|G_p\|_{p\to p}$$
$$\text{(here we are using (4), (5) in Proposition 1)}$$
$$\leqslant |\zeta|^{-1} + C_2|\zeta|^{-\frac{1}{2}-\frac{1}{2p'}}(1-m_dc_p\delta)^{-1}C_1|\zeta|^{-\frac{1}{2p}}$$
$$\leqslant C_p|\zeta|^{-1},\quad C_p := 1 + C_1C_2(1-m_dc_p\delta)^{-1}.$$

Proof of (ii). Put $\Theta_p\equiv\Theta_p(\mu,b_n)$, $Q_p\equiv Q_p(\mu,b_n)$, $T_p\equiv T_p(\mu,b_n)$, $P_p\equiv P_p(\mu,b_n)$. Since $\mu(\mu-\Delta)^{-1}\overset{s}{\to}1$, it suffices to show that $\mu\Theta_p - \mu(\mu-\Delta)^{-1}\overset{s}{\to}0$ in L^p. Since by (i) $\mu\Theta_p$ is uniformly (in μ) bounded in $\mathcal{B}(L^p)$, and C_c^∞ is dense in L^p, it suffices to show that $\mu\Theta_p h - \mu(\mu-\Delta)^{-1}h\to 0$ in L^p for every $h\in C_c^\infty$. Write

$$\Theta_p h - (\mu-\Delta)^{-1}h = -Q_p\big(1+T_p\big)^{-1} b_n^{\frac{1}{p}}\cdot(\mu-\Delta)^{-1}\nabla h.$$

By (4), $\quad \|\big(1+T_p\big)^{-1}\|_{p\to p} \leqslant \dfrac{1}{1-\|T_p\|_{p\to p}} \leqslant \dfrac{1}{1-m_dc_p\delta} < \infty,\quad$ by (5), $\quad \|Q_p\|_{p\to p}\leqslant C_2\mu^{-\frac{1}{2}-\frac{1}{2p}}.$

Again, by (5),

$$\begin{aligned}\|b_n^{\frac{1}{p}} \cdot (\mu - \Delta)^{-1} \nabla h\|_p &\leqslant \| |b_n|^{\frac{1}{p}} (\mu - \Delta)^{-1} |\nabla h| \|_p \\ &\leqslant \|P_p\|_{p \to p} \|\nabla h\|_p \\ &\leqslant C_3 \mu^{-\frac{1}{2} - \frac{1}{2p'}} \|\nabla h\|_p.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Theta_p h - (\mu - \Delta)^{-1} h\|_p &\leqslant \|Q_p\|_{p \to p} \|\big(1 + T_p\big)^{-1}\|_{p \to p} \|b_n^{\frac{1}{p}} \cdot (\mu - \Delta)^{-1} \nabla h\|_p \\ &\leqslant C_0 \mu^{-\frac{3}{2}} \|\nabla h\|_p\end{aligned}$$

for a $C_0 < \infty$ independent of n, which clearly implies (ii). □

Proposition 4. *For every $p \in \mathcal{I}$, and $n = 1, 2, \dots$, we have $\mathcal{O} \subset \rho(-\Lambda_p(b_n))$, the resolvent set of $-\Lambda_p(b_n)$. The operator-valued function $\Theta_p(\zeta, b_n)$ is the resolvent of $-\Lambda_p(b_n)$:*

$$\Theta_p(\zeta, b_n) = (\zeta + \Lambda_p(b_n))^{-1}, \quad \zeta \in \mathcal{O},$$

and

$$\|(\zeta + \Lambda_p(b_n))^{-1}\|_{p \to p} \leqslant C_p |\zeta|^{-1}, \quad \zeta \in \mathcal{O}.$$

Proof. By definition, we need to verify that, for every $\zeta \in \mathcal{O}$, $\Theta_p(\zeta, b_n)$ has dense image, and is the left and the right inverse of $\zeta + \Lambda_p(b_n)$. Indeed, Proposition 3(ii) implies that $\Theta_p(\zeta, b_n)$ has dense image. $\Lambda_p(b_n) := -\Delta + b_n \cdot \nabla$, $D(\Lambda_p(b_n)) = W^{2,p}$, is the generator of a C_0-semigroup $e^{-t\Lambda_p(b_n)}$ on L^p. Clearly, $\Theta_p(\zeta_n, b_n) = (\zeta_n + \Lambda_p(b_n))^{-1}$ for all sufficiently large ζ_n $(= \zeta(\|b_n\|_\infty))$, therefore, by Proposition 2,

