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ON UNIFORM SUBALGEBRAS OF L^∞ ON THE UNIT CIRCLE GENERATED BY ALMOST PERIODIC FUNCTIONS

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Abstract. Analogs of almost periodic functions for the unit circle are introduced. Certain uniform algebras generated by such functions are studied, the corona theorems for them are proved, and their maximal ideal spaces are described.

§1. Formulation of main results

1.1. The classical almost periodic functions on the real line, as first introduced by H. Bohr in the 1920s, play an important role in various areas of analysis. In the present paper we define analogs of almost periodic functions on the unit circle. We study certain uniform algebras generated by such functions. In particular, in these terms we describe some uniform subalgebras of the algebra H^∞ of bounded holomorphic functions on the open unit disk $\mathbb{D} \subset \mathbb{C}$ that, in a sense, have the weakest possible discontinuities on the boundary $\partial\mathbb{D}$.

To formulate the main results of the paper, we start with recalling the definition of an almost periodic function; see [B].

Definition 1.1. A continuous function $f : \mathbb{R} \to \mathbb{C}$ is said to be *almost periodic* if, for any $\epsilon > 0$, there exists $l(\epsilon) > 0$ such that for every $t_0 \in \mathbb{R}$ the interval $[t_0, t_0 + l(\epsilon)]$ contains at least one number τ for which

$$|f(t) - f(t+\tau)| < \epsilon \quad \text{for all} \quad t \in \mathbb{R}.$$

It is well known that every almost periodic function f is uniformly continuous and is the uniform limit of a sequence of exponential polynomials $\{q_n\}_{n\in\mathbb{N}}$, where $q_n(t) := \sum_{k=1}^{n} c_{kn} e^{i\lambda_{kn} t}$, $c_{kn} \in \mathbb{C}$, $\lambda_{kn} \in \mathbb{R}$, $1 \le k \le n$, and $i := \sqrt{-1}$.

In what follows we consider $\partial\mathbb{D}$ with the *counterclockwise orientation.* For $t_0 \in \mathbb{R}$, let $\gamma_{t_0^k}(s) := \{e^{i(t_0+kt)} \ : \ 0 < t < s \le 2\pi\} \subset \partial\mathbb{D}$, $k \in \{-1, 1\}$, be two open arcs having e^{it_0} as the right or the left endpoint relative to the chosen orientation, respectively.

We define almost periodic functions on open arcs of $\partial\mathbb{D}$.

Definition 1.2. A continuous function $f_k : \gamma_{t_0^k}(s) \to \mathbb{C}$, $k \in \{-1, 1\}$, is said to be *almost periodic* if the function $\widehat{f}_k : (-\infty, 0) \to \mathbb{C}$, $\widehat{f}_k(t) := f_k(e^{i(t_0+kse^t)})$, is the restriction of an almost periodic function on $\mathbb{R}$.

Example 1.3. In the sense of this definition, the function $e^{i\lambda \log_{t_0^k}}$, $\lambda \in \mathbb{R}$, where

$$\log_{t_0^k}(e^{i(t_0+kt)}) := \ln t, \quad 0 < t < 2\pi, \ \ k \in \{-1, 1\},$$

is almost periodic on $\gamma_{t_0^1}(2\pi) = \gamma_{t_0^{-1}}(2\pi)$.

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By $\mathrm{SAP}(\partial\mathbb{D}) \subset L^\infty(\partial\mathbb{D})$ we denote the uniform subalgebra of functions f such that for each t_0 and any $\epsilon > 0$ there is a number $s := s(t_0, \epsilon) \in (0, \pi)$ and almost periodic functions $f_k : \gamma_{t_0^k}(s) \to \mathbb{C}$, $k \in \{-1, 1\}$, such that

$$\operatorname*{ess\,sup}_{z\in\gamma_{t_0^1}(s)} |f(z) - f_1(z)| < \epsilon \quad\text{and}\quad \operatorname*{ess\,sup}_{z\in\gamma_{t_0^{-1}}(s)} |f(z) - f_{-1}(z)| < \epsilon. \tag{1.1}$$

Let $S \subset \partial\mathbb{D}$ be a nonempty closed subset. By $\mathrm{SAP}(S) \subset \mathrm{SAP}(\partial\mathbb{D})$ we denote the uniform algebra of functions belonging to $\mathrm{SAP}(\partial\mathbb{D})$ and continuous on $\partial\mathbb{D} \setminus S$.

Remark 1.4. Note that each algebra $\mathrm{SAP}(\{z\})$, $z \in S^1$, contains a proper subalgebra isomorphic to the algebra of semi-almost periodic functions first introduced by Sarason [S] (see also [Sp]).

We fix a real continuous function g on $\gamma_{t_0^1}(2\pi)$ such that

$$\lim_{t\to 0+} g(e^{it}) = 1, \qquad \lim_{t\to 2\pi-} g(e^{it}) = 0,$$

and $g(e^{it})$ is monotone decreasing for $0 < t < 2\pi$. We set $g_{t_0}(e^{it}) := g(e^{i(t_0+t)})$.

Theorem 1.5. *The algebra* $\mathrm{SAP}(S)$ *is the uniform closure in* $L^\infty(\partial\mathbb{D})$ *of the algebra of complex polynomials in variables* g_{t_0} *and* $e^{i\lambda \log_{t_0^k}}$, $\lambda \in \mathbb{R}$, $e^{it_0} \in S$, $k \in \{-1, 1\}$.

Let $\phi : \partial\mathbb{D} \to \partial\mathbb{D}$ be a C^1-diffeomorphism. We denote by $\phi^* : L^\infty(\partial\mathbb{D}) \to L^\infty(\partial\mathbb{D})$, $\phi^*(f) := f \circ \phi$, the pullback by ϕ and put $\widetilde{S} := \phi(S)$. As a consequence of Theorem 1.5, we obtain the following.

Corollary 1.6. ϕ^* *maps* $\mathrm{SAP}(\widetilde{S})$ *isomorphically onto* $\mathrm{SAP}(S)$.

1.2. We say that a complex-valued function $g \in L^\infty(\partial\mathbb{D})$ has a discontinuity of the *first kind* at x_0 if the one-sided limits of g at x_0 exist but have distinct values. For a closed subset $S \subset \partial\mathbb{D}$, we denote by $R_S \subset L^\infty(\partial\mathbb{D})$ the uniform algebra of complex functions allowing discontinuities of the first kind at points of S and continuous on $\partial\mathbb{D} \setminus S$. The elements of R_S are often referred to as *regulated functions* [D]. Clearly, $R_S \hookrightarrow \mathrm{SAP}(S)$. Also, we shall show (see Lemma 3.1 below) that R_S is the uniform closure of the algebra generated by all possible subalgebras R_F with finite $F \subset S$.

Let $\mathcal{M}(\mathrm{SAP}(S))$ be the *maximal ideal space* of $\mathrm{SAP}(S)$, that is, the space of all *characters* (= nonzero homomorphisms $\mathrm{SAP}(S) \to \mathbb{C}$) on $\mathrm{SAP}(S)$ equipped with the weak*-topology (also known as the *Gelfand topology*) inherited from $(\mathrm{SAP}(S))^*$. By definition, $\mathcal{M}(\mathrm{SAP}(S))$ is a compact Hausdorff space. The main result formulated in this section describes the topological structure of $\mathcal{M}(\mathrm{SAP}(S))$.

Consider the continuous embeddings of uniform algebras

$$C(\partial\mathbb{D}) \hookrightarrow R_S \hookrightarrow \mathrm{SAP}(S).$$

The maps dual to these embeddings determine continuous surjective maps of the corresponding maximal ideal spaces:

$$\mathcal{M}(\mathrm{SAP}(S)) \xrightarrow{r_S} \mathcal{M}(R_S) \xrightarrow{c_S} \mathcal{M}(C(\partial\mathbb{D})) \cong \partial\mathbb{D}.$$

Theorem 1.7. (1) *For each* $z \in S$, *the preimage* $c_S^{-1}(z)$ *consists of two points* z_+ *and* z_-, *which are identified naturally with the counterclockwise and clockwise orientations of* $\partial\mathbb{D}$ *at* z.

(2) $c_S : \mathcal{M}(R_S) \setminus c_S^{-1}(S) \to \partial\mathbb{D} \setminus S$ *is a homeomorphism.*

(3) $c_S^{-1}(S) \subset \mathcal{M}(R_S)$ *is a totally disconnected compact Hausdorff space.*

(4) *For each* $\xi \in c_S^{-1}(S)$, *the preimage* $r_S^{-1}(\xi)$ *is homeomorphic to the Bohr compactification* $b\mathbb{R}$ *of* $\mathbb{R}$.

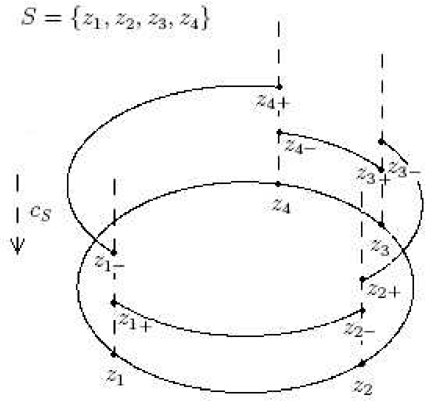

FIGURE 1. For a given $S = \{z_1, z_2, z_3, z_4\}$, c_S is a homeomorphism of $\mathcal{M}(R_S) \setminus \{z_{1+}, z_{1-}, z_{2+}, z_{2-}, z_{3+}, z_{3-}, z_{4+}, z_{4-}\}$ and $\partial\mathbb{D} \setminus \{z_1, z_2, z_3, z_4\}$, where $c_S(z_{i+}) = c_S(z_{i-}) = z_i$.

