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Schrödinger operators and unique continuation. Towards an optimal result

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Abstract

In this article we prove the property of unique continuation (also known for C^∞ functions as quasi-analyticity) for solutions of the differential inequality $|\Delta u| \leqslant |Vu|$ for V from a wide class of potentials (including $L^{d/2,\infty}_{\text{loc}}(\mathbb{R}^d)$ class) and u in a space of solutions Y_V containing all eigenfunctions of the corresponding self-adjoint Schrödinger operator. Motivating question: is it true that for potentials V, for which self-adjoint Schrödinger operator is well defined, the property of unique continuation holds?

Keywords: Unique continuation; Schrödinger operators

1. Introduction

Let Ω be an open set in $\mathbb{R}^d$ ($d \geqslant 3$), $X_p := L^p_{\text{loc}}(\Omega, dx)$ ($p \geqslant 1$), $H^{m,p}(\Omega)$ the standard Sobolev space and $\Delta := \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}$ the Laplace operator. Let $\mathcal{D}'(\Omega)$ be the space of distributions over $C_0^\infty(\Omega)$ and $\mathcal{L}^{2,1}_{\text{loc}}(\Omega) := \{f \in X_1{:}\ \Delta f \in \mathcal{D}'(\Omega) \cap X_1\}$.

Let now Ω be connected. For $Y_V \subset \mathcal{L}^{2,1}_{\text{loc}}(\Omega)$ a space of functions depending on $V \in X_1$ we say that the differential inequality

$$\left|\Delta u(x)\right| \leqslant \left|V(x)\right|\left|u(x)\right| \quad \text{a.e. in } \Omega \tag{1}$$

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has the property of *weak* unique continuation (WUC) in Y_V ($=: Y_V^{\text{weak}}$) provided that whenever u in Y_V satisfies inequality (1) and vanishes in an open subset of Ω it follows that $u \equiv 0$ in Ω. We also say that (1) has the property of *strong* unique continuation (SUC) in Y_V ($=: Y_V^{\text{str}}$) if whenever u in Y_V satisfies (1) and vanishes to an infinite order at a point $x_0 \in \Omega$, i.e.,

$$\lim_{\rho\to 0} \frac{1}{\rho^k} \int\limits_{|x-x_0|<\rho} \left|u(x)\right|^2 dx = 0, \quad \text{for all } k \in \mathbb{N},$$

it follows that $u \equiv 0$ in Ω.

The first result on unique continuation was obtained by T. Carleman [2]. He proved that (1) has the WUC property in the case $d = 2$, $V \in L^\infty_{\text{loc}}(\Omega)$. Since then, the properties of unique continuation were extensively studied by many authors (primarily following the original Carleman's approach), with the best possible for L^p_{loc}-potentials SUC result obtained by D. Jerison and C. Kenig ($p = \frac{d}{2}$, $Y_V^{\text{str}} = H^{2,\bar p}_{\text{loc}}$, $\bar p := \frac{2d}{d+2}$) [6], and its extension for $L^{d/2,\infty}_{\text{loc}}$-potentials obtained by E.M. Stein [21]. Further improvements of Stein's result were obtained in [3,17,23] where unique continuation is proved for potentials V locally in Campanato–Morrey class (see Section 3 for details), with $Y_V^{\text{str}} = H^{2,2}_{\text{loc}}$ or $H^{2,\bar p}_{\text{loc}}$. Before that, in 1984, E.T. Sawyer proved the SUC property in the case $d = 3$ for potentials from Kato class (see Section 3). Historically, the most important reason for establishing the WUC property is its application, discovered in 1959 by T. Kato [8], to the problem of absence of positive eigenvalues of self-adjoint Schrödinger operators. In what follows, we exploit this link. Our setting involves a 'local analogue' (for $d = 3$ and a subclass for $d \geqslant 4$) of the class of potentials for which the self-adjoint Schrödinger operator is defined in the sense of quadratic forms, as described below, and for each potential V a class of solutions Y_V containing all eigenfunctions of the corresponding Schrödinger operator. The latter allows us to use our WUC result to prove the absence of positive eigenvalues. Precisely, we prove that differential inequality (1) has WUC property in the space of solutions

$$Y_V^{\text{weak}} := \left\{u \in \mathcal{L}^{2,1}_{\text{loc}}\colon |V|^{\frac{1}{2}} u \in X_2\right\}$$

and, respectively, SUC property in

$$Y_V^{\text{str}} := Y_V^{\text{weak}} \cap H^{1,\bar p}_{\text{loc}}(\Omega).$$

Previously WUC and SUC properties were derived only for $Y_V = H^{2,\bar p}_{\text{loc}}(\Omega)$ (dependence of Y_V on V, i.e., $u \in Y_V$ implies $|V|^{\frac{1}{2}} u \in X_2$, is implicit in the papers cited above, see Section 3). Our 'abstract' form of the class of solutions leads to a substantially shorter and more transparent proof. (We note that the 'abstract' classes of potentials were previously considered, e.g., in [5,18,19].)

Following Carleman, most proofs of unique continuation rely on Carleman type estimates on the norms of the appropriate operators acting from L^p to L^q, for certain p and q (e.g., Theorem 2.1 in [6], Theorem 1 in [21]). Our method uses the $L^2 \mapsto L^2$ estimate of Proposition 1. The latter reduces an estimate of the 'singular' term I_1 in Carleman's expansion,

$$\mathbf{1}_{B(\rho)} V^{\frac{1}{2}} \varphi_{N+C(d)} u = I_1 + \cdots \quad (V \geqslant 1),$$

$$I_1 := \mathbf{1}_{B(\rho)} V^{\frac{1}{2}} \varphi_{N+C(d)} \left[(-\Delta)^{-1}\right]_N \varphi^{-1}_{N+C(d)} V^{\frac{1}{2}} \mathbf{1}_{B(\rho)} \varphi_{N+C(d)} \frac{-\Delta u}{V^{\frac{1}{2}}},$$

to the definition of the class of potentials (see the proof of Theorem 1 for details). Here u is assumed to be identically equal to 0 in the ball $B(a) := \{x \in \mathbb{R}^d\colon |x| < a\}$, $0 < a < \rho$, $\varphi_t(x) := |x|^{-t}$, the kernel $[(-\Delta)^{-1}]_N(x,y)$ is the kernel $[(-\Delta)^{-1}](x,y)$ (see the definition below) modified by subtracting its Taylor polynomial of order $N-1$, and $\mathbf{1}_{B(\rho)}$ is the characteristic function of the ball $B(\rho)$. In the case $d = 3$ we derive Proposition 1 using the classical pointwise estimate due to E.T. Sawyer [18] on the absolute value of $[(-\Delta)^{-1}]_N(x,y)$ in terms of $[(-\Delta)^{-1}](x,y)$ and of the ratio of the polynomial weights $\varphi_N(x)/\varphi_N(y)$ (our Lemma 1). This estimate allows one to interchange polynomial weights with the corresponding integral operators and thus to derive Proposition 1 which henceforth yields

$$\left\| \mathbf{1}_{B(\rho)} \frac{\varphi_{N+C(d)}(x)}{\varphi_{N+C(d)}(\rho)} u \right\|_2 \leqslant C' \quad \text{for all } N$$

with C' being independent of N; the latter inequality leads to a contradiction as $N \to \infty$, unless $u \equiv 0$ in $B(\rho)$ ($\supsetneq B(a)$). We reduce the case of $d \geqslant 4$ to the case of $d = 3$ at the cost of a more restrictive class of potentials: the proof uses Stein's interpolation theorem for analytic families of operators [22], and relies on Lemma 2 of [6] and our extension of the pointwise inequality of [18] mentioned above and of inequality from [21] (our Lemma 3, cf. Lemma 1 in [18], Lemma 5 in [21]).

Finally, we formulate the definition of the class of potentials for which we prove the uniqueness of continuation. Let $\mathbf{1}_S$ denote the characteristic function of a set $S \subset \mathbb{R}^d$, $B(x_0,\rho) := \{x \in \mathbb{R}^d\colon |x - x_0| < \rho\}$, $B_S(x_0,\rho) := B(x_0,\rho) \cap S$ (also set $B_S(\rho) := B_S(0,\rho)$), $\|A\|_{p\mapsto q}$ is the norm of operator $A : L^p(\mathbb{R}^d) \mapsto L^q(\mathbb{R}^d)$, $(-\Delta)^{-\frac{z}{2}}$, $0 < \mathrm{Re}(z) < d$, stands for the Riesz operator whose action on a function $f \in C_0^\infty(\mathbb{R}^d)$ is determined by the formula

$$(-\Delta)^{-\frac{z}{2}} f(x) = c_z \int_{\mathbb{R}^d} (-\Delta)^{-\frac{z}{2}}(x,y) f(y)\, dy,$$

where

$$(-\Delta)^{-\frac{z}{2}}(x,y) := |x-y|^{z-d}, \qquad c_z := \Gamma\left(\frac{d-z}{2}\right)\left(\pi^{d/2} 2^z \Gamma\left(\frac{z}{2}\right)\right)^{-1}$$

(see, e.g., [20]).