$$\Theta_p(\zeta, b_n) = (\zeta_n + \Lambda_p(b_n))^{-1} \big(1 + (\zeta_n - \zeta) \Theta_p(\zeta, b_n)\big), \quad \zeta \in \mathcal{O},$$

so $\Theta_p(\zeta, b_n) L^p \subset D(\Lambda_p(b_n)) = W^{2,p}$, and $(\zeta + \Lambda_p(b_n)) \Theta_p(\zeta, b_n) g = g$, $g \in L^p$, i.e. $\Theta_p(\zeta, b_n)$ is the right inverse of $\zeta + \Lambda_p(b_n)$ on $\mathcal{O}$. Similarly, it is seen that $\Theta(\zeta, b_n)$ is the left inverse of $\zeta + \Lambda_p(b_n)$ on $\mathcal{O}$.

Remark 9. Alternatively, we could verify conditions of the Kato theorem [Ka2]: in the reflexive space L^p, the pseudo-resolvent $\Theta_p(\zeta, b_n)$ (see Proposition 2) satisfying $\mu \Theta_p(\mu, b_n) \overset{s}{\to} 1$ in L^p as $\mu \uparrow \infty$ (see Proposition 3(ii)) is the resolvent of a densely defined closed operator on L^p. This operator coincides with $-\Lambda_p(b_n)$ (since $\Theta_p(\zeta_n, b_n) = (\zeta_n + \Lambda_p(b_n))^{-1}$ for all large ζ_n).

Now, $\|(\zeta + \Lambda_p(b_n))^{-1}\|_{p \to p} \leqslant C_p |\zeta|^{-1}$, $\zeta \in \mathcal{O}$, follows from Proposition 3(i). □

Proposition 5. *For every $\zeta \in \mathcal{O}$ and $p \in \mathcal{I}$,*

$$\Theta_p(\zeta, b_n) \overset{s}{\to} \Theta_p(\zeta, b) \text{ in } L^p,$$

Proof. Put $\Theta_p(b) \equiv \Theta_p(\zeta, b)$, $Q_p(b) \equiv Q_p(\zeta, b)$, $T_p(b) \equiv T_p(\zeta, b)$, $G_p(b) \equiv G_p(\zeta, b)$ (similarly for b_n's). It suffices to prove that

$$Q_p(b_n)(1 + T(b_n))^{-1} G_p(b_n) \overset{s}{\to} Q_p(b)(1 + T_p(b))^{-1} G_p(b).$$

Thus it suffices to prove consecutively that

$$G_p(b_n) \overset{s}{\to} G_p(b), \; (1 + T_p(b_n))^{-1} \overset{s}{\to} (1 + T_p(b))^{-1}, \; Q_p(b_n) \overset{s}{\to} Q_p(b).$$

In turn, since $(1+T_p(b_n))^{-1}-(1+T_p(b))^{-1} = (1+T_p(b_n))^{-1}(T_p(b)-T_p(b_n))(1+T_p(b))^{-1}$, it suffices to prove that $T_p(b_n) \overset{s}{\to} T_p(b)$. Finally,

$$T_p(b_n)-T_p(b) = T_p(b_n) - b_n^{\frac{1}{p}} \cdot \nabla(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}} + b_n^{\frac{1}{p}} \cdot \nabla(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}} - T_p(b),$$

and hence we have to prove that

$$b_n^{\frac{1}{p}} \cdot \nabla(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}} - T_p(b) := J_n^{(1)} \overset{s}{\to} 0 \text{ and } T_p(b_n) - b_n^{\frac{1}{p}} \cdot \nabla(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}} := J_n^{(2)} \overset{s}{\to} 0.$$

Now, by the Dominated Convergence Theorem (cf. the argument in the proof of (A.0)), $G_p(b_n) \overset{s}{\to} G_p(b)$, $J_n^{(1)}|_{\mathcal{E}} \overset{s}{\to} 0$. Also

$$\begin{aligned}\|J_n^{(2)} f\|_p =& \|G_p(b_n)(|b_n|^{\frac{1}{p'}} - |b|^{\frac{1}{p'}})f\|_p \\ &\leqslant \|G_p(b_n)\|_{p\to p}\|(|b_n|^{\frac{1}{p'}} - |b|^{\frac{1}{p'}})f\|_p \\ &\leqslant m_d(1+\delta)|\zeta|^{-\frac{1}{2p'}}\|(|b_n|^{\frac{1}{p'}} - |b|^{\frac{1}{p'}})f\|_p, \quad (f \in \mathcal{E}).\end{aligned}$$