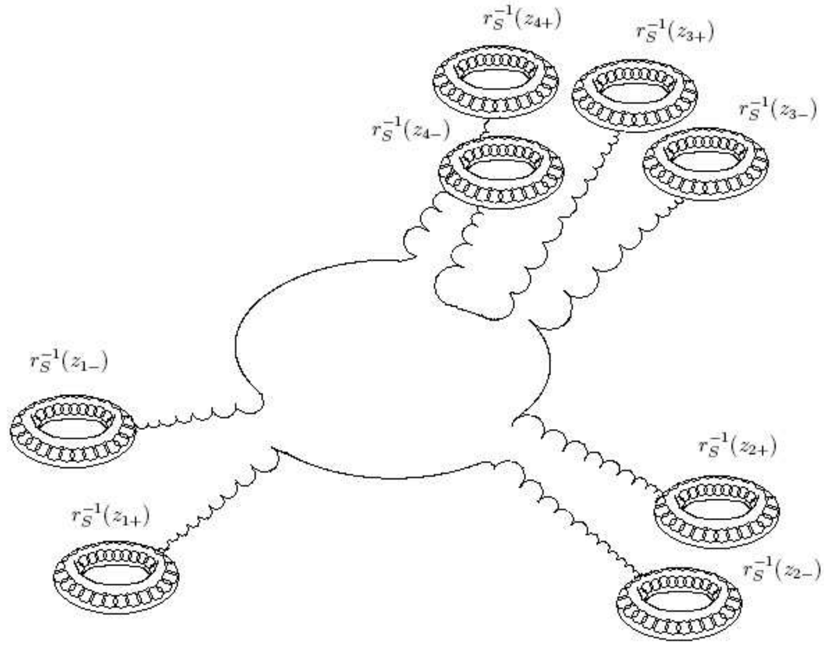

FIGURE 2. Given an n point set S, the maximal ideal space $\mathcal{M}(\mathrm{SAP}(S))$ is the union of $\partial\mathbb{D} \setminus S$ and $2n$ Bohr compactifications of $\mathbb{R}$ that can be viewed as infinite-dimensional "tori"; the spirals joining the arcs and the tori indicate (figuratively) that the topology of the Bohr compactifications affects the topology of the arcs.

(5) *The map* $r_S : \mathcal{M}(\mathrm{SAP}(S)) \setminus (c_S \circ r_S)^{-1}(S) \to \mathcal{M}(R_S) \setminus c_S^{-1}(S) \cong \partial\mathbb{D} \setminus S$ *is a homeomorphism.*

We recall that $b\mathbb{R}$ is a compact Abelian topological group homeomorphic to the maximal ideal space of the algebra of continuous almost periodic functions on $\mathbb{R}$. Also, (2)–(5) imply straitforwardly that

(6) the covering dimension of $\mathcal{M}(\mathrm{SAP}(S))$ is ∞;
(7) for a continuous map $\phi : T \to (c_S \circ r_S)^{-1}(S)$ of a connected topological space T, there is a point $\xi \in c_S^{-1}(S)$ such that $\phi(T) \subset r_S^{-1}(\xi)$.

1.3. Let $A_0 \subset H^\infty$ denote the *disk-algebra*, i.e., the algebra of functions continuous on the closure $\overline{\mathbb{D}}$ and holomorphic in $\mathbb{D}$. Also, by $f|_{\partial\mathbb{D}}$ we denote the boundary values of $f \in C(\mathbb{D})$ (in case they exist). In the present part we describe the uniform subalgebras of H^∞ generated by almost periodic functions. These subalgebras contain A_0 and have, in a sense, the weakest possible discontinuities on $\partial\mathbb{D}$.

Suppose that S contains at least 2 points. Let $A_S \subset H^\infty$ denote the uniform closure of the algebra generated by A_0 and by the holomorphic functions of the form e^f, where $\mathrm{Re} f|_{\partial\mathbb{D}}$ is a finite linear combination with real coefficients of characteristic functions of closed arcs whose endpoints belong to S. If S consists of a single point, we define $A_S \subset H^\infty$ to be the uniform closure of the algebra generated by A_0 and the functions $ge^{\lambda f}$, $\lambda \in \mathbb{R}$, where $\mathrm{Re} f|_{\partial\mathbb{D}}$ is the characteristic function of a closed arc with an endpoint at S and $g \in A_0$ is a function such that ge^f has discontinuity on S only. In the following result we naturally identify A_S and H^∞ with the algebras of their boundary values.

Theorem 1.8.

$$A_S = \mathrm{SAP}(S) \cap H^\infty.$$

Remark 1.9. Suppose that $F \subset \partial\mathbb{D}$ contains at least 2 points. Let $e^{\lambda f} \in A_S$, $\lambda \in \mathbb{R}$, where $\mathrm{Re}\, f$ is the characteristic function of an arc $[x, y]$ with $x, y \in S$. Let $\phi_{x,y} : \mathbb{D} \to \mathbb{H}_+$ be the bilinear map onto the upper half-plane that maps x to 0, the midpoint of the arc $[x, y]$ to 1, and y to ∞. Then there is a constant C such that

$$e^{\lambda f(z)} = e^{-\frac{i\lambda}{\pi}\mathrm{Log}\,\phi_{x,y}(z)+\lambda C}, \quad z \in \mathbb{D},$$

where Log denotes the principal branch of the logarithmic function. Thus, by Theorem 1.8, the algebra $SAP(S) \cap H^\infty$ is the uniform closure of the algebra generated by A_0 and the functions $e^{i\lambda(\mathrm{Log}\circ\phi_{x,y})}$, $\lambda \in \mathbb{R}$, $x, y \in S$.

The following example shows that if S is an infinite set, then A_S does not coincide with the uniform algebra generated by the functions e^f with $\mathrm{Re} f \in R_S$ (the corresponding arguments are presented in Subsection 4.3).

Example 1.10. Assume that a closed subset $S \subset \partial\mathbb{D}$ contains -1, 1, and a sequence $\{e^{it_k}\}_{k\in\mathbb{N}}$, $t_k \in (0, \pi/2)$, converging to 1. Let $\{\alpha_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers satisfying $\sum_{k=1}^\infty \alpha_k = 1$. Let χ_k denote the characteristic function of the arc $\gamma_k := \{e^{it} : t_k \le t \le \pi\}$. Consider the function

$$u(z) := \sum_{k=1}^\infty \alpha_k \chi_k(z), \quad z \in \partial\mathbb{D}.$$

Clearly, $u \in R_S$. Let h be a holomorphic function on $\mathbb{D}$ such that $\mathrm{Re}\, h|_{\partial\mathbb{D}} = u$. Then $e^h \in H^\infty \setminus A_S$. However, for any $f \in A_0$ with $f(1) = 0$ we have $fe^h \in A_S$.

Remark 1.11. It seems to be natural that a function $f \in H^\infty$ has the weakest possible discontinuities on $\partial\mathbb{D}$ if $f|_{\partial\mathbb{D}} \in R_S$. However, from the classical Lindelöf theorem [L] it follows that, in fact, any such f belongs to A_0. Moreover, the same conclusion is obtained even from the fact that $\mathrm{Re} f|_{\partial\mathbb{D}} \in R_S$ for $f \in H^\infty$. In particular, if f is holomorphic on $\mathbb{D}$ and $\mathrm{Re} f|_{\partial\mathbb{D}}$ is well defined and belongs to $R_S \setminus C(\partial\mathbb{D})$, then $f \notin H^\infty$.[1] Nevertheless, we have $e^f \in H^\infty$, which partly explains the choice of the object of our research.

Let $\mathcal{M}(A_S)$ be the maximal ideal space of A_S. Since the evaluation functionals $z(f) := f(z)$, $z \in \mathbb{D}$, $f \in A_S$, belong to $\mathcal{M}(A_S)$, and A_S separates the points on $\mathbb{D}$, there is a continuous embedding $i_S : \mathbb{D} \to \mathcal{M}(A_S)$. In the sequel we identify $\mathbb{D}$ with $i_S(\mathbb{D})$. Then the following corona theorem is true.

Theorem 1.12. *$\mathbb{D}$ is dense in $\mathcal{M}(A_S)$.*

Remark 1.13. Recall that the corona theorem is equivalent to the following statement (see, e.g., [G, Chapter V]):

For any collection of functions $f_1, \dots, f_n \in A_S$ satisfying the corona condition

$$\max_{1\le j\le n} |f_j(z)| \ge \delta > 0, \quad z \in \mathbb{D}, \tag{1.2}$$

there exist functions $g_1, \dots, g_n \in A_S$ such that

$$f_1 g_1 + \cdots + f_n g_n = 1. \tag{1.3}$$

Finally, we formulate some results on the structure of $\mathcal{M}(A_S)$. Since $A_0 \hookrightarrow A_S$, there is a continuous surjection

$$a_S : \mathcal{M}(A_S) \to \mathcal{M}(A_0) \cong \overline{\mathbb{D}}$$

of maximal ideal spaces. Recall that the *Šilov boundary* of A_S is the smallest compact subset $K \subset \mathcal{M}(A_S)$ such that

$$\sup_{z\in\mathcal{M}(A_S)} |f(z)| = \sup_{\xi\in K} |f(\xi)|$$

for each $f \in A_S$. Here we assume that every $f \in A_S$ is also defined on $\mathcal{M}(A_S)$; its extension to $\mathcal{M}(A_S) \setminus \mathbb{D}$ is given by the *Gelfand transform* $f(\xi) := \xi(f)$, $\xi \in \mathcal{M}(A_S)$.