Our class of potentials is

$$\mathcal{F}^d_{\beta,\mathrm{loc}} := \left\{ W \in X_{\frac{d-1}{2}}\colon \sup_{x_0 \in K} \varlimsup_{\rho\to 0} \left\| \mathbf{1}_{B_K(x_0,\rho)} |W|^{\frac{d-1}{4}} (-\Delta)^{-\frac{d-1}{2}} |W|^{\frac{d-1}{4}} \mathbf{1}_{B_K(x_0,\rho)} \right\|_{2\mapsto 2} \leqslant \beta \right\}$$

for all compacts $K \subset \Omega$. In 1959 T. Kato proved that if V has a compact support, then all eigenfunctions corresponding to positive eigenvalues must vanish outside of a ball of finite radius, hence by WUC must be identically equal to zero. We use our WUC result for (1) to prove the absence of positive eigenvalues of the self-adjoint Schrödinger operator $H \supset -\Delta + V$ in complex Hilbert space $\mathcal{H} := L^2(\mathbb{R}^d)$ defined in the sense of quadratic forms (see [9,16]), namely:

$$H := H_+ \dotplus (-V_-), \tag{2}$$

where $H_+ := H_0 \dotplus V_+$, $H_0 = (-\Delta|_{C^\infty(\mathbb{R}^d)})^*$, $D(H_0) = H^{2,2}(\mathbb{R}^d)$, $V = V_+ - V_-$, $V_\pm \geqslant 0$, $V_\pm \in L^1(\mathbb{R}^d)$ and

$$\inf_{\lambda>0} \left\| V_-^{\frac{1}{2}} (\lambda + H_+)^{-1} V_-^{\frac{1}{2}} \right\|_{2\mapsto 2} \leqslant \beta < 1. \tag{3}$$

The latter inequality guarantees the existence of the form sum (2) (see [9, Ch. VI]), and the inclusion $D(H) \subset Y_V^{\text{weak}}$ (see Section 2). The local nature of the problem of unique continuation and the form of differential inequality (1) lead to the definition of the following 'local analogue' of potentials satisfying (3):

$$F_{\beta,\text{loc}} := \left\{ W \in X_1\colon \sup_{x_0 \in K} \overline{\lim_{\rho\to 0}} \left\| \mathbf{1}_{B_K(x_0,\rho)} |W|^{\frac{1}{2}} (-\Delta)^{-1} |W|^{\frac{1}{2}} \mathbf{1}_{B_K(x_0,\rho)} \right\|_{2\mapsto 2} \leqslant \beta \right\} \tag{4}$$

for all compacts $K \subset \Omega$. This class coincides with $\mathcal{F}^d_{\beta,\text{loc}}$ if $d = 3$, and contains $\mathcal{F}^d_{\beta,\text{loc}}$ as a proper subclass if $d \geqslant 4$ (the latter easily follows from Heinz–Kato inequality, see, e.g., [7]). We believe that the results of this article can be extended to the larger class of potentials $F_{\beta,\text{loc}}$ for $d \geqslant 4$.

Class $\mathcal{F}^d_{\beta,\text{loc}}$ contains potentials considered in [3,6,18,21,23] as proper subclasses.

The results of this article have been announced in [11].

2. Main results

Our main results state that (1) has the WUC and SUC properties with potentials from $\mathcal{F}^d_{\beta,\text{loc}}$. The difference between the results is in the classes Y_V within which we look for solutions to (1).

Theorem 1. *There exists a sufficiently small constant $\beta < 1$ such that if $V \in \mathcal{F}^d_{\beta,\text{loc}}$ then* (1) *has the WUC property in Y_V^{weak}.*

Theorem 2. *There exists a sufficiently small constant $\beta < 1$ such that if $V \in \mathcal{F}^d_{\beta,\text{loc}}$, then* (1) *has the SUC property in Y_V^{str}.*

The proofs of Theorems 1 and 2 are given in Section 4. Concerning the eigenvalue problem, we have the following result.

Theorem 3. *Suppose that H is defined by* (2) *in assumption that* (3) *holds. Let us also assume that $V \in F^d_{\beta,\text{loc}}$ for $\beta < 1$ sufficiently small, and* supp(V) *is compact in $\mathbb{R}^d$. Then the only solution to the eigenvalue problem*

$$Hu = \lambda u, \quad u \in D(H),\ \lambda > 0, \tag{5}$$

is zero.

Proof. The following inclusions are immediate from the definition of operator H:

$$D(H) \subset H^{1,2}(\mathbb{R}^d) \cap D\big(V_+^{\frac{1}{2}}\big) \cap D\big(V_-^{\frac{1}{2}}\big),$$
$$D(H) \subset D(H_{\max}),$$

where

$$D(H_{\max}) := \{ f \in \mathcal{H}\colon\ \Delta f \in \mathcal{D}'(\mathbb{R}^d) \cap L^1_{\mathrm{loc}}(\mathbb{R}^d),\ Vf \in L^1_{\mathrm{loc}}(\mathbb{R}^d),\ -\Delta f + Vf \in \mathcal{H} \}.$$

Therefore, $D(H) \subset Y_V^{\mathrm{weak}}$ and if $u \in D(H)$ is a solution to (5), then

$$|\Delta u| = |(V-\lambda)u| \quad \text{a.e. in } \mathbb{R}^d.$$

By Kato's theorem [8] u has compact support. Now Theorem 3 follows from Theorem 1. $\square$

3. Historical context

1) D. Jerison and C. Keing [6] and E.M. Stein [21] proved the validity of the SUC property for potentials from classes $L^{\frac{d}{2}}_{\mathrm{loc}}(\Omega)$ and $L^{\frac{d}{2},\infty}_{\mathrm{loc}}(\Omega)$ (weak type $d/2$ Lorentz space), respectively. Below $\|\cdot\|_{p,\infty}$ denotes weak type p Lorentz norm. One has

$$L^{\frac{d}{2}}_{\mathrm{loc}}(\Omega) \subsetneq \bigcap_{\beta>0} \mathcal{F}^d_{\beta,\mathrm{loc}}, \tag{6}$$

$$L^{\frac{d}{2},\infty}_{\mathrm{loc}}(\Omega) \subsetneq \bigcup_{\beta>0} \mathcal{F}^d_{\beta,\mathrm{loc}}. \tag{7}$$

The first inclusion follows straightforwardly from the Sobolev embedding theorem. For the following proof of the second inclusion let us note first that

$$\left\| \mathbf{1}_{B(x_0,\rho)} |W|^{\frac{d-1}{4}} (-\Delta)^{-\frac{d-1}{2}} |W|^{\frac{d-1}{4}} \mathbf{1}_{B(x_0,\rho)} \right\|_{2\mapsto 2} = \left\| \mathbf{1}_{B(x_0,\rho)} |V|^{\frac{d-1}{4}} (-\Delta)^{-\frac{d-1}{4}} \right\|^2_{2\mapsto 2}.$$

Next, if $V \in L^{d/2,\infty}$, then

$$\left\| \mathbf{1}_{B(x_0,\rho)} |V|^{\frac{d-1}{4}} (-\Delta)^{-\frac{d-1}{4}} \right\|_{2\mapsto 2} \leqslant \left(\frac{2d^{-1}\pi^{\frac{d}{2}} c_{\frac{1}{2}}}{\Gamma(\frac{d}{2}) c_{\frac{d}{2}}} \right) \|\mathbf{1}_{B(x_0,\rho)} V\|^{\frac{d-1}{4}}_{\frac{d}{2},\infty}, \tag{8}$$

which is a special case of Strichartz inequality with sharp constants, proved in [13]. Inclusion (7) follows.

To see that the latter inclusion is strict we introduce a family of potentials

$$V(x) := \frac{C(\mathbf{1}_{B(1+\delta)}(x) - \mathbf{1}_{B(1-\delta)}(x))}{(|x|-1)^{\frac{2}{d-1}} (-\ln||x|-1|)^b}, \quad \text{where } b > \frac{2}{d-1},\ 0 < \delta < 1. \tag{9}$$

A straightforward computation shows that $V \in \mathcal{F}^d_{\beta,\mathrm{loc}}$, as well as $V \in L^{\frac{d-1}{2}}_{\mathrm{loc}}(\Omega) \setminus L^{\frac{d-1}{2}+\varepsilon}_{\mathrm{loc}}(\Omega)$ for any $\varepsilon > 0$, so that $V \notin L^{\frac{d}{2},\infty}_{\mathrm{loc}}(\Omega)$.

The result in [21] can be formulated as follows. Suppose that $d \geqslant 3$ and $V \in L^{\frac{d}{2},\infty}_{\mathrm{loc}}(\Omega)$. There exists a sufficiently small constant β such that if

$$\sup_{x_0 \in \Omega} \varlimsup_{\rho \to 0} \|\mathbf{1}_{B(x_0,\rho)} V\|_{\frac{d}{2},\infty} \leqslant \beta,$$

then (1) has the SUC property in $Y_V := H^{2,\bar{p}}_{\mathrm{loc}}(\Omega)$, where $\bar{p} := \frac{2d}{d+2}$. (It is known that the assumption of β being sufficiently small cannot be omitted, see [12].)