Thus, $J_n^{(2)}|_{\mathcal{E}} \overset{s}{\to} 0$. Since $\|J_n^{(2)}\|_{p\to p}, \|J_n^{(1)}\|_{p\to p} \leqslant m_d\delta$, we conclude that $T_p(b_n) \overset{s}{\to} T_p(b)$. It is clear now that $Q_p(b_n) \overset{s}{\to} Q_p(b)$. □

Now we are going to prove Theorem 1 using the Trotter approximation theorem [Ka1, IX.2.5]. Recall its conditions (in terms of $\Theta_p(\zeta, b_n)$ on the base of Proposition 4):

1) $\sup_{n\geqslant 1} \|\Theta_p(\zeta, b_n)\|_{p\to p} \leqslant C_p|\zeta|^{-1}$, $\zeta \in \mathcal{O}$.

2) $\mu\Theta_p(\mu, b_n) \overset{s}{\to} 1$ in L^p as $\mu \uparrow \infty$ uniformly in n.

3) There exists $s\text{-}L^p\text{-}\lim_n \Theta_p(\zeta, b_n)$ for some $\zeta \in \mathcal{O}$.

Now, 1) is the content of Proposition 3(i). 2) is Proposition 3(ii). Proposition 5 implies 3).

Therefore, by the Trotter approximation theorem, $\Theta_p(\zeta, b) = (\zeta + \Lambda_p(b))^{-1}$, $\zeta \in \mathcal{O}$, where $\Lambda_p(b)$ is the generator of the holomorphic C_0-semigroup $e^{-t\Lambda_p(b)}$ on L^p. (Note that, by Proposition 5, $\|\Theta_p(\zeta, b)\|_{p\to p} \leqslant C_p|\zeta|^{-1}$, $\zeta \in \mathcal{O}$. Hence, $\Theta_p(\zeta, b)$ can be extended to $\mathcal{O} \cup \{\zeta \in \mathbb{C} : |\operatorname{Arg}\zeta| < \frac{\pi}{2} + \varepsilon, |\zeta| > R\}$, $\varepsilon > 0$, for a sufficiently large $R > 0$, where it satisfies $\|\Theta_p(\zeta, b)\|_{p\to p} \leqslant C_p'|\zeta|^{-1}$, see the corresponding argument in [Yo, IX.10].)

Hence, the assertions (*i*), (*vi*) of Theorem 1 follow. (*ii*) follows from Proposition 3(i) and Proposition 5. (*iii*) is obvious from the definitions of the operators involved, cf. Proposition 1.

(*iii*) ⇒ (*iv*). In particular, if $p > d-1$, given a $0 < \gamma < 1 - \frac{d-1}{p}$, we can select $q > p$ sufficiently close to p so that by the Sobolev embedding theorem the Bessel space $\mathcal{W}^{1+\frac{1}{q},p}$ is embedded into $C^{0,\gamma}$.

(*v*) Let $\zeta \in \mathcal{O}$. By Proposition 5, $\Lambda_p(b_n)(\zeta + \Lambda_p(b_n))^{-1} \overset{s}{\to} \Lambda_p(b)(\zeta + \Lambda_p(b))^{-1}$ in L^p. Put $Q_p(b) \equiv Q_p(\zeta, b)$, $T_p(b) \equiv T_p(\zeta, b)$, $G_p(b) \equiv G_p(\zeta, b)$ (similarly for b_n's). Since $(\zeta + \Lambda_p(b))^{-1} = (\zeta-\Delta)^{-1} - Q_p(b)(1+T_p(b))^{-1}G_p(b)$, we have

$$b^{\frac{1}{p}} \cdot \nabla(\zeta + \Lambda_p(b))^{-1} = G_p(b) - T_p(b)(1+T_p(b))^{-1}G_p(b)$$

(similarly for b_n's). Since $G_p(b_n) \overset{s}{\to} G_p(b)$, $T_p(b_n) \overset{s}{\to} T_p(b)$ in L^p (see the proof of Proposition 5),

$$b_n^{\frac{1}{p}} \cdot \nabla(\zeta + \Lambda_p(b_n))^{-1} \overset{s}{\to} b^{\frac{1}{p}} \cdot \nabla(\zeta + \Lambda_p(b))^{-1} \text{ in } L^p. \qquad (**)$$