Theorem 1.14. (1) *$a_S : \mathcal{M}(A_S) \setminus a_S^{-1}(S) \to \overline{\mathbb{D}} \setminus S$ is a homeomorphism.*

(2) *The Šilov boundary K_S of A_S is naturally homeomorphic to $\mathcal{M}(\mathrm{SAP}(S))$. Under the identification of K_S and $\mathcal{M}(\mathrm{SAP}(S))$, we have $a_S|_{K_S} = r_S \circ c_S$.*

(3) *For each $z \in S$, the preimage $a_S^{-1}(z)$ is homeomorphic to the maximal ideal space of the algebra $AP_{\mathcal{O}}(\Sigma)$ of uniformly continuous almost periodic functions defined on the strip $\Sigma := \{z \in \mathbb{C} \;:\; \mathrm{Im}\, z \in [0, \pi]\}$ and holomorphic at the interior points of Σ.*

In the next section we describe the topological structure of the maximal ideal space $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ of $AP_{\mathcal{O}}(\Sigma)$. We show that this space is equipped with a natural "complex structure". Similarly, each fiber $a_S^{-1}(z)$, $z \in S$, has a natural complex structure so that the homeomorphisms in Theorem 1.14 (3) are analytic with respect to these structures.

In a forthcoming paper we shall present similar results for algebras of bounded holomorphic functions on an open polydisk that are generated by almost periodic functions.

[1] In this case $f|_{\partial\mathbb{D}} \in \mathrm{BMO}(\partial\mathbb{D})$ with $||f||_{\mathrm{BMO}(\partial\mathbb{D})} \le c||\mathrm{Re} f|_{\partial\mathbb{D}}||_{L^\infty(\partial\mathbb{D})}$ for some absolute constant $c > 0$.

§2. The maximal ideal space of the algebra $AP_{\mathcal{O}}(\Sigma)$

2.1. The construction presented below is rather general and makes sense for Galois coverings of complex manifolds with boundaries (cf. [Br]). However, we restrict ourselves to the case of coverings of annuli related to the subject of our paper.

Consider the annulus $R := \{z \in \mathbb{C} : \ e^{-2\pi^2} \le |z| \le 1\}$. Its universal covering can be identified with Σ so that $e : \Sigma \to R$, $e(z) := e^{2\pi i z}$, $z \in \Sigma$, is the covering map. We can also regard Σ as a principal bundle on R with fiber $\mathbb{Z}$ (see, e.g., [H] for the corresponding topological definitions). To specify, consider a cover of R by compact simply connected sets U_1 and U_2. Then $e^{-1}(U_k)$ can be identified with $U_k \times \mathbb{Z}$, $k = 1, 2$. Also, there is a continuous map $c_{12} : U_1 \cap U_2 \to \mathbb{Z}$ such that R is isomorphic (in the category of complex manifolds with boundaries) to the quotient space of $(U_1 \times \mathbb{Z}) \sqcup (U_2 \times \mathbb{Z})$ under the following equivalence relation:

$$U_1 \times \mathbb{Z} \ni (z, n) \sim (z, n + c_{12}(z)) \in U_2 \times \mathbb{Z} \quad \text{for all} \ \ z \in U_1 \cap U_2 \ \ \text{and} \ \ n \in \mathbb{Z}.$$

Let $b\mathbb{Z}$ be the Bohr compactification of $\mathbb{Z}$. Then the action of $\mathbb{Z}$ on itself by translations extends naturally to the action on $b\mathbb{Z}$: $\xi \mapsto \xi + n$, $\xi \in b\mathbb{Z}$, $n \in \mathbb{Z}$. Let $E(R, b\mathbb{Z})$ denote the principal bundle on R with fiber $b\mathbb{Z}$ defined as the quotient of $(U_1 \times b\mathbb{Z}) \sqcup (U_2 \times b\mathbb{Z})$ under the following equivalence relation:

$$U_1 \times b\mathbb{Z} \ni (z, \xi) \sim (z, \xi + c_{12}(z)) \in U_2 \times b\mathbb{Z} \ \text{ for all } \ z \in U_1 \cap U_2 \ \text{ and } \ \xi \in b\mathbb{Z}.$$

Clearly, $E(R, b\mathbb{Z})$ is a compact Hausdorff space in the quotient topology induced by that of $(U_1 \times b\mathbb{Z}) \sqcup (U_2 \times b\mathbb{Z})$. Also, the projection $p : E(R, b\mathbb{Z}) \to R$ is determined by the natural projections $U_k \times b\mathbb{Z} \to U_k$, $k = 1, 2$, onto the first coordinate.

Next, the natural local injections $U_k \times \mathbb{Z} \hookrightarrow U_k \times b\mathbb{Z}$, $k = 1, 2$, determine an injection $i_0 : \Sigma \hookrightarrow E(R, b\mathbb{Z})$ such that $p \circ i_0 = e$. Moreover, $i_0(\Sigma)$ is dense in $E(R, b\mathbb{Z})$, because $\mathbb{Z}$ is dense in $b\mathbb{Z}$ (in the topology of $b\mathbb{Z}$). Similarly, we can define an injection $i_\xi : \Sigma \hookrightarrow E(R, b\mathbb{Z})$, $\xi \in b\mathbb{Z}$, by the formula $i_\xi((z, n)) := (z, \xi + n)$, $z \in U_k$, $n \in \mathbb{Z}$, $k = 1, 2$. Since, by definition, $\xi + \mathbb{Z}$ is dense in $b\mathbb{Z}$, the image $i_\xi(\Sigma)$ is dense in $E(R, b\mathbb{Z})$ for any ξ. Moreover, $E(R, b\mathbb{Z}) = \sqcup_\xi i_\xi(\Sigma)$, where the union is taken over all ξ whose images in the quotient group $b\mathbb{Z}/\mathbb{Z}$ are mutually distinct. Observe also that every i_ξ is a continuous map and locally is an embedding.

Definition 2.1. A continuous function f on $E(R, b\mathbb{Z})$ is said to be *holomorphic* if every function $f \circ i_\xi$ is holomorphic at the interior points of Σ.

We denote by $\mathcal{O}(E(R, b\mathbb{Z})) \subset C(E(R, b\mathbb{Z}))$ the Banach algebra of holomorphic functions on $E(R, b\mathbb{Z})$.

Remark 2.2. By using a normal family argument and the fact that $i_\xi(\Sigma)$ is dense in $E(R, b\mathbb{Z})$, it can easily be shown that $f \in C(E(R, b\mathbb{Z}))$ is holomorphic if and only if there is $\xi \in b\mathbb{Z}$ such that $f \circ i_\xi$ is holomorphic at the interior points of Σ.

2.2. We recall that $f \in AP_{\mathcal{O}}(\Sigma)$ if f is uniformly continuous with respect to the Euclidean metric on Σ, holomorphic at the interior points of Σ, and its restriction to each straight line parallel to the x-axis is almost periodic. In the following lemma we identify Σ with $i_0(\Sigma) \subset E(R, b\mathbb{Z})$.

Lemma 2.3. *Every $f \in AP_{\mathcal{O}}(\Sigma)$ admits a continuous extension up to a holomorphic function $\widehat{f}$ on $E(R, b\mathbb{Z})$. Moreover, the correspondence $AP_{\mathcal{O}}(\Sigma) \to \mathcal{O}(E(R, b\mathbb{Z}))$, $f \mapsto \widehat{f}$, determines an isomorphism of Banach algebras.*

Proof. By using the Pontryagin duality [P], it is easy to show that the closure of $\mathbb{Z}$ in $b\mathbb{R}$, the Bohr compactification of $\mathbb{R}$, is isomorphic to $b\mathbb{Z}$. In particular, the restrictions to $\mathbb{Z}$ of almost periodic functions on $\mathbb{R}$ are almost periodic functions on $\mathbb{Z}$, and the algebra generated by the extensions of such functions to $b\mathbb{Z}$ separates points on $b\mathbb{Z}$. In a natural way, we identify every $p^{-1}(U_k)$ with $U_k \times b\mathbb{Z}$ and every $e^{-1}(U_k)$ with $U_k \times \mathbb{Z}$ ($\subset U_k \times b\mathbb{Z}$), $k = 1, 2$, and regard $f|_{e^{-1}(U_k)}$, $f \in AP_{\mathcal{O}}(\Sigma)$, as a bounded function $f_k \in C(U_k \times \mathbb{Z})$. Then:

(a) f_k is holomorphic at the interior points of $U_k \times \mathbb{Z}$;
(b) f_k is uniformly continuous on $U_k \times \mathbb{Z}$ with respect to the metric $r(v_1, v_2) := |z_1 - z_2| + |n_1 - n_2|$ on $U_k \times \mathbb{Z}$, where $v_1 = (z_1, n_1)$, $v_2 = (z_2, n_2) \in U_k \times \mathbb{Z}$, and $|\cdot|$ is the Euclidean norm on $\mathbb{C}$;
(c) $f_k|_{\{z\}\times\mathbb{Z}}$ is almost periodic on $\mathbb{Z}$ for every $z \in U_k$.