In view of (6), (7), the results in [21] and in [6] follow from Theorem 2 provided that we show $|V|^{\frac{1}{2}} u \in X_2$. Indeed, let $L^{q,p}$ be the (q,p) Lorentz space (see [22]). By Sobolev embedding theorem for Lorentz spaces $H^{2,\bar{p}}_{\mathrm{loc}}(\Omega) \hookrightarrow L^{\bar{q},\bar{p}}_{\mathrm{loc}}(\Omega)$ with $\bar{q} := \frac{2d}{d-2}$ [22]. Hence, by Hölder inequality in Lorentz spaces $|V|^{\frac{1}{2}} u \in X_2$ whenever $u \in L^{\bar{q},\bar{p}}_{\mathrm{loc}}(\Omega)$ and $V \in L^{d/2,\infty}_{\mathrm{loc}}$. Also, $H^{2,\bar{p}}_{\mathrm{loc}}(\Omega) \hookrightarrow H^{1,\bar{p}}_{\mathrm{loc}}(\Omega)$, so $H^{2,\bar{p}}_{\mathrm{loc}}(\Omega) \subset Y_V^{\mathrm{str}}$, as required.

2) E.T. Sawyer [18] proved uniqueness of continuation for the case $d = 3$ and potential V from the local Kato-class

$$\mathcal{K}_{\beta,\mathrm{loc}} := \Big\{ W \in L^1_{\mathrm{loc}}(\Omega)\colon \sup_K \varlimsup_{\rho \to 0} \sup_{x_0 \in K} \big\| (-\Delta)^{-1} \mathbf{1}_{B_K(x_0,\rho)} |W| \big\|_\infty \leqslant \beta \Big\},$$

where K is a compact subset of Ω. It is easy to see that

$$\mathcal{K}_{\beta,\mathrm{loc}} \subsetneq F_{\beta,\mathrm{loc}}.$$

To see that the latter inclusion is strict consider, for instance, potential

$$V_\beta(x) := \beta v_0, \quad v_0 := \left(\frac{d-2}{2}\right)^2 |x|^{-2}.$$

By Hardy's inequality, $V_\beta \in F_{\beta,\mathrm{loc}}$. At the same time, $\|(-\Delta)^{-1} v_0 \mathbf{1}_{B(\rho)}\|_\infty = \infty$ for all $\rho > 0$, hence $V_\beta \notin \mathcal{K}_{\beta,\mathrm{loc}}$ for all $\beta \neq 0$.

The next statement is essentially due to E.T. Sawyer [18].

Theorem 4. *Let $d = 3$. There exists a constant $\beta < 1$ such that if $V \in \mathcal{K}_{\beta,\mathrm{loc}}$ then* (1) *has the WUC property in $Y_V^{\mathcal{K}} := \{f \in X_1\colon\ \Delta f \in X_1,\ Vf \in X_1\}$.*

The proof of Theorem 4 is provided in Section 5.

Despite the embedding $\mathcal{K}_{\beta,\mathrm{loc}} \hookrightarrow F_{\beta,\mathrm{loc}}$, Theorem 1 does not imply Theorem 4. The reason is simple: $Y_V^{\mathcal{K}} \not\subset Y_V^{\mathrm{weak}}$.

3) S. Chanillo and E.T. Sawyer showed in [3] the validity of the SUC property for (1) in $Y_V = H^{2,2}_{\mathrm{loc}}(\Omega)$ $(d \geqslant 3)$ for potentials V locally small in Campanato–Morrey class M^p $(p > \frac{d-1}{2})$,

$$M^p := \Big\{ W \in L^p\colon \|W\|_{M^p} := \sup_{x \in \Omega,\ r > 0} r^{2-\frac{d}{p}} \|\mathbf{1}_{B(x,r)} W\|_p < \infty \Big\}.$$

Note that for $p > \frac{d-1}{2}$

$$M^p_{\mathrm{loc}} \subsetneq \bigcup_{\beta>0} \mathcal{F}^d_{\beta,\mathrm{loc}}$$

(see [3,4,10]). To see that the above inclusion is strict one may consider, for instance, potential defined in (9).

It is easy to see, using Hölder inequality, that if $u \in H^{2,2}_{\mathrm{loc}}(\Omega)$ and $V \in M^p_{\mathrm{loc}}$ ($p > \frac{d-1}{2}$), then $|V|^{\frac{1}{2}} u \in X_2$, i.e., $u \in Y_V^{\mathrm{weak}}$. However, the assumption '$u \in H^{2,2}_{\mathrm{loc}}$' is in general too restrictive for application of this result to the problem of absence of positive eigenvalues (see Remark 1).

Remark 1. Below we make several comments about $H^{2,q}$-properties of the eigenfunctions of the self-adjoint Schrödinger operator $H = (-\Delta \dotplus V_+) \dotplus (-V_-)$, $V = V_+ - V_-$, defined by (2) in the assumption that condition

$$V_- \leqslant \beta(H_0 \dotplus V_+) + c_\beta, \quad \beta < 1,\ c_\beta < \infty, \tag{10}$$

is satisfied. (Note that (10) implies condition (3). We say that (10) is satisfied with $\beta = 0$ if (10) holds for any $\beta > 0$ arbitrarily close to 0, for an appropriate $c_\beta < \infty$.)

Let $u \in D(H)$ and $Hu = \mu u$. Then

$$e^{-tH} u = e^{-t\mu} u, \quad t > 0.$$

As is shown in [14], for every $2 \leqslant r < \frac{2d}{d-2}\frac{1}{1-\sqrt{1-\beta}}$ there exists a constant $c = c(r,\beta) > 0$ such that

$$\left\| e^{-tH} f \right\|_r \leqslant c t^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{r})} \|f\|_2, \tag{11}$$

where $f \in L^2 = L^2(\mathbb{R}^d)$. Let us now consider several possible L^p and $L^{p,\infty}$ (as well as L^p_{loc} and $L^{p,\infty}_{\mathrm{loc}}$) conditions on potential V.

(A) Suppose in addition to (10) that $V \in L^p_{\mathrm{loc}}$ for some $1 \leqslant p < \frac{d}{2}$. Then by Hölder inequality and (11) $Vu \in L^q_{\mathrm{loc}}$ and, due to inclusion $D(H) \subset D(H_{\max})$, $\Delta u \in L^q_{\mathrm{loc}}$ for any q such that

$$\frac{1}{q} > \frac{1}{p} + \frac{d-2}{d}\frac{1-\sqrt{1-\beta}}{2}.$$

The latter implies that $q < \bar{p}$ in general, i.e., when β in (10) is close to 1. Hence, in general the assumption '$u \in H^{2,\bar{p}}_{\mathrm{loc}}$' (and, moreover, '$u \in H^{2,2}_{\mathrm{loc}}$') is too restrictive for applications to the problem of absence of positive eigenvalues even under additional hypothesis of the type $V \in L^p_{\mathrm{com}}$, $\frac{d-1}{2} \leqslant p < \frac{d}{2}$ or $V \in M^p_{\mathrm{com}}$, $\frac{d-1}{2} \leqslant p < \frac{d}{2}$ (cf. [3,17]).

(B1) If $V = V_1 + V_2 \in L^p + L^\infty$, $p > \frac{d}{2}$, then (10) holds with $\beta = 0$ and $u \in L^\infty$, therefore, $|V|^{\frac{1}{2}} u \in L^2$ (cf. $D(H)$ and Y_V^{weak}).

It follows that $u \in C^{0,\alpha}$ for any $\alpha \in (0, 1 - \frac{2}{d}]$. Therefore, $u \in H^{2,p}_{\mathrm{loc}}$ and, in particular, for $d \geqslant 4$, $u \in H^{2,2}$.

(B2) Assume in addition to (10) that $V \in L^p_{\mathrm{loc}}$, $p > \frac{d}{2}$, and $\beta = 0$. Then $u \in H^{2,\underline{p}}_{\mathrm{loc}}$, $\underline{p} > \frac{d}{2}$. Using Hölder inequality, one immediately obtains $|V|^{\frac{1}{2}} u \in X_2$.

If $d = 3$, and $p > \frac{d}{2}$ is close to $\frac{d}{2}$, then $u \notin H^{2,2}_{\mathrm{loc}}$, but of course $u \in H^{2,\bar{p}}_{\mathrm{loc}}$, $\bar{p} = \frac{2d}{d+2}$ ($< \underline{p}$, cf. remark in [1, p. 166]).

(B3) If $V = V_1 + V_2 \in L^{\frac{d}{2}} + L^\infty$, then (10) is satisfied with $\beta = 0$ and $u \in \bigcap_{2 \leqslant r < \infty} L^r$. Therefore $u \in H^{2,q}_{\mathrm{loc}}$, $q < \frac{d}{2}$. In particular, $u \in H^{2,\bar{p}}_{\mathrm{loc}}$ (cf. [6]). By Hölder inequality, $|V|^{\frac{1}{2}} u \in L^2$.