Clearly, $|b|^{\frac{1}{p'}} \in L^{p'}_{\rm loc}$, for $|b| \in L^1_{\rm loc}$ by the definition of class $\mathbf{F}_\delta^{\frac{1}{2}}$. Now, given $u \in D(\Lambda_p(b))$, we have $u = (\zeta + \Lambda_p(b))^{-1} g$ for some $g \in L^p$, and so, for every $v \in C_c^\infty$,

$$\begin{aligned}\langle \Lambda_p(b)u, v\rangle &= \langle \Lambda_p(b)(\zeta + \Lambda_p(b))^{-1}g, v\rangle \\ &= \lim_n \langle \Lambda_p(b_n)(\zeta + \Lambda_p(b_n))^{-1}g, v\rangle \\ &= \lim_n \langle (\zeta + \Lambda_p(b_n))^{-1}g, -\Delta v\rangle + \lim_n \langle b_n^{\frac{1}{p}} \cdot \nabla(\zeta + \Lambda_p(b_n))^{-1}g, |b_n|^{\frac{1}{p'}} v\rangle \\ &\text{(here we are using } (**) \text{ and the fact that } |b_n|^{\frac{1}{p'}} v \to |b|^{\frac{1}{p'}} v \text{ in } L^{p'}) \\ &= \langle (\zeta + \Lambda_p(b))^{-1}g, -\Delta v\rangle + \langle b^{\frac{1}{p}} \cdot \nabla(\zeta + \Lambda_p(b))^{-1}g, |b|^{\frac{1}{p'}} v\rangle \\ &= \langle u, -\Delta v\rangle + \langle b^{\frac{1}{p}} \cdot \nabla u, |b|^{\frac{1}{p'}} v\rangle.\end{aligned}$$

Next, since for $u \in D(\Lambda_p(b))$, $b^{\frac{1}{p}} \cdot \nabla u \in L^p$, it follows that $b \cdot \nabla u = |b|^{\frac{1}{p'}}\, b^{\frac{1}{p}} \cdot \nabla u \in L^1_{\rm loc}$. Also, $\Lambda_p(b)u \in L^p$, and hence $\langle \Lambda_p(b)u, v\rangle = \langle u, -\Delta v\rangle + \langle b \cdot \nabla u, v\rangle$. Therefore, Δu (understood in the sense of distributions) $= -\Lambda_p(b)u + b \cdot \nabla u \in L^1_{\rm loc}$, i.e. $u \in \mathcal{W}^{2,1}_{\rm loc}$. The proof of (v) is completed.

For the proof of $(viii)$ see the argument in [Se2, p. 415-416].

The proof of Theorem 1 is completed.

2. Proof of Theorem 2

It is easily seen that, due to the strict inequality $m_d\delta < 4\frac{d-2}{(d-1)^2}$, for every $\tilde{\delta} > \delta$ such that $m_d\tilde{\delta} < 4\frac{d-2}{(d-1)^2}$ there exists $\{\varepsilon_n\}$, $\varepsilon_n \downarrow 0$, such that

$$\tilde{b}_n := \eta_{\varepsilon_n} * b_n \in \mathbf{F}_{\tilde{\delta}}^{\frac{1}{2}}, \quad n = 1, 2, \dots$$

(i) We verify conditions of the Trotter approximation theorem:

1°) $\sup_n \|(\mu + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}\|_{\infty\to\infty} \leqslant \mu^{-1}$, $\mu \geqslant \kappa_d\lambda$.

2°) $\mu(\mu + \Lambda_{C_\infty}(\tilde{b}_n))^{-1} \to 1$ in C_∞ as $\mu \uparrow \infty$ uniformly in n.

3°) There exists s-C_∞-$\lim_n(\mu + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}$ for some $\mu \geqslant \kappa_d\lambda$.

The condition 1°) is immediate. In view of 1°), it suffices to verify 2°), 3°) on $\mathcal{S}$, the L. Schwartz space of test functions. Fix $p \in \mathcal{I}$, $p > d-1$ (such p exists since $m_d\tilde{\delta} < 4\frac{d-2}{(d-1)^2}$).

Proposition 6. *For every $\mu \geqslant \kappa_d\lambda$, $n = 1, 2, \dots$, $\Theta_p(\mu, \tilde{b}_n)\mathcal{S} \subset \mathcal{S}$, and*

$$(\mu + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}|_{\mathcal{S}} = \Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}}.$$