To prove the lemma, it suffices to show that

(1) *there is a continuous function $\widehat{f}_k$ on $U_k \times b\mathbb{Z}$ such that $\widehat{f}_k|_{U_k\times\mathbb{Z}} = f_k$, for every $\xi \in b\mathbb{Z}$ the function $\widehat{f}_k|_{U_k\times\{\xi\}}$ is holomorphic at the interior points of U_k, and* $\sup_{U_k\times b\mathbb{Z}} |\widehat{f}_k| = \sup_{U_k\times\mathbb{Z}} |f_k|$;
(2) *if $f \in \mathcal{O}(E(R, b\mathbb{Z}))$, then $f|_\Sigma \in AP_{\mathcal{O}}(\Sigma)$.*

(1) Since for every $z \in U_k$ the function $f_{kz}(n) := f_k(z, n)$ is almost periodic on $\mathbb{Z}$, there is a continuous function $\widehat{f}_{kz}$ on $b\mathbb{Z}$ that extends f_{kz}. We set $\widehat{f}_k(z, \xi) := \widehat{f}_{kz}(\xi)$, $\xi \in b\mathbb{Z}$, and prove that $\widehat{f}_k$ is continuous. Indeed, take a point $w = (z, \xi) \in U_k \times b\mathbb{Z}$ and a number $\epsilon > 0$. By the uniform continuity of f_k, there is $\delta > 0$ such that for any pair of points $v_1 = (z_1, n)$ and $v_2 = (z_2, n)$ in $U_k \times \mathbb{Z}$ with $|z_1 - z_2| < \delta$ we have $|f(z_1, n) - f(z_2, n)| < \epsilon/3$. We define a neighborhood U_z of $z \in U_k$ by $U_z := \{z' \in U_k : |z - z'| < \delta\}$. By the definition of $\widehat{f}_{kz}$, there is a neighborhood $U_\xi \subset b\mathbb{Z}$ of ξ such that $|\widehat{f}_{kz}(\eta) - \widehat{f}_{kz}(\xi)| < \epsilon/3$ for any $\eta \in U_\xi$. Consider $U_w := U_z \times U_\xi$. Then U_w is an open neighborhood of $w \in U_k \times b\mathbb{Z}$. Note that $f_{kz} - f_{kz'}$ is an almost periodic function on $\mathbb{Z}$ and for any $z' \in U_z$ its supremum norm is less than $\epsilon/3$. This implies that $|\widehat{f}_{kz}(\eta) - \widehat{f}_{kz'}(\eta)| < \epsilon/2$ for each $\eta \in b\mathbb{Z}$. In particular, for any $(x, \eta) \in U_w$ we have

$$|\widehat{f}_k(x, \eta) - \widehat{f}_k(z, \xi)| \le |\widehat{f}_{kx}(\eta) - \widehat{f}_{kz}(\eta)| + |\widehat{f}_{kz}(\eta) - \widehat{f}_{kz}(\xi)| < \epsilon.$$

This shows that $\widehat{f}_k$ is continuous at every $w \in U_k \times b\mathbb{Z}$.

Now, we show that $\widehat{f}|_{U_k\times\{\xi\}}$ is holomorphic at the interior points of U_k for every $\xi \in b\mathbb{Z}$.

Since $\widehat{f}_k$ is uniformly continuous on the compact set $U_k \times b\mathbb{Z}$, for any $\epsilon > 0$ there is $n_\epsilon \in \mathbb{Z}$ such that $\sup_{z\in U_k} |\widehat{f}_k(z, \xi) - f_k(z, n_\epsilon)| < \epsilon$. In particular, $\widehat{f}_k(\cdot, \xi)$ is the limit in $C(U_k)$ of the sequence $\{f_k(\cdot, n_{1/l})\}_{l\ge1}$ of bounded continuous functions holomorphic on the interior of U_k. Thus, $\widehat{f}_k|_{U_k\times\{\xi\}}$ is also holomorphic on the interior of U_k.

The identity $\sup_{U_k\times b\mathbb{Z}} |\widehat{f}_k| = \sup_{U_k\times\mathbb{Z}} |f_k|$ follows directly from the definition of $\widehat{f}_k$. This completes the proof of (1).

(2) Suppose that $f \in \mathcal{O}(E(R, b\mathbb{Z}))$. We must show that $f|_\Sigma \in AP_{\mathcal{O}}(\Sigma)$. For this, it suffices to show that the function $f|_L$ is almost periodic for every line $L := \{z \in \mathbb{C} : \operatorname{Im} z = t \in [0, \pi]\}$. (The uniform continuity of $f|_\Sigma$ with respect to the Euclidean metric on $\mathbb{C}$ follows easily from the uniform continuity of f on $E(R, b\mathbb{Z})$.) By definition, the image $S := e(L) \subset R$ is a circle, and $e|_L : L \to S$ is the universal covering. Consider the compact set $p^{-1}(S) \subset E(R, b\mathbb{Z})$. Since the function $f|_{p^{-1}(S)}$ is continuous and $b\mathbb{Z} \subset b\mathbb{R}$, for any $\epsilon > 0$ a finite open cover $(V_n)_{1\le n\le m}$ of S by sets homeomorphic to open intervals in $\mathbb{R}$ can be found, together with continuous almost periodic functions f_n on L, $1 \le n \le m$, such that

$$\sup_{\substack{z\in e^{-1}(V_n)\\ 1\le n\le m}} |f_n(z) - f(z)| < \epsilon. \tag{2.1}$$

Let $\{\rho_n\}_{1\le n\le m}$ be a continuous partition of unity subordinate to $(V_n)_{1\le n\le m}$. We pull it back to L by e and denote by $\widetilde{\rho}_n$ $(:= e^*\rho_n)$, $1\le n\le m$, the resulting functions. Since each $\widetilde{\rho}_n$ is periodic on L, it is almost periodic. We define a function f_ϵ on L by the formula

$$f_\epsilon(z) := \sum_{n=1}^{m}\widetilde{\rho}_n(z)f_n(z).$$

Then, clearly, f_ϵ is a continuous almost periodic function on L, and

$$\sup_{z\in L}|f_\epsilon(z)-f(z)|<\epsilon$$

by (2.1). This shows that f admits uniform approximation on L by continuous almost periodic functions, whence $f|_L$ is almost periodic.

The lemma is proved. □

2.3. In this part we prove the corona theorem for the algebra $AP_{\mathcal{O}}(\Sigma)$. Recall that $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ stands for its maximal ideal space. It is well known that every $f\in AP_{\mathcal{O}}(\Sigma)$ can be approximated uniformly on Σ by polynomials in $e^{i\lambda z}$, $\lambda\in\mathbb{R}$ (see, e.g., [JT]). Then, using the inverse limit construction for maximal ideal spaces of uniform algebras (see [R]), we see that a base of the topology of $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ is generated by the functions $e^{i\lambda z}$. Namely, the base of the topology on $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ consists of open sets of the form

$$U(\lambda_1,\dots,\lambda_l,\xi,\epsilon) := \{\eta\in\mathcal{M}(AP_{\mathcal{O}}(\Sigma))\ :\ \max_{1\le k\le l}|e_{\lambda_k}(\eta)-e_{\lambda_k}(\xi)|<\epsilon\},$$

where e_λ is the extension of $e^{i\lambda z}$ to $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ via the Gelfand transformation.

Theorem 2.4. *Σ is dense in $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ in the Gelfand topology.*

Proof. Assume that the corona theorem is not true for $AP_{\mathcal{O}}(\Sigma)$, that is, Σ is not dense in $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$. Then there exists $\xi\in\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ and its neighborhood $U(\lambda_1,\dots,\lambda_l,\xi,\epsilon)$ such that

$$U(\lambda_1,\dots,\lambda_l,\xi,\epsilon)\cap cl(\Sigma)=\varnothing;$$

here $cl(\Sigma)$ is the closure of Σ in $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$. Denoting $c_k := e_{\lambda_k}(\xi)$, $1\le k\le l$, we have

$$\max_{1\le k\le l}|e^{i\lambda_k z}-c_k|\ge\epsilon>0\ \text{ for all }\ z\in\Sigma. \tag{2.2}$$

Clearly, every function $e^{i\lambda_k z}-c_k$, $1\le k\le l$, has at least one zero in Σ. (Indeed, otherwise, if, say, $e^{i\lambda_k z}-c_k$ has no zeros on Σ, then the function $g_k(z):=1/(e^{i\lambda_k z}-c_k)$, $z\in\Sigma$, obviously belongs to $AP_{\mathcal{O}}(\Sigma)$ and $g_k(z)(e^{i\lambda_k z}-c_k)=1$ for all $z\in\Sigma$, which contradicts our assumption.) In particular, since the solutions of the equation

$$e^{i\lambda_k z}=c_k,\quad \lambda_k\ne 0,$$

are given by

$$z=-\frac{i\ln|c_k|}{\lambda_k}+\frac{\operatorname{Arg}c_k+2\pi s}{\lambda_k},\quad s\in\mathbb{Z},$$

they all belong to Σ. Next, without loss of generality we may assume that all λ_k are positive. Indeed, if some λ_k is strictly negative, we can replace the function $e^{i\lambda_k z}-c_k$ by $e^{-i\lambda_k z}-\frac{1}{c_k}$ (observe that $c_k\ne 0$ by the above argument); then the new family of functions also satisfies (2.2) (possibly with a different ϵ) and the extensions of these functions to $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ vanish at ξ. Since all these functions have zeros in Σ and satisfy (2.2) there, we have

$$\max_{1\le k\le l}|e^{i\lambda_k z}-c_k|\ge\widetilde{\epsilon}>0\ \text{ for all }\ z\in\mathbb{H}_+,$$

where $\mathbb{H}_+ \subset \mathbb{C}$ is the open upper half-plane, and all $e^{i\lambda_k z} - c_k$, $\lambda_k \in \mathbb{R}_+$, are almost periodic on $\overline{\mathbb{H}}_+$. The above inequality and the Böttcher corona theorem (see [Bö]) imply that there exist holomorphic almost periodic functions $g_1, \dots, g_l$ on $\overline{\mathbb{H}}_+$ such that

$$\sum_{k=1}^{l} g_k(z)(e^{i\lambda_k z} - c_k) = 1 \quad \text{for all} \quad z \in \overline{\mathbb{H}}_+.$$

Thus, taking the restrictions of these functions to Σ, we obtain a contradiction to our assumption.