(B4) Finally, suppose that $V = V_1 + V_2 \in L^{\frac{d}{2},\infty} + L^\infty$ is such that

$$\beta := \left(\frac{d^{-1} \pi^{\frac{d}{2}} \Gamma(\frac{d}{4} - \frac{1}{2})}{\Gamma(\frac{d}{2}) \Gamma(\frac{d}{4} + \frac{1}{2})} \right) \| V_1 \|_{\frac{d}{2},\infty} < 1.$$

Then we have

$$|V| \leqslant \beta H_0 + c_\beta, \quad c_\beta < \infty \tag{12}$$

and, at the same time,

$$\left\| V (\lambda + H_{0,\bar{p}})^{-1} \right\|_{\bar{p} \mapsto \bar{p}} \leqslant \beta, \quad \lambda \geqslant \frac{c_\beta}{\beta}$$

(see [13]), where $H_{0,\bar{p}}$ stands for the extension of $-\Delta$ in $L^{\bar{p}}$ with $D(H_{0,\bar{p}}) = H^{2,\bar{p}}$. The first inequality implies condition (10) and, hence, allows us to conclude that the form sum $H := H_0 \dotplus V$ is well defined. In turn, the second inequality implies that the algebraic sum $\hat{H}_{\bar{p}} := H_{0,\bar{p}} + V$ defined in $L^{\bar{p}}$ with $D(\hat{H}_{\bar{p}}) = H^{2,\bar{p}}$ coincides with H on the intersection of domains $D(H) \cap H^{2,\bar{p}}$ and is a generator of a semigroup. By making use of the representation

$$(\lambda + \hat{H}_{\bar{p}})^{-1} = (\lambda + H_{0,\bar{p}})^{-1} \left(1 + V (\lambda + H_{0,\bar{p}})^{-1}\right)^{-1}, \quad \lambda > \frac{c_\beta}{\beta},$$

one immediately obtains that $(\lambda + \hat{H}_{\bar{p}})^{-1} : L^{\bar{p}} \mapsto L^2$, i.e., any eigenfunction of operator $\hat{H}_{\bar{p}}$ belongs to L^2. Furthermore, an analogous representation for $(\lambda + H)^{-1}$ yields the identity

$$(\lambda + H)^{-1} f = (\lambda + \hat{H}_{\bar{p}})^{-1} f, \quad f \in L^2 \cap L^{\bar{p}}.$$

Therefore, any eigenfunction of $\hat{H}_{\bar{p}}$ is an eigenfunction of H (cf. [21]). The converse statement is valid, e.g., for eigenfunctions having compact support.

If $V \in L^{\frac{d}{2},\infty}_{\mathrm{loc}}$ and (12) holds, then $u \in H^{2,q_0}_{\mathrm{loc}}$ for some $q_0 > \bar{p}$. Indeed, we have $V \in L^r_{\mathrm{loc}}$ for any $r < \frac{d}{2}$, and so by (11) $u \in L^p$ for some $p > \frac{2d}{d-2}$. Thus, $Vu \in L^{q_0}_{\mathrm{loc}}$ for a certain $q_0 > \bar{p}$, hence $u \in H^{2,q_0}_{\mathrm{loc}}$ and, therefore, $|V|^{\frac{1}{2}} u \in X_2$. The latter confirms that the result in [21] applies to the problem of absence of positive eigenvalues (cf. $D(H)$).

4. Proofs of Theorems 1 and 2

Let us introduce some notations. In what follows, we omit index K in $B_K(x_0,\rho)$, and write simply $B(x_0,\rho)$.

Let $W \in X_{\frac{d-1}{2}}$, $x_0 \in \Omega$, $\rho > 0$, $d \geqslant 3$, define

$$\tau(W, x_0, \rho) := \left\| \mathbf{1}_{B(x_0,\rho)} |W|^{\frac{d-1}{4}} (-\Delta)^{-\frac{d-1}{2}} |W|^{\frac{d-1}{4}} \mathbf{1}_{B(x_0,\rho)} \right\|_{2 \mapsto 2}. \tag{13}$$

Note that if V is a potential from $\mathcal{F}^d_{\beta,\mathrm{loc}}$, and $V_1 := |V| + 1$, then

$$\tau(V_1, x_0, \rho) \leqslant \tau(V, x_0, \rho) + \varepsilon(\rho), \tag{14}$$

where $\varepsilon(\rho) \to 0$ as $\rho \to 0$.

Let $\mathbf{1}_{B(\rho\backslash a)}$ be the characteristic function of set $B(0,\rho) \setminus B(0,a)$, where $0 < a < \rho$. We define integral operator

$$\left[(-\Delta)^{-\frac{z}{2}}\right]_N f(x) := \int_{\mathbb{R}^d} \left[(-\Delta)^{-\frac{z}{2}}\right]_N (x,y) f(y)\, dy, \quad 0 \leqslant \mathrm{Re}(z) \leqslant d-1$$

whose kernel $[(-\Delta)^{-\frac{z}{2}}]_N(x,y)$ is defined by subtracting Taylor polynomial of degree $N-1$ at $x = 0$ of function $x \mapsto |x-y|^{z-d}$,

$$\left[(-\Delta)^{-\frac{z}{2}}\right]_N (x,y) := c_z \left(|x-y|^{z-d} - \sum_{k=0}^{N-1} \frac{(x\cdot\nabla)^k}{k!} |0-y|^{z-d} \right),$$

where $(x\cdot\nabla)^k := \sum_{|\alpha|=k} \frac{k!}{\alpha_1!\ldots\alpha_d!} x^\alpha \frac{\partial^k}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}$ is the multinomial expansion of $(x\cdot\nabla)^k$. Define, further,

$$\left[(-\Delta)^{-\frac{z}{2}}\right]_{N,t} := \varphi_t \left[(-\Delta)^{-\frac{z}{2}}\right]_N \varphi_t^{-1},$$

where $\varphi_t(x) := |x|^{-t}$.

4.1. Proof of Theorem 1

Our proof is based on the inequalities of Proposition 1 and Lemma 1.

Proposition 1. *If $\tau(V,0,\rho) < \infty$, then there exists a constant $C = C(\rho,\delta,d) > 0$ such that*

$$\left\| \mathbf{1}_{B(\rho\backslash a)} |V|^{\frac{1}{2}} \left[(-\Delta)^{-1}\right]_{N,N_d^\delta} |V|^{\frac{1}{2}} \mathbf{1}_{B(\rho\backslash a)} \right\|_{2\mapsto 2} \leqslant C\tau(V,0,\rho)^{\frac{2}{d-1}},$$

for all positive integers N, where $0 < \delta < 1/2$ and

$$N_d^\delta := N + \left(\frac{d}{2} - \delta\right)\frac{d-3}{d-1}.$$

Lemma 1. *There exists a constant $C = C(d)$ such that*

$$\left|\left[(-\Delta)^{-1}\right]_N(x,y)\right| \leqslant C N^{d-3}\left(\frac{|x|}{|y|}\right)^N (-\Delta)^{-1}(x,y)$$

for all $x, y \in \mathbb{R}^d$ and all positive integers N.

Lemma 1 is a simple consequence of Lemma 3 below for $\gamma = 0$. Lemmas 2 and 3 are required for analytic interpolation procedure used in the proof of Proposition 1 when $d \geqslant 4$.

Lemma 2. *(See [6].) There exist constants $C_2 = C_2(\rho_1, \rho_2, \delta, d)$ and $c_2 = c_2(\rho_1, \rho_2, \delta, d) > 0$ such that*

$$\left\|\mathbf{1}_{B(\rho_1\backslash a)}\left[(-\Delta)^{-i\gamma}\right]_{N,N+\frac{d}{2}-\delta}\mathbf{1}_{B(\rho_2\backslash a)}\right\|_{2\mapsto 2} \leqslant C_2 e^{c_2|\gamma|},$$

where $0 < \delta < 1/2$, for all $\gamma \in \mathbb{R}$ and all positive integers N.

Lemma 3. *There exist constants $C_1 = C_1(d)$ and $c_1 = c_1(d) > 0$ such that*

$$\left|\left[(-\Delta)^{-\frac{d-1}{2}(1+i\gamma)}\right]_N(x,y)\right| \leqslant C_1 e^{c_1\gamma^2}\left(\frac{|x|}{|y|}\right)^N (-\Delta)^{-\frac{d-1}{2}}(x,y)$$

for all $x, y \in \mathbb{R}^d$, all $\gamma \in \mathbb{R}$ and all positive integers N.

We prove Lemma 3 at the end of this section.

Proof of Proposition 1. If $d = 3$, then Proposition 1 follows immediately from Lemma 1. Suppose that $d \geqslant 4$. Consider the operator-valued function

$$F(z) := \mathbf{1}_{B(\rho\backslash a)}|V|^{\frac{d-1}{4}z}\varphi_{N+(\frac{d}{2}-\delta)(1-z)}\left[(-\Delta)^{-\frac{d-1}{2}z}\right]_N\varphi^{-1}_{N+(\frac{d}{2}-\delta)(1-z)}|V|^{\frac{d-1}{4}z}\mathbf{1}_{B(\rho\backslash a)}$$

defined on the strip $\{z \in \mathbb{C}\colon\ 0 \leqslant \mathrm{Re}(z) \leqslant 1\}$ and acting on L^2. By Lemma 2,

$$\left\|F(i\gamma)\right\|_{2\mapsto 2} \leqslant C_2 e^{c_2|\gamma|}, \quad \gamma \in \mathbb{R},$$

and by Lemma 3,

$$\left\|F(1+i\gamma)\right\|_{2\mapsto 2} \leqslant \tau(V,0,\rho)C_1 e^{c_1\gamma^2}, \quad \gamma \in \mathbb{R}.$$

Together with obvious observations about analyticity of F this implies that F satisfies all conditions of Stein's interpolation theorem. In particular, $F(\frac{2}{d-1})\colon L^2 \mapsto L^2$ is bounded, which completes the proof. □

Proof of Theorem 1. Let $u \in Y_V^{\mathrm{weak}}$. Without loss of generality we may assume $u \equiv 0$ on $B(0,a)$ for $a > 0$ sufficiently small, such that there exists $\rho > a$ with the properties $\rho < 1$ and $\bar{B}(0,3\rho) \subset \Omega$. In order to prove that u vanishes on Ω it suffices to show that $u \equiv 0$ on $B(0,\rho)$ for any such ρ.