Proof. The inclusion $\Theta_p(\mu, \tilde{b}_n)\mathcal{S} \subset \mathcal{S}$ is clear. Clearly, $\Theta_p(\mu_n, \tilde{b}_n)|_{\mathcal{S}} = (\mu_n + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}|_{\mathcal{S}}$ for all sufficiently large μ_n $(= \mu(\|\tilde{b}_n\|_\infty))$. By $\Theta_p(\mu, \tilde{b}_n)\mathcal{S} \subset \mathcal{S}$ and Proposition 2, $\Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}}$ satisfies the resolvent identity on $\mu \geqslant \kappa_d\lambda$,

$$\Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}} = (\mu_n + \Lambda_{C_\infty}(\tilde{b}_n))^{-1}\big(1 + (\mu_n - \mu)\Theta_p(\mu, \tilde{b}_n)\big)|_{\mathcal{S}}, \quad \mu \geqslant \kappa_d\lambda,$$

so $\Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}}$ is the right inverse of $\mu + \Lambda_{C_\infty}(\tilde{b}_n)|_{\mathcal{S}}$ on $\mu \geqslant \kappa_d\lambda$. Similarly, it is seen that $\Theta_p(\mu, \tilde{b}_n)|_{\mathcal{S}}$ is the left inverse of $\mu + \Lambda_{C_\infty}(\tilde{b}_n)|_{\mathcal{S}}$ on $\mu \geqslant \kappa_d\lambda$. □

Proposition 7. *For every $\mu \geqslant \kappa_d\lambda$, $\Theta_p(\mu, b)\mathcal{S} \subset C_\infty$, and*

$$\Theta_p(\mu, \tilde{b}_n) \overset{s}{\to} \Theta_p(\mu, b) \text{ in } C_\infty.$$

Proof. By Theorem 1(*iv*), since $p > d-1$, $\Theta_p(\mu,b)L^p \subset C_\infty$. Put

$$Q_p(q,b) \equiv Q_p(q,\mu,b), \quad T_p(b) \equiv T_p(\mu,b), \quad G_p(b) \equiv G_p(\mu,b).$$

To establish the required convergence, it suffices to prove that

$$(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}}Q_p(q,\tilde{b}_n)(1+T_p(\tilde{b}_n))^{-1}G_p(\tilde{b}_n) \overset{s}{\to} (\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}}Q_p(q,b)(1+T_p(b))^{-1}G_p(b) \text{ in } C_\infty.$$

We choose q $(> p)$ close to $d-1$ so that $(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}}L^p \hookrightarrow C_\infty$. Thus it suffices to prove that

$$G_p(\tilde{b}_n) \overset{s}{\to} G_p(b),\ (1+T_p(\tilde{b}_n))^{-1} \overset{s}{\to} (1+T_p(b))^{-1},\ Q_p(q,\tilde{b}_n) \overset{s}{\to} Q_p(q,b) \text{ in } L^p,$$

which can be done by repeating the arguments in the proof of Proposition 5. □

Proposition 8.

$$\mu\Theta_p(\mu,\tilde{b}_n) \overset{s}{\to} 1 \text{ as } \mu\uparrow\infty \text{ in } C_\infty \text{ uniformly in } n. \tag{7}$$

Proof. Put $\Theta_p \equiv \Theta_p(\mu,\tilde{b}_n)$, $T_p \equiv T_p(\mu,\tilde{b}_n)$. Since $\mu(\mu-\Delta)^{-1} \overset{s}{\to} 1$ in C_∞, and $\mathcal{S}$ is dense in C_∞, it suffices to show that $\|\mu\Theta_p f - \mu(\mu-\Delta)^{-1}f\|_\infty \to 0$ for every $f \in \mathcal{S}$. For each $f \in \mathcal{S}$ there is $h \in \mathcal{S}$ such that $f = (\lambda-\Delta)^{-\frac{1}{2}}h$, where $\lambda = \lambda_\delta > 0$. Let $q > p$. Write

$$\Theta_p f - (\mu-\Delta)^{-1}f = -(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}}Q_p(q)\big(1+T_p\big)^{-1}b^{\frac{1}{p}}(\lambda-\Delta)^{-\frac{1}{2}}\cdot(\mu-\Delta)^{-1}\nabla h.$$

Now, arguing as in the proof of Proposition 3(i), but using estimates

$$\|(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}}\|_{p\to\infty} \leqslant c\mu^{-\frac{1}{2}+\frac{d}{2p}-\frac{1}{2q}}, \quad c<\infty, \quad \text{and} \quad \|Q_p(q)\|_{p\to p} \leqslant \tilde{K}_{2,q} < \infty \quad (\text{see } (5)),$$

we obtain

$$\|\Theta_p f - (\mu-\Delta)^{-1}f\|_\infty \leqslant C\mu^{-\frac{1}{2}+\frac{d}{2p}-\frac{1}{2q}}\mu^{-1}\|\nabla h\|_p.$$

Since $p > d-1$, choosing q sufficiently close to p, we obtain

$$-\frac{1}{2}+\frac{d}{2p}-\frac{1}{2q}-1 < -1,$$

so $\mu\Theta_p - \mu(\mu-\Delta)^{-1} \overset{s}{\to} 0$ in C_∞, as needed. □

Now, Proposition 7 verifies condition 3°), and Proposition 8 verifies condition 2°). The assertion (*i*) of Theorem 2 now follows from the Trotter approximation theorem.