This completes the proof of the corona theorem for $AP_{\mathcal{O}}(\Sigma)$. □

Corollary 2.5. *$E(R, b\mathbb{Z})$ is homeomorphic to $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$.*

Proof. By Lemma 2.3, there exists a continuous embedding i of $E(R, b\mathbb{Z})$ in $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$. Since Σ is dense in $E(R, b\mathbb{Z})$ and $i(\Sigma)$ is dense in $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ by the corona theorem, i is a homeomorphism. □

Remark 2.6. (1) In accordance with our construction, the closure of $\mathbb{R}$ in $E(R, b\mathbb{Z})$ coincides with $b\mathbb{R}$. In fact, this closure is $E(\partial\mathbb{D}, b\mathbb{Z})$, the principal bundle on $\partial\mathbb{D}$ with fiber $b\mathbb{Z}$ obtained as the restriction of $E(R, b\mathbb{Z})$ to $\partial\mathbb{D}$. Since R is homotopically equivalent to $\partial\mathbb{D}$, the covering homotopy theorem shows that $E(R, b\mathbb{Z})$ is homotopically equivalent to $b\mathbb{R} = E(\partial\mathbb{D}, b\mathbb{Z})$.

(2) Observe also that the covering dimension of $b\mathbb{R}$ is ∞, because $b\mathbb{R}$ is the inverse limit of real tori whose dimensions go to ∞. Therefore, the covering dimension of $\mathcal{M}(AP_{\mathcal{O}}(\Sigma)) = E(R, b\mathbb{Z})$ is also ∞.

(3) Finally, it is easy to show that the Šilov boundary of $AP_{\mathcal{O}}(\Sigma)$ is $E(R, b\mathbb{Z})|_{\partial R}$, the restriction of $E(R, b\mathbb{Z})$ to the boundary ∂R of R, and is homeomorphic to $b\mathbb{R} \sqcup b\mathbb{R}$.

§3. Proofs of Theorems 1.5, 1.7 and Corollary 1.6

3.1. Proof of Theorem 1.5. Let $f \in \mathrm{SAP}(S)$. Since $\partial\mathbb{D}$ is a compact set, for any $\epsilon > 0$ there are finitely many points $z_l := e^{it_l} \in \partial\mathbb{D}$, numbers $s_l \in (0, \pi)$, and almost periodic functions $f_k^l : \gamma_{t_l^k}(s_l) \to \mathbb{C}$, $k \in \{-1, 1\}$, $1 \le l \le n$, such that

$$\begin{gathered}\bigcup_{l=1}^{n} (\gamma_{t_l^{-1}}(s_l) \cup \gamma_{t_l^1}(s_l)) = \partial\mathbb{D} \setminus \{z_1, \dots, z_n\} \quad \text{and} \\ \operatorname*{ess\,sup}_{z \in \gamma_{t_l^1}(s_l)} |f(z) - f_1^l(z)| < \frac{\epsilon}{2}, \qquad \operatorname*{ess\,sup}_{z \in \gamma_{t_l^{-1}}(s_l)} |f(z) - f_{-1}^l(z)| < \frac{\epsilon}{2}\end{gathered} \tag{3.1}$$

for all $1 \le l \le n$. We set $U_l := \gamma_{t_l^1}(s_l) \cup \gamma_{t_l^{-1}}(s_l) \cup \{z_l\}$. Then $U = (U_l)_{l=1}^n$ is a finite open cover of $\partial\mathbb{D}$. Let $\{\rho_l\}_{l=1}^n$ be a continuous partition of unity subordinate to U and such that $\operatorname{supp} \rho_l \subset\subset U_l$ and $\rho_l(z_l) = 1$, $1 \le l \le n$. Consider the functions f_l on $\partial\mathbb{D} \setminus \{z_l\}$ defined by the formulas

$$f_l(z) := \begin{cases} \rho_l(z) f_{-1}^l(z) & \text{if } \ z \in \gamma_{t_l^{-1}}(s_l), \\ \rho_l(z) f_1^l(z) & \text{if } \ z \in \gamma_{t_l^1}(s_l). \end{cases}$$

Since f_l is continuous outside z_l and coincides with f_{-1}^l and f_1^l in a neighborhood of z_l, f_l belongs to $SAP(S)$. Moreover,

$$||f - \sum_{l=1}^{n} f_l||_{L^\infty(\partial\mathbb{D})} < \frac{\epsilon}{2}. \tag{3.2}$$

Thus, to prove the theorem it suffices to approximate every f_l by polynomials in g_{t_l} and $e^{i\lambda \log_{t_l^k}}$, $k \in \{-1, 1\}$, $\lambda \in \mathbb{R}$.

First, suppose that $z_l \notin S$. Since f is continuous outside the compact set S, we can choose the above cover U and the family of functions $\{f_k^m\}_{1\le m\le n}$, $k \in \{-1,1\}$, so that in U_l the two functions f_k^l, $k \in \{-1,1\}$, have the same limit at z_l. This implies the continuity of f_l on $\partial\mathbb{D}$. Next, consider the uniform algebra $\mathbb{C}(g_{t_l})$ over $\mathbb{C}$ generated by the function g_{t_l}. Since by our definition g_{t_l} separates points on $\partial\mathbb{D}\setminus\{z_l\}$, the maximal ideal space of $\mathbb{C}(g_{t_l})$ is homeomorphic to the closed interval $(\partial\mathbb{D}\setminus\{z_l\})\cup\{z_{l^k}\}$, $k\in\{-1,1\}$, with endpoints z_{l^1} and $z_{l^{-1}}$ identified with the counterclockwise and clockwise orientations at z_l. Clearly, every continuous function on $\partial\mathbb{D}$ extends to the maximal ideal space of $\mathbb{C}(g_{t_l})$ as a continuous function having the same values at z_{l^1} and $z_{l^{-1}}$. Thus, by the Stone–Weierstrass theorem, f_l can be uniformly approximated on $\partial\mathbb{D}\setminus\{z_l\}$ by complex polynomials in g_{t_l}.

Now, suppose that $z_l \in S$. Choose some $s \in (0, s_l)$. Let $\mathrm{SAP}_{\{z_l\}}(s)$ denote the uniform algebra of complex continuous functions on $\partial\mathbb{D}\setminus\{z_l\}$ almost periodic on the open arcs $\gamma_{t_l^k}(s)$, $k\in\{-1,1\}$. (Since $s\in(0,\pi)$, the closures of these arcs are disjoint.) We denote by $\mathcal{M}_{\{z_l\}}(s)$ the maximal ideal space of $\mathrm{SAP}_{\{z_l\}}(s)$. Then $\partial\mathbb{D}\setminus\{z_l\}$ is dense in $\mathcal{M}_{\{z_l\}}(s)$ (in the Gelfand topology). Note that the space $\mathcal{M}_{\{z_l\}}(s)$ is constructed as follows.

Consider the Bohr compactification $b\mathbb{R}$ of $\mathbb{R}$. We identify the negative ray $\mathbb{R}_-$ in $\mathbb{R}\subset b\mathbb{R}$ with $\gamma_{t_{l^1}}(s)\subset\partial\mathbb{D}$ via the map $t\mapsto e^{i(t_l+se^t)}$, $t\in\mathbb{R}_-$. Similarly, consider another copy of $b\mathbb{R}$ and identify $\mathbb{R}_-(\subset\mathbb{R})$ in this copy with $\gamma_{t_{l^{-1}}}(s)\subset\partial\mathbb{D}$ via the map $t\mapsto e^{i(t_l-se^t)}$, $t\in\mathbb{R}_-$. The identified sets are equipped with the topology induced from $b\mathbb{R}$, and on $\partial\mathbb{D}\setminus(\gamma_{t_{l^{-1}}}(s)\cup\gamma_{t_{l^1}}(s))$ the topology is induced from $\partial\mathbb{D}$. Then, under these identifications, the quotient space of $b\mathbb{R}\sqcup b\mathbb{R}\sqcup\partial\mathbb{D}$ coincides with $\mathcal{M}_{\{z_l\}}(s)$.