Let $\eta \in C_0^\infty(\Omega)$ be such that $0 \leqslant \eta \leqslant 1$, $\eta \equiv 1$ on $B(0, 2\rho)$, $\eta \equiv 0$ on $\Omega \setminus B(0, 3\rho)$, $|\nabla\eta| \leqslant \frac{c}{\rho}$, $|\Delta\eta| \leqslant \frac{c}{\rho^2}$. Let $E_\eta(u) := 2\nabla\eta\nabla u + u\Delta\eta \in X_1$. Denote $u_\eta := u\eta$. Since $\mathcal{L}^{2,1}_{\mathrm{loc}}(\Omega) \subset H^{1,p}_{\mathrm{loc}}(\Omega)$, $p < \frac{d}{d-1}$, we have $E_\eta(u) \in L^1_{\mathrm{com}}(\Omega)$ and hence

$$\Delta u_\eta = \eta\Delta u + E_\eta(u)$$

implies $\Delta u_\eta \in L^1_{\mathrm{com}}(\Omega)$. Thus, we can write

$$u_\eta = (-\Delta)^{-1}(-\Delta u_\eta).$$

The standard limiting argument (involving consideration of C_0^∞-mollifiers, subtraction of Taylor polynomial of degree $N-1$ at 0 of function u_η and interchanging the signs of differentiation and integration) allows us to conclude further

$$u_\eta = \left[(-\Delta)^{-1}\right]_N(-\Delta u_\eta). \tag{15}$$

Let us denote $\mathbf{1}^c_{B(\rho)} := 1 - \mathbf{1}_{B(\rho)}$, so that $\Delta u_\eta = (\mathbf{1}_{B(\rho\setminus a)} + \mathbf{1}^c_{B(\rho)})\Delta u_\eta$. Observe that

$$\operatorname{supp}\eta\Delta u \subset \bar{B}(0, 3\rho) \setminus B(0, a), \qquad \operatorname{supp} E_\eta(u) \subset \bar{B}(0, 3\rho) \setminus B(0, 2\rho)$$

and, thus, $\mathbf{1}^c_{B(\rho)}\eta\Delta u = \mathbf{1}_{B(3\rho\setminus\rho)}\eta\Delta u$, $\mathbf{1}^c_{B(\rho)}E_\eta(u) = \mathbf{1}_{B(3\rho\setminus 2\rho)}E_\eta(u)$. Identity (15) implies then

$$\begin{aligned}
\mathbf{1}_{B(\rho)}V_1^{\frac{1}{2}}\varphi_{N_d^\delta}u = {} & \mathbf{1}_{B(\rho)}V_1^{\frac{1}{2}}\left[(-\Delta)^{-1}\right]_{N,N_d^\delta}V_1^{\frac{1}{2}}\mathbf{1}_{B(\rho\setminus a)}\varphi_{N_d^\delta}\frac{-\Delta u}{V_1^{\frac{1}{2}}} \\
& + \mathbf{1}_{B(\rho)}V_1^{\frac{1}{2}}\left[(-\Delta)^{-1}\right]_{N,N_d^\delta}V_1^{\frac{1}{2}}\mathbf{1}^c_{B(\rho)}\varphi_{N_d^\delta}\frac{-\eta\Delta u}{V_1^{\frac{1}{2}}} \\
& + \mathbf{1}_{B(\rho)}V_1^{\frac{1}{2}}\left[(-\Delta)^{-1}\right]_{N,N_d^\delta}\mathbf{1}_{B(3\rho\setminus 2\rho)}\varphi_{N_d^\delta}\bigl(-E_\eta(u)\bigr)
\end{aligned}$$

(we assume that $0 < \delta < 1/2$ is fixed throughout the proof) or, letting I to denote the left hand side and, respectively, I_1, I_1^c and I_2 the three summands of the right hand side of the last equality, we rewrite the latter as

$$I = I_1 + I_1^c + I_2.$$

We would like to emphasize that a priori $I \notin L^2$, but only $I \in L^s$, $s < d/(d-2)$. Hence, we must first prove that I_1, I_1^c and I_2 are in L^2, so that $I \in L^2$ as well. After this done, we obtain the estimates $\|I_1^c\|_2 \leqslant c_1\varphi_{N_d^\delta}(\rho)$, $\|I_2\|_2 \leqslant c_2\varphi_{N_d^\delta}(\rho)$ and $\|I_1\|_2 \leqslant \alpha\|I\|_2$, $\alpha < 1$, and conclude that $(1-\alpha)\|I\|_2 \leqslant (c_1 + c_2)\varphi_{N_d^\delta}(\rho)$, and therefore that

$$\left\|\mathbf{1}_{B(\rho\setminus a)}\frac{\varphi_{N_d^\delta}}{\varphi_{N_d^\delta}(\rho)}u\right\|_2 \leqslant \frac{c_1 + c_2}{1-\alpha}.$$

Letting $N \to \infty$, we derive identity $u \equiv 0$ in $B(0, \rho)$.

1) *Proof of* $I_1 \in L^2$ *and* $\|I_1\|_2 \leqslant \alpha\|I\|_2$, $\alpha < 1$. Observe that

$$\mathbf{1}_{B(\rho\setminus a)}\frac{|\Delta u|}{V_1^{1/2}} \leqslant \mathbf{1}_{B(\rho)}\frac{|V||u|}{V_1^{1/2}} \leqslant \mathbf{1}_{B(\rho)}|V|^{1/2}|u| \in X_2 \quad \big(\text{since } u \in Y_V^{\text{weak}}\big),$$

and hence, according to Proposition 1,

$$\begin{aligned}\|I_1\|_2 &\leqslant \big\|\mathbf{1}_{B(\rho\setminus a)}V_1^{\frac{1}{2}}\big[(-\Delta)^{-1}\big]_{N,N_d^\delta}V_1^{\frac{1}{2}}\mathbf{1}_{B(\rho\setminus a)}\big\|_{2\mapsto 2}\big\|\mathbf{1}_{B(\rho)}\varphi_{N_d^\delta}|V|^{\frac{1}{2}}u\big\|_2\\ &\leqslant \beta_1\big\|\mathbf{1}_{B(\rho)}\varphi_{N_d^\delta}|V|^{\frac{1}{2}}u\big\|_2.\end{aligned}$$

Here $\beta_1 := C\tau(V_1, 0, \rho)^{\frac{2}{d-1}}$, where C is the constant in formulation of Proposition 1. We may assume that $\beta_1 < 1$ (see (14)).

2) *Proof of* $\|I_1^c\|_2 \leqslant c_1\varphi_{N_d^\delta}(\rho)$. By Proposition 1,

$$\begin{aligned}\big\|I_1^c\big\|_2 &\leqslant \big\|\mathbf{1}_{B(\rho\setminus a)}V_1^{\frac{1}{2}}\big[(-\Delta)^{-1}\big]_{N,N_d^\delta}V_1^{\frac{1}{2}}\mathbf{1}_{B(3\rho\setminus\rho)}\big\|_{2\mapsto 2}\big\|\mathbf{1}_{B(3\rho\setminus\rho)}\varphi_{N_d^\delta}|V|^{\frac{1}{2}}u\big\|_2\\ &\leqslant \beta_2\varphi_{N_d^\delta}(\rho)\big\|\mathbf{1}_{B(3\rho)}|V|^{1/2}u\big\|_2,\end{aligned}$$

where $\beta_2 := C\tau(V_1, 0, 3\rho)^{\frac{2}{d-1}} < \infty$.

3) *Proof of* $\|I_2\|_2 \leqslant c_2\varphi_{N_d^\delta}(\rho)$. We need to derive an estimate of the form

$$\|I_2\|_2 \leqslant C\varphi_{N_d^\delta}(\rho)\big\|E_\eta(u)\big\|_1,$$

where C can depend on d, δ, a, ρ, $\|\mathbf{1}_{B(\rho)}V\|_1$, but not on N. We have

$$\begin{aligned}\|I_2\|_2 &\leqslant \big\|\mathbf{1}_{B(\rho\setminus a)}V_1^{1/2}\big[(-\Delta)^{-1}\big]_{N,N_d^\delta}\mathbf{1}_{B(3\rho\setminus 2\rho)}\big\|_{1\mapsto 2}\big\|\mathbf{1}_{B(3\rho\setminus 2\rho)}\varphi_{N_d^\delta}E_\eta(u)\big\|_1\\ &\leqslant \big\|\mathbf{1}_{B(\rho\setminus a)}V_1^{1/2}\big[(-\Delta)^{-1}\big]_{N,N_d^\delta}\mathbf{1}_{B(3\rho\setminus 2\rho)}\big\|_{1\mapsto 2}2^{-N}\varphi_{N_d^\delta}(\rho)\big\|E_\eta(u)\big\|_1.\end{aligned}$$