Assertion (*ii*) of Theorem 2 follows from Theorem 1(*iii*).

The proof of assertion (*iii*) is standard, and is omitted.

Remark 10. We could construct $e^{-t\Lambda_{C_\infty}(b)}$ alternatively as follows:

$$e^{-t\Lambda_{C_\infty}(b)} := \big(e^{-t\Lambda_p(b)}|_{C_\infty\cap L^p}\big)^{\text{clos}}_{C_\infty} \text{ (after a change on a set of measure zero)}, \quad t>0,$$

where $p \in \big(d-1, \frac{2}{1-\sqrt{1-m_d\delta}}\big)$.

Appendix A

Define $I_n := \|(b-b_n)\cdot\nabla(\zeta-\Delta)^{-1}f\|_1$.

1. *Let* $b \in \mathbf{K}^{d+1}_\delta$. *For every* $f \in L^1$ *and* $\operatorname{Re}\zeta \geqslant \kappa_d\lambda$,

$$I_n \to 0 \text{ as } n\uparrow\infty. \tag{A.0}$$

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Proof of (A.0). Since $I_n \leqslant 2m_d \| |b|(\lambda-\Delta)^{-\frac{1}{2}}|f|\|_1 \leqslant 2m_d\delta\|f\|_1$, it suffices to prove (A.0) for each $f \in L^1 \cap L^\infty$. Let $f \in L^1 \cap L^\infty$, $\lambda > 0$ and b be fixed. Since $|b|(\lambda-\Delta)^{-\frac{1}{2}}|f| \in L^1$, for a given $\epsilon > 0$, there exists $\mathcal{K}$, a compact, such that

$$\|(\mathbf{1}-\mathbf{1}_\mathcal{K})|b|(\lambda-\Delta)^{-\frac{1}{2}}|f|\|_1 \leqslant \epsilon,$$

where $\mathbf{1}_\mathcal{K}$ is the characteristic function of $\mathcal{K}$. Define $I_{\mathcal{K},n} := \|\mathbf{1}_\mathcal{K}|b-b_n|(\lambda-\Delta)^{-\frac{1}{2}}|f|\|_1$. Clearly,

$$I_{\mathcal{K},n} \leqslant \lambda^{-\frac{1}{2}}\|f\|_\infty \|\mathbf{1}_\mathcal{K}|b-b_n|\|_1.$$

Since $|b| \in L^1_{\rm loc}$ and $\mathcal{K}$ independent of $n = 1, 2, \dots$,

$$\|\mathbf{1}_\mathcal{K}|b-b_n|\|_1 \leqslant \|\mathbf{1}_{|b|\geqslant n}(\mathbf{1}_\mathcal{K}|b|)\|_1 \to 0 \text{ as } n \uparrow \infty.$$

Therefore, for a given ϵ, there exists $n_0 = n_0(\epsilon) \geqslant 1$, such that $I_{\mathcal{K},n} \leqslant \epsilon$ whenever $n \geqslant n_0$, and so

$$I_n \leqslant 3m_d\epsilon \qquad \forall n \geqslant n_0.$$

□

We use the following pointwise estimates ($x, y \in \mathbb{R}^d$, $x \neq y$).

2. *For every* $\mathrm{Re}\,\zeta > 0$,

$$|\nabla(\zeta-\Delta)^{-1}(x,y)| \leqslant m_d(\kappa_d^{-1}\mathrm{Re}\,\zeta - \Delta)^{-\frac{1}{2}}(x,y), \tag{A.1}$$

where $m_d^2 := \pi(2e)^{-1}d^d(d-1)^{1-d}$, $\kappa_d := \frac{d}{d-1}$.

For every $r \in (1,\infty]$ *there exists a constant* $m_{r,d} < \infty$ *such that for all* $\mathrm{Re}\,\zeta > 0$,

$$|\nabla(\zeta-\Delta)^{-1+\frac{1}{2r}}(x,y)| \leqslant m_{r,d}(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{2}+\frac{1}{2r}}(x,y). \tag{A.2}$$

3. *For every* $\mathrm{Re}\,\zeta > 0$,

$$|\nabla(\zeta-\Delta)^{-1}(x,y)| \leqslant 2^{\frac{d}{4}}m_d\Big(\kappa_d^{-1}2^{-1}|\zeta|-\Delta\Big)^{-\frac{1}{2}}(x,y), \tag{A.3}$$

$$|(\zeta-\Delta)^{-\frac{1}{2}}(x,y)| \leqslant 2^{\frac{d}{4}+\frac{1}{4}}\Big(2^{-1}|\zeta|-\Delta\Big)^{-\frac{1}{2}}(x,y). \tag{A.4}$$