Next, we recall that, by definition, the algebra $\mathrm{SAP}(\{z_l\})$ is the uniform closure in $C(\partial\mathbb{D}\setminus\{z_l\})$ of the algebra generated by the algebras $\mathrm{SAP}_{\{z_l\}}(s)$, $s\in(0,s_l)$. Let $\mathcal{M}(\mathrm{SAP}(\{z_l\}))$ denote the maximal ideal space of $\mathrm{SAP}(\{z_l\})$. Since for any $s''<s'$ we have inclusions $i_{s''s'}:\mathrm{SAP}_{\{z_l\}}(s')\hookrightarrow\mathrm{SAP}_{\{z_l\}}(s'')$, the space $\mathcal{M}(\mathrm{SAP}(\{z_l\}))$ is the inverse limit of the spaces $\mathcal{M}_{\{z_l\}}(s)$ (here the maps $p_{s''s'}:\mathcal{M}_{\{z_l\}}(s'')\to\mathcal{M}_{\{z_l\}}(s')$ in the definition of this limit are defined as the maps dual to $i_{s''s'}$). Also, $\partial\mathbb{D}\setminus\{z_l\}$ is dense in $\mathcal{M}(\mathrm{SAP}(\{z_l\}))$ in the Gelfand topology. Since, by definition, the functions f_l, $e^{i\lambda\log_{t_{l^k}}}$ and g_{t_l} admit continuous extensions (denoted by the same symbols) to $\mathcal{M}(\mathrm{SAP}(\{z_l\}))$, it suffices to show that the extended functions $e^{i\lambda\log_{t_{l^k}}}$, g_{t_l} separate points on $\mathcal{M}(\mathrm{SAP}(\{z_l\}))$. Then we can apply the Stone–Weierstrass theorem to get a complex polynomial p_l in the variables $e^{i\lambda\log_{t_l^k}}$ and g_{t_l} that approximates f_l on $\mathcal{M}(\mathrm{SAP}(\{z_l\}))$ uniformly with an error less that $\epsilon/2n$. Therefore, $\sum_{l=1}^n p_l$ will approximate f in $L^\infty(\partial\mathbb{D})$ with an error less than ϵ.

So, we show that the functions $e^{i\lambda\log_{t_l^k}}$, g_{t_l} separate points on $\mathcal{M}(\mathrm{SAP}(\{z_l\}))$. Let $p_s:\mathcal{M}(\mathrm{SAP}(\{z_l\}))\to\mathcal{M}_{\{z_l\}}(s)$ denote the continuous surjection determined by the inverse limit construction. First, we consider distinct points $x,y\in\mathcal{M}(\mathrm{SAP}(\{z_l\}))$ for which there is $s\in(0,s_l)$ such that $p_s(x)$ and $p_s(y)$ are distinct and belong to one of the Bohr compactifications of $\mathbb{R}$ in $\mathcal{M}_{\{z_l\}}(s)$, say, to the compactification obtained by gluing $\mathbb{R}_-$ with $\gamma_{t_{l^1}}(s)$. Since in this case the functions $e^{i\lambda\log_{t_{l^1}}}$ extended to $b\mathbb{R}$ are identified with the extensions to $b\mathbb{R}$ of the functions $c_{\lambda s}e^{i\lambda t}$ on $\mathbb{R}$, $c_{\lambda s}:=e^{i\lambda\ln s}$, the classical Bohr theorem says that for some $\lambda_0\in\mathbb{R}$ the extension of $e^{i\lambda_0\log_{t_{l^1}}}$ to $b\mathbb{R}$ separates $p_s(x)$ and $p_s(y)$. Thus, the extension of $e^{i\lambda_0\log_{t_{l^1}}}$ to $\mathcal{M}(\mathrm{SAP}(\{z_l\}))$ separates x and y.

Suppose now that x and y are such that $p_s(x)$ and $p_s(y)$ belong to different Bohr compactifications of $\mathbb{R}$ for all s. This implies that x and y are limit points of the sets $\gamma_{t_l^{k(x)}}(s)$ and $\gamma_{t_l^{k(y)}}(s)$ for some $s\in(0,s_l)$, with $k(x)\neq k(y)$ and $k(x),k(y)\in\{-1,1\}$. Then the function g_{t_l} is equal to 1 at one of the points x, y and to 0 at the other point.

Finally, assume that $x \in \mathcal{M}(\mathrm{SAP}(\{z_l\})) \setminus \partial\mathbb{D}$ and $y \in \partial\mathbb{D} \setminus \{z_l\}$. Then $g_{t_l}(x)$ is equal either to 0 or to 1, and $g_{t_l}(y)$ differs from these numbers because g_{t_l} is monotone decreasing on $\partial\mathbb{D} \setminus \{z_l\}$.

Thus, we have proved that the family of functions $e^{i\lambda \log_{t_l^k}}$, g_{t_l} separates points on $\mathcal{M}(\mathrm{SAP}(\{z_l\}))$. This completes the proof of the theorem. □

3.2. Proof of Corollary 1.6. By Theorem 1.5, it suffices to check that for the functions g_{t_0} and $e^{i\lambda \log_{t_0^k}}$, $\lambda \in \mathbb{R}$, $z_0 := e^{it_0} \in \widetilde{S} := \phi(S)$, $k \in \{-1, 1\}$, the corresponding functions $\phi^*(g_{t_0})$ and $f := \phi^*(e^{i\lambda \log_{t_0^k}})$ belong to $\mathrm{SAP}(S)$. Since $g_{t_0} \in R_{\{z_0\}}$, the statement is trivial for $\phi^*(g_{t_0})$. Without loss of generality we may assume that ϕ preserves the orientation on $\partial\mathbb{D}$. Let $e^{i\tilde{t}_0} := \phi^{-1}(z_0)$. Then we have

$$\phi(e^{i(\tilde{t}_0+t)}) = e^{i(t_0+\tilde{\phi}(t))}, \quad t \in [0, 2\pi],$$

where $\tilde{\phi} : [0, 2\pi] \to [0, 2\pi]$ is a C^1-diffeomorphism and $\tilde{\phi}(0) = 0$. We set $c := \tilde{\phi}'(0)$. By the definition on the open arcs $\gamma_{\tilde{t}_0^k}(s)$, for $t \in (-\infty, 0)$ we obtain

$$\widehat{f}(t) := f(e^{i(\tilde{t}_0+kse^t)}) = e^{i\lambda \ln(\tilde{\phi}(se^t))} = e^{i\lambda \ln(cse^t+o(se^t))} = e^{i(\lambda \ln(cs)+o(1))}e^{i\lambda t} \quad \text{as} \quad s \to 0.$$

Since f is continuous outside $e^{i\tilde{t}_0}$, this implies that $f \in \mathrm{SAP}(S)$. □

3.3. Proof of Theorem 1.7. We begin with the following statement.

Lemma 3.1. *The algebra R_S is the uniform closure of the algebra generated by the algebras $R_{\{z\}}$ for all possible $z \in S$.*

(For $S = \partial\mathbb{D}$, a similar statement was proved for the first time by Dieudonne [D].)

Proof. Consider a regulated function $f \in R_S$. Since $\partial\mathbb{D}$ is a compact set, the definition of R_S implies that for any $\epsilon > 0$ there are finitely many points $z_l := e^{it_l} \in \partial\mathbb{D}$, numbers $s_l \in (0, \pi)$, and constant functions $f_k^l : \gamma_{t_l^k}(s_l) \to \mathbb{C}$, $k \in \{-1, 1\}$, $1 \le l \le n$, such that

$$\bigcup_{l=1}^n (\gamma_{t_l^{-1}}(s_l) \cup \gamma_{t_l^1}(s_l)) = \partial\mathbb{D} \setminus \{z_1, \dots, z_n\} \quad \text{and}$$

$$\operatorname*{ess\,sup}_{z \in \gamma_{t_l^1}(s_l)} |f(z) - f_1^l(z)| < \epsilon, \qquad \operatorname*{ess\,sup}_{z \in \gamma_{t_l^{-1}}(s_l)} |f(z) - f_{-1}^l(z)| < \epsilon$$

for all $1 \le l \le n$. We set $U_l := \gamma_{t_l^1}(s_l) \cup \gamma_{t_l^{-1}}(s_l) \cup \{z_l\}$. Then $U = (U_l)_{l=1}^n$ is a finite open cover of $\partial\mathbb{D}$. Let $\{\rho_l\}_{l=1}^n$ be a continuous partition of unity subordinate to U and such that $\operatorname{supp} \rho_l \subset\subset U_l$ and $\rho_l(z_l) = 1$, $1 \le l \le n$. Consider the functions f_l on $\partial\mathbb{D} \setminus \{z_l\}$ defined by the formulas

$$f_l(z) := \begin{cases} \rho_l(z) f_{-1}^l(z) & \text{if } z \in \gamma_{t_l^{-1}}(s_l), \\ \rho_l(z) f_1^l(z) & \text{if } z \in \gamma_{t_l^1}(s_l). \end{cases}$$

If $z_l \in S$, then $f_l \in R_{\{z_l\}}$ by definition. If $z_l \notin S$, then, since f is continuous outside the compact set S, we can choose the above cover U and the family of functions $\{f_k^m\}_{1 \le m \le n}$, $k \in \{-1, 1\}$, so that in U_l the two functions f_k^l, $k \in \{-1, 1\}$, have the same limit at z_l. This implies the continuity of f_l on $\partial\mathbb{D}$, i.e., $f_l \in R_{\{z_l\}}$. Also, we have

$$||f - \sum_{l=1}^n f_l||_{L^\infty(\partial\mathbb{D})} < \epsilon.$$

This completes the proof of the lemma. □

Let $F_1 \subset F_2$ be finite subsets of S. Then we have the natural injection $i_{F_1F_2} : R_{F_1} \hookrightarrow R_{F_2}$. Passing to the map dual to $i_{F_1F_2}$, we obtain a continuous map $p_{F_1F_2} : \mathcal{M}(R_{F_2}) \to \mathcal{M}(R_{F_1})$ of the corresponding maximal ideal spaces. Now the family $\{(\mathcal{M}(R_{F_1}), \mathcal{M}(R_{F_2}), p_{F_1F_2})\}_{F_1 \subset F_2 \subset S}$ determines an inverse limit system whose limit coincides with $\mathcal{M}(R_S)$ by Lemma 3.1. By $p_F : \mathcal{M}(R_S) \to \mathcal{M}(R_F)$, where $F \subset S$ is finite, we denote the limit maps determined by this limit. Then

$$c_F \circ p_F = c_S. \tag{3.3}$$

Suppose F consists of n points. Then $\partial\mathbb{D} \setminus F$ is a disjoint union of open arcs γ_k, $1 \le k \le n$. Consider a real function g_F on $\partial\mathbb{D}$ that has discontinuities of the first kind at points of F, is continuous outside F, and is such that $g_F : \partial\mathbb{D} \setminus F \to \mathbb{R}$ is an injection and $g_F(\partial\mathbb{D} \setminus F)$ is the union of open intervals whose closures are mutually disjoint. Then an argument similar to that in the proof of Theorem 1.5 (see the case of $z_l \notin S$ there) shows that $R_F = \mathbb{C}(g_F)$, the uniform algebra in $L^\infty(\partial\mathbb{D})$ generated by g_F. This implies that $\mathcal{M}(R_F)$ is naturally homeomorphic to the disjoint union of the closures $\overline{\gamma}_k$ of γ_k, $1 \le k \le n$, and $c_F : \mathcal{M}(R_F) \to \partial\mathbb{D}$ maps every γ_k in this union identically to $\gamma_k \subset \partial\mathbb{D}$. Since c_F is continuous, for every $z \in F$ the preimage $c_F^{-1}(z)$ consists of two points z_+ and z_- that can be identified naturally with the counterclockwise and clockwise orientations of $\partial\mathbb{D}$ at z. Thus, we obtain proofs of statements (1)–(3) of the theorem for a finite subset $F \subset S$. To prove the general case, we use (3.3).