Now for $h \in L^1(\mathbb{R}^d)$, in virtue of Lemma 1,

$$\begin{aligned}&\big\|\mathbf{1}_{B(\rho\setminus a)}V_1^{1/2}\big[(-\Delta)^{-1}\big]_{N,N_d^\delta}\mathbf{1}_{B(3\rho\setminus 2\rho)}h\big\|_2\\ &\quad\leqslant \big\|\mathbf{1}_{B(\rho)}V_1^{1/2}\big\|_2\big\|\mathbf{1}_{B(\rho\setminus a)}\big[(-\Delta)^{-1}\big]_{N,N_d^\delta}\mathbf{1}_{B(3\rho\setminus 2\rho)}h\big\|_\infty\\ &\quad\leqslant \big\|\mathbf{1}_{B(\rho)}V_1^{1/2}\big\|_2CN^{d-3}\varphi_{(\frac{d}{2}-\delta)\frac{d-3}{d-1}}(a)\varphi^{-1}_{(\frac{d}{2}-\delta)\frac{d-3}{d-1}}(3\rho)\big\|\mathbf{1}_{B(\rho)}(-\Delta)^{-1}\mathbf{1}_{B(3\rho\setminus 2\rho)}h\big\|_\infty\\ &\quad\leqslant \big(\|\mathbf{1}_{B(\rho)}\|_1 + \|\mathbf{1}_{B(\rho)}V\|_1\big)^{1/2}CN^{d-3}\left(\frac{3\rho}{a}\right)^{(\frac{d}{2}-\delta)\frac{d-3}{d-1}}M_\rho,\end{aligned}$$

where

$$M_\rho := C_2\,\operatorname{esssup}_{x\in B(0,\rho)}\int\limits_{2\rho\leqslant|y|\leqslant 3\rho}|x-y|^{2-d}\big|h(y)\big|\,dy \leqslant C_2\rho^{2-d}\|h\|_1.$$

Therefore

$$\begin{aligned}&\left\|\mathbf{1}_{B(\rho\backslash a)}V_1^{1/2}\left[(-\Delta)^{-1}\right]_{N,N_d^\delta}\mathbf{1}_{B(3\rho\backslash 2\rho)}\right\|_{1\mapsto 2}\\&\quad\leqslant\left(\left\|\mathbf{1}_B(\rho)\right\|_1+\|\mathbf{1}_{B(\rho)}V\|_1\right)^{1/2}CC_2N^{d-3}\left(\frac{3\rho}{a}\right)^{(\frac{d}{2}-\delta)\frac{d-3}{d-1}}\rho^{2-d}.\end{aligned}$$

Hence, there exists a constant $\hat{C}=\hat{C}(d,\delta,a,\rho,\|\mathbf{1}_{B(\rho)}V\|_1)$ such that

$$\|I_2\|_2\leqslant\hat{C}N^{d-3}2^{-N}\varphi_{N_d^\delta}(\rho)\left\|E_\eta(u)\right\|_1,$$

which implies the required estimate. □

Proof of Lemma 3. The proof essentially follows the argument in [18]. Put

$$\begin{bmatrix}-\frac{1}{2}+\frac{i\gamma}{2}\\k\end{bmatrix}:=\prod_{j=1}^{k}\left(1+\frac{-\frac{1}{2}+\frac{i\gamma}{2}}{j}\right).$$

Then

$$\begin{aligned}\left|\begin{bmatrix}-\frac{1}{2}+\frac{i\gamma}{2}\\k\end{bmatrix}\right|&=\prod_{j=1}^{k}\left(1-\frac{1}{2j}\right)\prod_{j=1}^{k}\sqrt{1+\frac{\gamma^2}{(2j-1)^2}}\\&\leqslant\prod_{j=1}^{k}\left(1-\frac{1}{2j}\right)e^{\gamma^2\sum_{j=1}^{k}\frac{1}{(2j-1)^2}}\leqslant\prod_{j=1}^{k}\left(1-\frac{1}{2j}\right)e^{\gamma^2c},\quad c=\frac{\pi^2}{48}.\end{aligned}\tag{16}$$

We may assume, after a dilation and rotation, that $x=(x_1,x_2,0,\dots,0)$, $y=(1,0,\dots,0)$. Thus, passing to polar coordinates $(x_1,x_2)=te^{i\theta}$, we reduce our inequality to inequality

$$\left|\left|1-te^{i\theta}\right|^{-1-i\gamma}-P_{N-1}(t,\theta)\right|\leqslant Ce^{c\gamma^2}t^N\left|1-te^{i\theta}\right|^{-1},\quad\text{for all }\gamma\in\mathbb{R},$$

and for appropriate $C>0$, $c>0$. Here $P_{N-1}(t,\theta)$ denotes the Taylor polynomial of degree $N-1$ at point $z=0$ of function $z=te^{i\theta}\mapsto|1-z|^{-1}$. Similarly to [18], via summation of geometric series we obtain a representation

$$P_{N-1}(t,\theta)=\sum_{m=0}^{N-1}a_m^\gamma(\theta)t^m,$$

where

$$a_m^\gamma(\theta):=\sum_{k+l=m}\begin{bmatrix}-\frac{1}{2}+\frac{i\gamma}{2}\\l\end{bmatrix}\begin{bmatrix}-\frac{1}{2}+\frac{i\gamma}{2}\\k\end{bmatrix}e^{i(k-l)\theta}.$$

Note that

$$a_m^0(0)=\sum_{k+l=m}\begin{bmatrix}-\frac{1}{2}\\ l\end{bmatrix}\begin{bmatrix}-\frac{1}{2}\\ k\end{bmatrix}=1$$

since

$$\sum_{m=0}^{\infty}a_m^0(0)t^m=(1-t)^{-1}=\sum_{m=0}^{\infty}t^m.$$

Now estimate (16) and identity $a_m^0(0)=1$ yield

$$\left|a_m^{\gamma}(\theta)\right|\leqslant\sum_{k+l=m}\left|\begin{bmatrix}-\frac{1}{2}\\ l\end{bmatrix}\right|\left|\begin{bmatrix}-\frac{1}{2}\\ k\end{bmatrix}\right|e^{2c\gamma^2}=e^{2c\gamma^2}.$$

We have to distinguish between four cases $t\geqslant 2$, $1<t<2$, $0\leqslant t\leqslant\frac{1}{2}$ and $\frac{1}{2}<t<1$. Below we consider only the cases $t\geqslant 2$ and $1<t<2$ (proofs in two other cases are similar).

If $t\geqslant 2$, then

$$\left|P_{N-1}(t,\theta)\right|\leqslant\sum_{m=0}^{N-1}\left|a_m^{\gamma}(\theta)\right|t^m\leqslant e^{2c\gamma^2}t^N\leqslant\frac{3}{2}e^{2c\gamma^2}t^N\left|1-te^{i\theta}\right|^{-1}$$

since $1\leqslant\frac{3}{2}t|1-te^{i\theta}|^{-1}$. Hence, using $||1-te^{i\theta}|^{-1-i\gamma}|\leqslant t^N|1-te^{i\theta}|^{-1}$, it follows

$$\begin{aligned}\left|\left|1-te^{i\theta}\right|^{-1-i\gamma}-P_{N-1}(t,\theta)\right|&\leqslant t^N\left|1-te^{i\theta}\right|^{-1}+\frac{3}{2}e^{2c\gamma^2}t^N\left|1-te^{i\theta}\right|^{-1}\\&\leqslant Ce^{2c\gamma^2}t^N\left|1-te^{i\theta}\right|^{-1}\end{aligned}$$

for an appropriate $C>0$, as required.

If $1<t<2$, then, after two summations by parts, we derive

$$\begin{aligned}P_{N-1}(t,\theta)&=\sum_{l=0}^{N-3}S\begin{bmatrix}-\frac{1}{2}+\frac{i\gamma}{2}\\ l\end{bmatrix}D_l(\bar z)\sum_{k=0}^{N-l-3}S\begin{bmatrix}-\frac{1}{2}+\frac{i\gamma}{2}\\ k\end{bmatrix}D_k(z)\\&\quad+\sum_{l=0}^{N-2}S\begin{bmatrix}-\frac{1}{2}+\frac{i\gamma}{2}\\ l\end{bmatrix}\begin{bmatrix}-\frac{1}{2}+\frac{i\gamma}{2}\\ N-l-2\end{bmatrix}D_l(\bar z)D_{N-l-2}(z)\\&\quad+\sum_{k=0}^{N-1}\begin{bmatrix}-\frac{1}{2}+\frac{i\gamma}{2}\\ k\end{bmatrix}\begin{bmatrix}-\frac{1}{2}+\frac{i\gamma}{2}\\ N-k-1\end{bmatrix}z^kD_{N-1-k}(z)=J_1+J_2+J_3,\end{aligned}$$

where

$$S\begin{bmatrix}\delta\\ k\end{bmatrix}:=\begin{bmatrix}\delta\\ k\end{bmatrix}-\begin{bmatrix}\delta\\ k+1\end{bmatrix},\qquad D_k(z):=\sum_{j=0}^{k}z^j.$$

We use estimate

$$\left|S\begin{bmatrix} -\frac{1}{2}+\frac{i\gamma}{2} \\ k \end{bmatrix}\right| = \left|\begin{bmatrix} -\frac{1}{2}+\frac{i\gamma}{2} \\ k \end{bmatrix}\left(\frac{-\frac{1}{2}+\frac{i\gamma}{2}}{1+k}\right)\right| \leqslant C(k+1)^{-\frac{1}{2}}e^{c\gamma^2}$$

to obtain, following an argument in [18], that each J_i $(i=1,2,3)$ is majorized by $Ce^{c\gamma^2}t^N|1-te^{i\theta}|^{-1}$ for some $C>0$. Since $||1-te^{i\theta}|^{-1-i\gamma}| \leqslant t^N|1-te^{i\theta}|^{-1}$, Lemma 3 follows. □

4.2. Proof of Theorem 2

Choose $\Psi_j \in C^\infty(\Omega)$ in such a way that $0 \leqslant \Psi_j \leqslant 1$, $\Psi_j(x)=1$ for $|x|>\frac{2}{j}$, $\Psi_j(x)=0$ for $|x|<\frac{1}{j}$, $|\nabla\Psi_j(x)| \leqslant c'j$, $|\Delta\Psi_j(x)| \leqslant c'j^2$.