Proof of (A.1). Let $\alpha \in (0,1)$. Set $c(\alpha) := \sup_{\xi>0}\xi e^{-(1-\alpha)\xi^2}$ $\Big(= \frac{1}{\sqrt{2}}(1-\alpha)^{-\frac{1}{2}}e^{-\frac{1}{2}}\Big)$, so that

$$\xi e^{-\xi^2} \leqslant c(\alpha)e^{-\alpha\xi^2} \qquad \text{for all } \xi > 0. \tag{$\star$}$$

We use the well known formula

$$(\zeta-\Delta)^{-\frac{\gamma}{2}}(x,y) = \frac{1}{\Gamma(\frac{\gamma}{2})}\int_0^\infty e^{-\zeta t}t^{\frac{\gamma}{2}-1}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}dt, \quad 0<\gamma\leqslant 2,$$

first with $\gamma = 2$, and then with $\gamma = 1$, to obtain:

$$\begin{aligned}|\nabla(\zeta-\Delta)^{-1}(x,y)| &\leqslant \int_0^\infty e^{-t\mathrm{Re}\,\zeta}(4\pi t)^{-\frac{d}{2}}\frac{|x-y|}{2t}e^{-\frac{|x-y|^2}{4t}}dt \\ &\leqslant c(\alpha)\int_0^\infty e^{-t\mathrm{Re}\,\zeta}t^{-\frac{1}{2}}(4\pi t)^{-\frac{d}{2}}e^{-\alpha\frac{|x-y|^2}{4t}}dt \qquad \left(\text{By } (\star) \text{ with } \xi := \frac{|x-y|}{2\sqrt{t}}\right) \\ &\leqslant c(\alpha)\alpha^{-\frac{1}{2}-\frac{d}{2}+1}\int_0^\infty e^{-(\mathrm{Re}\,\zeta)\alpha t}t^{-\frac{1}{2}}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}dt \qquad \left(\text{change } t/\alpha \text{ to } t\right) \\ &= c(\alpha)\alpha^{\frac{1}{2}-\frac{d}{2}}\Gamma\left(\frac{1}{2}\right)\left(\alpha\mathrm{Re}\,\zeta-\Delta\right)^{-\frac{1}{2}}(x,y).\end{aligned}$$

Now, we minimize $c(\alpha)\alpha^{\frac{1}{2}-\frac{d}{2}}\Gamma(\frac{1}{2})$ in $\alpha \in (0,1)$. The minimum is attained at $\alpha_d = \frac{d-1}{d}$ $(=: \kappa_d^{-1})$, and is equal to m_d.

The proof of (A.2) is similar. □

Proof of (A.3). First, suppose that Im $\zeta \leqslant 0$. By Cauchy's theorem,

$$(\zeta-\Delta)^{-1}(x,y) = \int_0^\infty e^{-\zeta t}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}dt = \int_0^\infty e^{-\zeta r e^{i\frac{\pi}{4}}}e^{-i\frac{\pi}{4}\frac{d}{2}}(4\pi r)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4re^{i\frac{\pi}{4}}}}e^{i\frac{\pi}{4}}dr,$$

(i.e. we have changed the contour of integration from $\{t : t \geqslant 0\}$ to $\{re^{i\frac{\pi}{4}} : r \geqslant 0\}$). Thus,

$$|\nabla(\zeta-\Delta)^{-1}(x,y)| \leqslant \int_0^\infty \left|e^{-\zeta r e^{i\frac{\pi}{4}}}\right|(4\pi r)^{-\frac{d}{2}}\left|\frac{x-y}{2r}\right|\left|e^{-\frac{|x-y|^2}{4re^{i\frac{\pi}{4}}}}\right|dr.$$

We have

$$|e^{-\zeta r e^{i\frac{\pi}{4}}}| \leqslant e^{-r\frac{1}{\sqrt{2}}(\mathrm{Re}\,\zeta-\mathrm{Im}\,\zeta)}, \quad \left|e^{-\frac{|x-y|^2}{4re^{i\frac{\pi}{4}}}}\right| \leqslant e^{-\frac{|x-y|^2}{4r}\frac{1}{\sqrt{2}}}, \quad \mathrm{Re}\,\zeta-\mathrm{Im}\,\zeta \geqslant |\zeta|.$$