Assume that for some $z \in \partial\mathbb{D}$ the preimage $c_S^{-1}(z)$ contains at least three points x_1, x_2, and x_3. Then, by the definition of the inverse limit, there is a finite subset $F \subset S$ such that $p_F(x_1)$, $p_F(x_2)$, and $p_F(x_3)$ are distinct points in $\mathcal{M}(R_F)$. Since by (3.3) the images of these points under c_F coincide with z, the case settled above for $\mathcal{M}(R_F)$ shows that $c_F^{-1}(z)$ consists of at most two points, a contradiction. Thus, $c_S^{-1}(z)$ consists of at most 2 points for every $z \in \partial\mathbb{D}$.

Now, assume that $z \in S$. Let $F \subset S$ be a finite subset containing z. Then the preimage $c_F^{-1}(z)$ consists of two points. Since $c_S^{-1}(z) = p_F^{-1}(c_F^{-1}(z))$ by (3.3), the preimage $c_S^{-1}(z)$ also consists of two points. (As before, they can be identified naturally with the counterclockwise and clockwise orientations of $\partial\mathbb{D}$ at z.)

This proves statement (1) of the theorem.

Next, $c_S^{-1}(z)$ is a single point for $z \notin S$. (Otherwise, for some finite F the set $c_F^{-1}(z)$ consists of at least 2 points, a contradiction.) Since $c_S^{-1}(S)$ is compact, the latter implies that $c_S : \mathcal{M}(R_S) \setminus c_S^{-1}(S) \to \partial\mathbb{D} \setminus S$ is a homeomorphism.

The proof of statement (2) is complete.

To prove (3), we assume that S is infinite (for finite S the statement has already been proved). Let $F \subset S$ be a finite subset consisting of n points, $n \ge 2$. Let $\partial\mathbb{D} \setminus F$ be the disjoint union of open arcs γ_k, $1 \le k \le n$. Let χ_{γ_k} denote the characteristic function of γ_k. Then every χ_{γ_k} belongs to R_F. The same symbols will denote the continuous extensions of χ_{γ_k} to $\mathcal{M}(R_F)$ via the Gelfand transformation. We define a continuous map $K_F : \mathcal{M}(R_F) \to \mathbb{Z}_2(F) := \{0,1\}^n$ by the formula

$$K_F(m) := (\chi_{\gamma_1}(m), \dots, \chi_{\gamma_n}(m)), \quad m \in \mathcal{M}(R_F).$$

Setting $\mathcal{Z}(S) := \prod_{F \subset S} \mathbb{Z}_2(F)$, we introduce a map $\mathcal{K}_S : \mathcal{M}(R_S) \to \mathcal{Z}(S)$ by the formula

$$\mathcal{K}_S(m) := \{K_F(p_F(m))\}_{F \subset S}.$$

We equip $\mathcal{Z}(S)$ with the *Tychonoff topology.* Then $\mathcal{Z}(S)$ is a totally disconnected compact Hausdorff space, and the map $\mathcal{K}_S$ is continuous.

We show that $\mathcal{K}_S|_{c_S^{-1}(S)} : c_S^{-1}(S) \to \mathcal{Z}(S)$ is an injection. Indeed, for distinct points x, y in $c_S^{-1}(S)$ there exists a finite subset $F \subset S$ consisting of at least two points and such that $p_F(x) \neq p_F(y)$ and $p_F(x), p_F(y) \in c_F^{-1}(F)$. Then, by the definition of the map K_F, we have

$$K_F(p_F(x)) \neq K_F(p_F(y)).$$

This implies that $\mathcal{K}_S(x) \neq \mathcal{K}_S(y)$. Since $c_S^{-1}(S)$ is a compact set, injectivity implies that $c_S^{-1}(S)$ is homeomorphic to $\mathcal{K}_S(c_S^{-1}(S))$. The latter space is totally disconnected, being a compact subset of the totally disconnected space $\mathcal{Z}(S)$.

This completes the proof of (3).

(4) In accordance with (3.2), the maximal ideal space $\mathcal{M}(\mathrm{SAP}(S))$ of the algebra $\mathrm{SAP}(S)$ is homeomorphic to the inverse limit of the compact spaces $\mathcal{M}(\mathrm{SAP}(F))$ with $F \subset S$ finite. Let $\widetilde{p}_{F_1F_2} : \mathcal{M}(\mathrm{SAP}(F_2)) \to \mathcal{M}(\mathrm{SAP}(F_1))$, $F_1 \subset F_2$, be continuous maps determining this limit, and let $\widetilde{p}_F : \mathcal{M}(\mathrm{SAP}(S)) \to \mathcal{M}(\mathrm{SAP}(F))$ be the corresponding limit maps. Since each $\mathrm{SAP}(F)$ is a selfadjoint algebra, $\partial\mathbb{D} \setminus F$ is dense in $\mathcal{M}(\mathrm{SAP}(F))$ by the Stone–Weierstrass theorem. Hence, $\widetilde{p}_{F_1F_2}$ and $\widetilde{p}_F$ are surjective maps.

We begin with the description of $\mathcal{M}(\mathrm{SAP}(F))$. Suppose that $F := \{z_1, \dots, z_n\}$ and $F_i := F \setminus \{z_i\}$, $1 \le i \le n$. Consider the disjoint union

$$X = \bigsqcup_{1 \le i \le n} (\mathcal{M}(\mathrm{SAP}(\{z_i\})) \setminus F_i).$$

Note that each component of X contains $\partial\mathbb{D} \setminus F$ as an open subset. By $h_i : \partial\mathbb{D} \setminus F \hookrightarrow \mathcal{M}(\mathrm{SAP}(\{z_i\})) \setminus F_i$ we denote the corresponding embeddings. Then for each $z \in \partial\mathbb{D} \setminus F$ we sew together the points $h_i(z)$, $1 \le i \le n$, and identify the resulting point with z. As a result, we obtain a quotient space $\widetilde{X}$ of X and a "sewing" map $\pi : X \to \widetilde{X}$. We equip $\widetilde{X}$ with the quotient topology:

$$U \subset \widetilde{X} \ \text{ is open} \quad \Longleftrightarrow \quad \pi^{-1}(U) \subset X \ \text{ is open.}$$

Lemma 3.2. *$\widetilde{X}$ is homeomorphic to $\mathcal{M}(\mathrm{SAP}(F))$.*

Proof. By definition, each $V_i := \pi(\mathcal{M}(\mathrm{SAP}(\{z_i\})) \setminus F_i)$ is an open subset of $\widetilde{X}$ homeomorphic to $\mathcal{M}(\mathrm{SAP}(\{z_i\})) \setminus F_i$. Since the latter spaces are Hausdorff, $\widetilde{X}$ is also Hausdorff. We cover $\partial\mathbb{D}$ by closed arcs $\gamma_1, \dots, \gamma_n$ such that $\gamma_i \cap F = \{z_i\}$, $1 \le i \le n$, and denote by $\widetilde{\gamma}_i$ the closure of γ_i in $\mathcal{M}(\mathrm{SAP}(\{z_i\}))$. Then $\widetilde{\gamma}_i$ is a compact subset of $\mathcal{M}(\mathrm{SAP}(\{z_i\})) \setminus F_i$, and $U_i := \pi(\widetilde{\gamma}_i)$ is a compact subset of V_i. It is easily seen that $\widetilde{X} = \cup_{1 \le i \le n} U_i$. Thus, $\widetilde{X}$ is a compact space. Next, by (3.2), each function in $\mathrm{SAP}(F)$ extends continuously to $\widetilde{X}$, and the algebra of these extensions separates points on $\widetilde{X}$. Hence, $\widetilde{X}$ is homeomorphic to $\mathcal{M}(\mathrm{SAP}(F))$ by the Stone–Weierstrass theorem. □

As a consequence of the above construction, we immediately obtain the following statement.