Proposition 2. *Let $\tau(V,0,\rho)<\infty$. There exists a constant $C=C(\rho,\delta,d)>0$ such that for all positive integers N and j*

$$\left\|\mathbf{1}_{B(\rho)}\Psi_j|V|^{\frac{1}{2}}\left[(-\Delta)^{-1}\right]_{N,N_d^\delta}|V|^{\frac{1}{2}}\Psi_j\mathbf{1}_{B(\rho)}\right\|_{2\mapsto 2} \leqslant C\tau(V,0,\rho)^{\frac{2}{d-1}}, \tag{E1}$$

$$\left\|\mathbf{1}_{B(\rho)}\Psi_j|V|^{\frac{1}{2}}\left[(-\Delta)^{-1}\right]_{N,N_d^\delta}|V|^{\frac{1}{2}}\mathbf{1}_{B(3\rho\setminus\rho)}\right\|_{2\mapsto 2} \leqslant C\tau(V,0,3\rho)^{\frac{2}{d-1}}, \tag{E2}$$

$$\left\|\mathbf{1}_{B(\rho)}\Psi_j|V|^{\frac{1}{2}}\left[(-\Delta)^{-1}\right]_{N,N_d^\delta}\mathbf{1}_{B(\frac{2}{j}\setminus\frac{1}{j})}\right\|_{p\mapsto 2} \leqslant C\tau(V,0,\rho)^{\frac{1}{d-1}}, \tag{E3}$$

$$\left\|\mathbf{1}_{B(\rho)}\Psi_j|V|^{\frac{1}{2}}\left[(-\Delta)^{-1}\right]_{N,N_d^\delta}\mathbf{1}_{B(3\rho\setminus 2\rho)}\right\|_{p\mapsto 2} \leqslant C\tau(V,0,3\rho)^{\frac{1}{d-1}}, \tag{E4}$$

where $p=\frac{2d}{d+2}$.

We prove Proposition 2 at the end of this section.

Proof of Theorem 2. We use the same notations as in the proof of Theorem 1. Suppose that $u \in Y_V^{\mathrm{str}}$ satisfies (1) and vanishes to an infinite order at $0 \in \Omega$. We wish to obtain an estimate of the form

$$\left\|\mathbf{1}_{B(\rho)}\frac{\varphi_{N_d^\delta}}{\varphi_{N_d^\delta}(\rho)}u\right\|_2 \leqslant C. \tag{17}$$

Then, letting $N \to \infty$, we would derive the required identity: $u \equiv 0$ in $B(0,\rho)$.

The same argument as in the proof of Theorem 1 leads us to an identity

$$u_{\eta_j} = (-\Delta)^{-1}(-\Delta u_{\eta_j}), \quad \eta_j = \eta\Psi_j,$$

which, in turn, implies

$$\begin{aligned}&\mathbf{1}_{B(\rho)}\Psi_j V_1^{\frac{1}{2}}\varphi_{N_d^\delta}u \\ &\quad = \mathbf{1}_{B(\rho)}\Psi_j V_1^{\frac{1}{2}}\left[(-\Delta)^{-1}\right]_{N,N_d^\delta}V_1^{\frac{1}{2}}\varphi_{N_d^\delta}\frac{-\eta_j\Delta u}{V_1^{\frac{1}{2}}} + \mathbf{1}_{B(\rho)}\Psi_j V_1^{\frac{1}{2}}\left[(-\Delta)^{-1}\right]_{N,N_d^\delta}\varphi_{N_d^\delta}E_j(u).\end{aligned}$$

Here $0 < \delta < 1/2$ is fixed, $2/j \leqslant \rho$, $\Delta u_{\eta_j} = \eta_j \Delta u + E_j(u)$ and

$$E_j(u) := 2\nabla\eta_j \nabla u + (\Delta\eta_j)u.$$

Letting I to denote the left hand side of the previous identity, and, respectively, I_1 and I_2 the two summands of the right hand side, we rewrite the latter as

$$I = I_1 + I_2.$$

Note that $I \in L^2$, since $H^{1,p}_{\mathrm{loc}}(\Omega) \subset X_2$ by Sobolev embedding theorem, and $|V|^{\frac{1}{2}}u \in X_2$ by the definition of Y_V^{str}.

Next, we expand I_1 as a sum $I_{11} + I_{11}^c$, where

$$I_{11} := \mathbf{1}_{B(\rho)} \Psi_j V_1^{\frac{1}{2}} \big[(-\Delta)^{-1}\big]_{N,N_d^\delta} V_1^{\frac{1}{2}} \mathbf{1}_{B(\rho)} \varphi_{N_d^\delta} \frac{-\Psi_j \Delta u}{V_1^{\frac{1}{2}}}$$

and

$$I_{11}^c := \mathbf{1}_{B(\rho)} \Psi_j V_1^{\frac{1}{2}} \big[(-\Delta)^{-1}\big]_{N,N_d^\delta} V_1^{\frac{1}{2}} \mathbf{1}^c_{B(\rho)} \varphi_{N_d^\delta} \frac{-\eta \Delta u}{V_1^{\frac{1}{2}}}.$$

Proposition 2 and inequalities (E1) and (E2) imply the required estimates:

$$\|I_{11}\|_2 \leqslant C\tau(V_1, 0, \rho)^{\frac{2}{d-1}} \|I\|_2$$

and

$$\big\|I_{11}^c\big\|_2 \leqslant C\varphi_{N_d^\delta}(\rho)\tau(V_1, 0, 3\rho)^{\frac{2}{d-1}} \big\|\mathbf{1}_{B(3\rho)}|V|^{\frac{1}{2}}u\big\|_2.$$

Finally, we represent I_2 as a sum $I_{21} + I_{22}$, where

$$I_{21} := \mathbf{1}_{B(\rho)} \Psi_j V_1^{\frac{1}{2}} \big[(-\Delta)^{-1}\big]_{N,N_d^\delta} \mathbf{1}_{B(\frac{2}{j}\setminus\frac{1}{j})} \varphi_{N_d^\delta} E_j^{(1)}(u)$$

and

$$I_{22} := \mathbf{1}_{B(\rho)} \Psi_j V_1^{\frac{1}{2}} \big[(-\Delta)^{-1}\big]_{N,N_d^\delta} \mathbf{1}_{B(3\rho\setminus 2\rho)} \varphi_{N_d^\delta} E_j^{(2)}(u).$$

Here

$$E_j^{(1)}(u) := -2\nabla\Psi_j \nabla u - (\Delta\Psi_j)u, \qquad E^{(2)}(u) := -2\nabla\eta\nabla u - (\Delta\eta)u.$$

In order to derive an estimate on $\|I_{21}\|_2$, we expand

$$I_{21} = I_{21}' + I_{21}'',$$

where

$$I_{21}' := \mathbf{1}_{B(\rho)}\Psi_j V_1^{\frac{1}{2}}\big[(-\Delta)^{-1}\big]_{N,N_d^\delta}\mathbf{1}_{B(\frac{2}{j}\setminus\frac{1}{j})}\varphi_{N_d^\delta}(-\Delta\Psi_j)u,$$

$$I_{21}'' := \mathbf{1}_{B(\rho)}\Psi_j V_1^{\frac{1}{2}}\big[(-\Delta)^{-1}\big]_{N,N_d^\delta}\mathbf{1}_{B(\frac{2}{j}\setminus\frac{1}{j})}\varphi_{N_d^\delta}(-2\nabla\eta\nabla u).$$

1) Term I_{21}' presents no problem: by (E3),

$$\begin{aligned}\|I_{21}'\|_2 &\leqslant \big\|\mathbf{1}_{B(\rho)}\Psi_j V_1^{\frac{1}{2}}\big[(-\Delta)^{-1}\big]_{N,N_d^\delta}\mathbf{1}_{B(\frac{2}{j}\setminus\frac{1}{j})}\big\|_{p\mapsto 2}\big\|\mathbf{1}_{B(\frac{2}{j}\setminus\frac{1}{j})}\varphi_{N_d^\delta}(\Delta\Psi_j)u\big\|_2\\ &\leqslant C\tau(V_1,0,\rho)^{\frac{1}{d-1}}\big\|\mathbf{1}_{B(\frac{2}{j}\setminus\frac{1}{j})}\varphi_{N_d^\delta}(\Delta\Psi_j)u\big\|_2,\end{aligned}$$

where

$$\big\|\mathbf{1}_{B(\frac{2}{j}\setminus\frac{1}{j})}\varphi_{N_d^\delta}(\Delta\Psi_j)u\big\|_2 \leqslant Cj^{N_d^\delta+2}\|\mathbf{1}_{B(\frac{2}{j})}u\|_2 \to 0 \quad \text{as } j\to\infty,$$

by the definition of the SUC property.