Therefore,

$$\begin{aligned}|\nabla(\zeta-\Delta)^{-1}(x,y)| &\leqslant \int_0^\infty e^{-r\frac{1}{\sqrt{2}}|\zeta|}(4\pi r)^{-\frac{d}{2}}\left|\frac{x-y}{2r}\right|e^{-\frac{|x-y|^2}{4r}\frac{1}{\sqrt{2}}}dr \qquad (\text{change } r\sqrt{2} \text{ to } r) \\ &= 2^{\frac{d}{4}}\int_0^\infty e^{-r\frac{1}{2}|\zeta|}(4\pi r)^{-\frac{d}{2}}\left|\frac{x-y}{2r}\right|e^{-\frac{|x-y|^2}{4r}}dr \\ &\leqslant \frac{2^{\frac{d}{4}}m_d}{\Gamma(\frac{1}{2})}\int_0^\infty e^{-r\kappa_d^{-1}\frac{1}{2}|\zeta|}(4\pi r)^{-\frac{d}{2}}r^{-\frac{1}{2}}e^{-\frac{|x-y|^2}{4r}}dr \quad (\text{cf. proof of (A.1)}) \\ &= 2^{\frac{d}{4}}m_d\left(\kappa_d^{-1}2^{-1}|\zeta|-\Delta\right)^{-\frac{1}{2}}(x,y)\end{aligned}$$

which yields (A.3) for Im $\zeta \leqslant 0$. The case Im $\zeta > 0$ is treated analogously. □

Proof of (A.4). First, suppose that Im $\zeta \leqslant 0$. By Cauchy's theorem,

$$\begin{aligned}(\zeta-\Delta)^{-\frac{1}{2}}(x,y) &= \int_0^\infty e^{-\zeta t}t^{-\frac{1}{2}}(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4t}}dt \\ &= \int_0^\infty e^{-\zeta r e^{i\frac{\pi}{4}}}r^{-\frac{1}{2}}e^{-i\frac{\pi}{8}}e^{-i\frac{\pi}{4}\frac{d}{2}}(4\pi r)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4re^{i\frac{\pi}{4}}}}e^{i\frac{\pi}{4}}dr,\end{aligned}$$

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so we estimate as above:

$$|(\zeta-\Delta)^{-\frac{1}{2}}(x,y)| \leqslant \int_0^\infty e^{-r\frac{1}{\sqrt{2}}|\zeta|} r^{-\frac{1}{2}}(4\pi r)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4r}\frac{1}{\sqrt{2}}}dr$$
$$= 2^{\frac{d}{4}+\frac{1}{4}}\left(2^{-1}|\zeta|-\Delta\right)^{-\frac{1}{2}}(x,y).$$

The case $\operatorname{Im}\zeta > 0$ is treated analogously. □

4. *In the proof of Proposition 1 we need the following formula: for every $0<\alpha<1$, $\operatorname{Re}\zeta>0$,*

$$(\zeta-\Delta)^{-\alpha} = \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(t+\zeta-\Delta)^{-1}dt. \tag{A.5}$$

Appendix B

Proof of (1). Let $b \in \mathbf{K}_\delta^{d+1}$, i.e. $\||b|(\lambda-\Delta)^{-\frac{1}{2}}\|_{1\to 1} \leqslant \delta$, $\|(\lambda-\Delta)^{-\frac{1}{2}}|b|\|_\infty \leqslant \delta$ (by duality). Then, using e.g. interpolation, $\||b|^{\frac{1}{2}}(\lambda-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{2}}\|_{2\to 2} \leqslant \delta$, i.e. $b \in \mathbf{F}_\delta^{\frac{1}{2}}$. The first inclusion is proved. (Here, the proof depends crucially of the fact that $(\lambda-\Delta)^{-\frac{1}{2}}$ is an integral operator with a symmetric kernel.)

The second inclusion $\mathbf{F}_{\delta_1} \subsetneq \mathbf{F}_\delta^{\frac{1}{2}}$, $\delta=\sqrt{\delta_1}$, follows e.g. by the Heinz inequality [He]. The last assertion now follows from

$$b \in \mathbf{F}^{\frac{1}{2}}_{\sqrt{\delta_1}}, \mathsf{f} \in \mathbf{F}^{\frac{1}{2}}_{\delta_2} \Rightarrow b+\mathsf{f} \in \mathbf{F}^{\frac{1}{2}}_\delta,$$

where we have used $(|b|+|\mathsf{f}|)^{\frac{1}{2}} \leqslant |b|^{\frac{1}{2}}+|\mathsf{f}|^{\frac{1}{2}}$. □

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