2) In order to derive an estimate on I_{21}'', we once again use inequality (E3):

$$\begin{aligned}\|I_{21}''\|_2 &\leqslant \big\|\mathbf{1}_{B(\rho)}\Psi_j V_1^{\frac{1}{2}}\big[(-\Delta)^{-1}\big]_{N,N_d^\delta}\mathbf{1}_{B(\frac{2}{j}\setminus\frac{1}{j})}\big\|_{p\mapsto 2}\|\mathbf{1}_{B(\frac{2}{j})}\varphi_{N_d^\delta}\nabla\Psi_j\nabla u\|_p\\ &\leqslant C\tau(V_1,0,\rho)^{\frac{1}{d-1}}\|\mathbf{1}_{B(\frac{2}{j})}\varphi_{N_d^\delta}\nabla\Psi_j\nabla u\|_p \leqslant \tilde{C}j^{N_d^\delta+1}\|\mathbf{1}_{B(\frac{2}{j})}\nabla u\|_p,\end{aligned}$$

where $p := \frac{2d}{d+2}$. We must estimate $\|\mathbf{1}_{B(\frac{2}{j})}\nabla u\|_2$ by $\|\mathbf{1}_{B(\frac{4}{j})}u\|_2$ in order to apply the SUC property. For this purpose, we make use of the following well-known interpolation inequality

$$\|\mathbf{1}_{B(\frac{2}{j})}\nabla u\|_p \leqslant Cj^{\frac{d}{p}}\big(C'j^{\frac{d}{2}-1}\|\mathbf{1}_{B(\frac{4}{j})}u\|_2 + j^{\frac{d+6}{2}}\|\mathbf{1}_{B(\frac{4}{j})}\Delta u\|_r\big),$$

where $r := \frac{2d}{d+4}$ (see [15]). Using differential inequality (1), we reduce the problem to the problem of finding an estimate on $\|\mathbf{1}_{B(\frac{4}{j})}Vu\|_r$ in terms of $\|\mathbf{1}_{B(\frac{4}{j})}u\|_2^{\mu}$, $\mu>0$. By Hölder inequality,

$$\|\mathbf{1}_{B(\frac{4}{j})}Vu\|_r \leqslant \big\|\mathbf{1}_{B(\frac{4}{j})}|V|^{\frac{1}{2}}u\big\|_2^{\frac{2}{d}}\|\mathbf{1}_{B(\frac{4}{j})}V\|_{\frac{d-1}{2}}^{\frac{d-1}{d}}\|\mathbf{1}_{B(\frac{4}{j})}u\|_2^{1-\frac{2}{d}},$$

as required.

As the last step of the proof, we use inequality (E4) to derive an estimate on term I_{22}:

$$\|I_{22}\|_2 \leqslant C\tau(V_1,0,3\rho)^{\frac{1}{d-1}}\varphi_{N_d^\delta}(\rho)\big\|E_j^{(2)}(u)\big\|_p.$$

This estimate and the estimates obtained above imply (17). $\square$

Proof of Proposition 2. Estimates (E1) and (E2) follow straightforwardly from Proposition 1. In order to prove estimate (E3), we introduce the following interpolation function:

$$F_1(z) := \mathbf{1}_{B(\rho)}\Psi_j|V|^{\frac{d-1}{4}z}\varphi_{N+(\frac{d}{2}-\delta)(1-z)}\big[(-\Delta)^{-\frac{d-1}{2}z}\big]_N\varphi_{N+(\frac{d}{2}-\delta)(1-z)}^{-1}\mathbf{1}_{B(\frac{2}{j}\setminus\frac{1}{j})}, \quad 0\leqslant \mathrm{Re}(z)\leqslant 1.$$

According to Lemma 2, $\|F_1(i\gamma)\|_{2\mapsto 2} \leqslant C_1 e^{c_1|\gamma|}$ for appropriate C_1, $c_1 > 0$. Further, according to Lemma 3,

$$\begin{aligned}\left\|F_1(1+i\gamma)\right\|_{\frac{2d}{2d-1}\mapsto 2} &\leqslant C_2 e^{c_2\gamma^2}\left\|\mathbf{1}_{B(\rho)}|V|^{\frac{d-1}{4}}(-\Delta)^{-\frac{d-1}{2}}\right\|_{\frac{2d}{2d-1}\mapsto 2}\\ &\leqslant C_2 e^{c_2\gamma^2}\left\|\mathbf{1}_{B(\rho)}|V|^{\frac{d-1}{4}}(-\Delta)^{-\frac{d-1}{4}}\right\|_{2\mapsto 2}\left\|(-\Delta)^{-\frac{d-1}{4}}\right\|_{\frac{2d}{2d-1}\mapsto 2}\\ &\leqslant C_2 e^{c_2\gamma^2}\tau(V,x_0,\rho)^{\frac{1}{2}}\left\|(-\Delta)^{-\frac{d-1}{4}}\right\|_{\frac{2d}{2d-1}\mapsto 2}\end{aligned}$$

for appropriate C_2, $c_2 > 0$, where, clearly, $\|(-\Delta)^{-\frac{d-1}{4}}\|_{\frac{2d}{2d-1}\mapsto 2} < \infty$. Therefore, by Stein's interpolation theorem,

$$\left\|F_1\left(\frac{2}{d-1}\right)\right\|_{p\mapsto 2} \leqslant C\tau(V,x_0,\rho)^{\frac{1}{d-1}}.$$

The latter inequality implies (E3).

The proof of estimate (E4) is similar: it suffices to consider interpolation function

$$F_2(z) := \mathbf{1}_{B(\rho)}\Psi_j|V|^{\frac{d-1}{4}z}\varphi_{N+(\frac{d}{2}-\delta)(1-z)}\left[(-\Delta)^{-\frac{d-1}{2}z}\right]_N\varphi^{-1}_{N+(\frac{d}{2}-\delta)(1-z)}\mathbf{1}_{B(3\rho\setminus 2\rho)}$$

for $0 \leqslant \mathrm{Re}(z) \leqslant 1$. $\square$

5. Proof of Theorem 4

Proof of Theorem 4. Let $u \in Y_V^{\mathcal{K}}$. Suppose that $u \equiv 0$ in some neighborhood of 0. Assume that $\rho > 0$ is sufficiently small, so that $\bar{B}(0,2\rho) \subset \Omega$, and let $\eta \in C^\infty(\Omega)$ be such that $\eta \equiv 1$ on $B(0,\rho)$, $\eta \equiv 0$ on $\Omega \setminus B(0,2\rho)$. We may assume, without loss of generality, that $V \geqslant 1$. The standard limiting argument implies the following identity:

$$\mathbf{1}_{B(\rho)}u = \mathbf{1}_{B(\rho)}\left[(-\Delta)^{-1}\right]_N(-\Delta u_\eta).$$

Therefore, we can write

$$\begin{aligned}&\mathbf{1}_{B(\rho)}\varphi_N Vu\\ &\quad= \mathbf{1}_{B(\rho)}\varphi_N V\left[(-\Delta)^{-1}\right]_N\varphi_N^{-1}\mathbf{1}_{B(\rho)}\varphi_N(-\Delta u) + \mathbf{1}_{B(\rho)}\varphi_N V\left[(-\Delta)^{-1}\right]_N\varphi_N^{-1}\mathbf{1}^c_{B(\rho)}\varphi_N(-\Delta u_\eta),\end{aligned}$$

or, letting K to denote the left hand side and, respectively, K_1 and K_2 the two summands of the right hand side of the last equality, we rewrite the latter as

$$K = K_1 + K_2.$$

Note that $K \in L^1(\mathbb{R}^d)$, as follows from definition of space $Y_V^{\mathcal{K}}$. Lemma 1 implies that

$$\left\|\mathbf{1}_{B(\rho)}\varphi_N V\left[(-\Delta)^{-1}\right]_N\varphi_N^{-1}f\right\|_1 \leqslant C\left\|\mathbf{1}_{B(\rho)}V(-\Delta)^{-1}f\right\|_1 \leqslant C\beta\|f\|_1,$$

for all $f \in L^1(\Omega)$, which implies an estimate on K_1:

$$\|K_1\|_1 \leqslant C\beta\|K\|_1.$$

In order to estimate K_2, we first note that $\mathbf{1}^c_{B(\rho)}(-\Delta u_\eta) = \mathbf{1}_{B(2\rho\setminus\rho)}(-\Delta u_\eta)$. According to Lemma 1 there exists a constant $\hat{C} > 0$ such that

$$\left\|\mathbf{1}_{B(2\rho)}\varphi_N V\left[(-\Delta)^{-1}\right]_N \varphi_N^{-1}\right\|_{1\mapsto 1} \leqslant \hat{C}.$$

Hence,

$$\|K_2\|_1 \leqslant \hat{C}\left\|\mathbf{1}_{B(2\rho\setminus\rho)}\varphi_N(-\Delta u_\eta)\right\|_1 \leqslant \hat{C}\rho^{-N}\|\Delta u_\eta\|_1.$$

Let us choose $\beta > 0$ such that $C\beta < 1$. Then the estimates above imply

$$(1 - C\beta)\left\|\mathbf{1}_{B(\rho)}\rho^N\varphi_N u\right\|_1 \leqslant (1 - C\beta)\left\|\rho^N K\right\|_1 \leqslant \left\|\rho^N K_2\right\|_1 \leqslant \hat{C}\|\Delta u_\eta\|_1.$$

Letting $N \to \infty$, we obtain $u \equiv 0$ in $B(0, \rho)$. □

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