Let $F_1 \subset F_2$ be finite subsets of S. Consider the commutative diagram

$$\begin{array}{ccccc}
\mathcal{M}(\mathrm{SAP}(S)) & \xrightarrow{r_S} & \mathcal{M}(R_S) & \xrightarrow{c_S} & \partial\mathbb{D} \\
{\scriptstyle \widetilde{p}_{F_2}}\downarrow & & {\scriptstyle p_{F_2}}\downarrow & & || \\
\mathcal{M}(\mathrm{SAP}(F_2)) & \xrightarrow{r_{F_2}} & \mathcal{M}(R_{F_2}) & \xrightarrow{c_{F_2}} & \partial\mathbb{D} \\
{\scriptstyle \widetilde{p}_{F_1F_2}}\downarrow & & {\scriptstyle p_{F_1F_2}}\downarrow & & || \\
\mathcal{M}(\mathrm{SAP}(F_1)) & \xrightarrow{r_{F_1}} & \mathcal{M}(R_{F_1}) & \xrightarrow{c_{F_1}} & \partial\mathbb{D}.
\end{array} \tag{3.4}$$

Here $\widetilde{p}_{F_1} := \widetilde{p}_{F_1F_2} \circ \widetilde{p}_{F_2}$ and $p_{F_1} := p_{F_1F_2} \circ p_{F_2}$. We set $\widetilde{F}_1 := (c_{F_1} \circ r_{F_1})^{-1}(F_1)$ and $\widetilde{S}_i := (c_{F_i} \circ r_{F_i})^{-1}(S)$, $i = 1, 2$. Then

(A)

$$\widetilde{p}_{F_1F_2} : \widetilde{p}_{F_1F_2}^{-1}(\widetilde{F}_1) \to \widetilde{F}_1$$

is a homeomorphism; and

(B)

$$\mathcal{M}(\mathrm{SAP}(F_2)) \setminus \widetilde{S}_2 \overset{\widetilde{p}_{F_1F_2}}{\longrightarrow} \mathcal{M}(\mathrm{SAP}(F_1)) \setminus \widetilde{S}_1 \overset{c_{F_1}\circ r_{F_1}}{\longrightarrow} \partial\mathbb{D} \setminus S$$

are the identity maps.

Recalling the definition of the inverse limit, we see that

(A1)

$$\widetilde{p}_{F_1} : \widetilde{p}_{F_1}^{-1}(\widetilde{F}_1) \to \widetilde{F}_1$$

is a homeomorphism; and

(B1)

$$\mathcal{M}(\mathrm{SAP}(S)) \setminus (c_S \circ r_S)^{-1}(S) \overset{\widetilde{p}_{F_1}}{\longrightarrow} \mathcal{M}(\mathrm{SAP}(F_1)) \setminus \widetilde{S}_1 \overset{c_{F_1}\circ r_{F_1}}{\longrightarrow} \partial\mathbb{D} \setminus S$$

are the identity maps.

Now, assume that $F_1 = \{z\} \subset S$. Then, by Theorem 1.5 (1), the set $c_{F_1}^{-1}(F_1) = c_S^{-1}(F_1)$ consists of two points $\{z_+\}$ and $\{z_-\}$ identified with the counterclockwise and clockwise orientations at z. Thus, to prove (4) we must show (by (3.4) and statements (A), (A1)) that each set $r_{\{z\}}^{-1}(z_\pm)$ is homeomorphic to $b\mathbb{R}$.

For this, we recall that in the proof of Theorem 1.5 we established that $\mathcal{M}(\mathrm{SAP}(\{z\}))$ is the inverse limit of the maximal ideal spaces $\mathcal{M}_{\{z\}}(s)$ of the algebras $\mathrm{SAP}_{\{z\}}(s)$ of continuous functions on $\partial\mathbb{D} \setminus \{z\}$ almost periodic on the open arcs $\gamma_{t^k}(s)$, where $z := e^{it}$ and $s \in (0, \pi)$. Also in that proof, the structure of each $\mathcal{M}_{\{z\}}(s)$ was described.

For every pair $0 < s'' < s' < \pi$, let $p_{s''s'} : \mathcal{M}_{\{z\}}(s'') \to \mathcal{M}_{\{z\}}(s')$ be the continuous surjective map dual to the embedding $i_{s''s'} : \mathrm{SAP}_{\{z\}}(s') \hookrightarrow \mathrm{SAP}_{\{z\}}(s'')$. From the proof of Theorem 1.5 we know that every $\mathcal{M}_{\{z\}}(s)$ is obtained by gluing $\partial\mathbb{D} \setminus \{z\}$ with two copies of $b\mathbb{R}$, where one copy (denoted by $b\mathbb{R}_1$) is obtained by gluing with $\gamma_{t^1}(s)$ and another copy (denoted by $b\mathbb{R}_{-1}$) is obtained by gluing with $\gamma_{t^{-1}}(s)$. Suppose that $\xi \in b\mathbb{R}_1 \subset \mathcal{M}_{\{z\}}(s'')$. We compute $p_{s''s'}(\xi) \in b\mathbb{R}_1 \subset \mathcal{M}_{\{z\}}(s')$. Let $\{z_\alpha\} \subset \gamma_{t^1}(s'')$ be a net converging to ξ. This means that the net $\{\phi_{s''}(z_\alpha)\} \subset \mathbb{R}_-$ converges to ξ in the topology of the Bohr compactification on $b\mathbb{R}$; here $\phi_{s''}$ is the map inverse to the map $\psi_{s''} : \mathbb{R}_- \to \gamma_{t^1}(s'')$, $x \mapsto e^{i(t+s''e^x)}$. Next, by definition, the net $\{\phi_{s'}(z_\alpha)\}$ converges to $p_{s''s'}(\xi)$. A straightforward computation shows that

$$\phi_{s'}(z_\alpha) = \phi_{s''}(z_\alpha) + \ln\left(\frac{s'}{s''}\right) \quad \text{for all} \quad z_\alpha.$$

Thus, we have

$$p_{s''s'}(\xi) = \xi + \ln\left(\frac{s'}{s''}\right), \quad \xi \in b\mathbb{R}_1. \tag{3.5}$$

Here the sum refers to the group operation on $b\mathbb{R}$. Similarly,

$$p_{s''s'}(\xi) = \xi + \ln\left(\frac{s'}{s''}\right), \quad \xi \in b\mathbb{R}_{-1}. \tag{3.6}$$

Using these formulas, now we prove that each $r_{\{z\}}^{-1}(z_\pm)$ is homeomorphic to $b\mathbb{R}$. We shall prove the statement for z_+ (for z_- the argument is similar).

For a fixed $s_0 \in (0, \pi)$, consider the limit map $p_{s_0} : \mathcal{M}(\mathrm{SAP}(\{z\})) \to \mathcal{M}_{\{z\}}(s_0)$. Then p_{s_0} maps $r_{\{z\}}^{-1}(z_+)$ into $X_{s_0} := b\mathbb{R}_1 \subset \mathcal{M}_{\{z\}}(s_0)$. Moreover, by definition, $r_{\{z\}}^{-1}(z_+)$ is the inverse limit of the system $\{(X_{s''}, X_{s'}, p_{s''s'})\}$, where we write X_s for $b\mathbb{R}_1 \subset \mathcal{M}_{\{z\}}(s)$. Since, in accordance with (3.5), every $p_{s''s'} : X_{s''} \to X_{s'}$ is a homeomorphism (even an automorphism of $b\mathbb{R}$), the definition of the inverse limit shows that $p_{s_0} : r_{\{z\}}^{-1}(z_+) \to X_{s_0}$ is a homeomorphism.

This completes the proof of statement (4).

(5) This follows from (B1) and Theorem 1.7 (2).

The proof of Theorem 1.7 is complete. □

Remark 3.3. It is well known that the covering dimension of $b\mathbb{R}$ is ∞, because this group is the inverse limit of compact Abelian Lie groups whose dimensions tend to ∞. Since $b\mathbb{R} \subset \mathcal{M}(\mathrm{SAP}(S))$, the covering dimension of $\mathcal{M}(\mathrm{SAP}(S))$ is also infinite (statement (6)). Also, statement (7) follows from the fact that $c_S^{-1}(S)$ is totally disconnected. Indeed, the image of $r_S \circ \phi$ is a single point for a continuous map $\phi : T \to (c_S \circ r_S)^{-1}(S)$ of a connected topological space T. This implies the claim.

§4. Proofs of Theorem 1.8 and Example 1.10

4.1. In this section we formulate and prove some auxiliary results used in the proof of the theorem.

Notation. Let $z_0 \in \partial\mathbb{D}$, and let U_{z_0} be the intersection of an open disk of radius not exceeding 1 centered at z_0 with $\overline{\mathbb{D}} \setminus \{z_0\}$. We call such U_{z_0} a *circular neighborhood* of z_0.

Next, we define almost periodic functions continuous on a circular neighborhood U_{z_0} of z_0 and holomorphic in the interior of U_{z_0} as follows.

Let $\phi_{z_0} : \mathbb{D} \to \mathbb{H}_+$,

$$\phi_{z_0}(z) := \frac{2i(z_0 - z)}{z_0 + z}, \quad z \in \mathbb{D}, \tag{4.1}$$

be a conformal map of $\mathbb{D}$ onto the upper half-plane $\mathbb{H}_+$. Then ϕ_{z_0} is continuous on $\partial\mathbb{D} \setminus \{-z_0\}$ and maps this set diffeomorphically onto $\mathbb{R}$ (the boundary of $\mathbb{H}_+$) so that $\phi_{z_0}(z_0) = 0$. Let Σ_0 be the interior of the strip $\Sigma := \{z \in \mathbb{C} \ : \ \mathrm{Im}\ z \in [0, \pi]\}$. Consider the conformal map $\mathrm{Log} : \mathbb{H}_+ \to \Sigma_0$, $z \mapsto \mathrm{Log}(z) := \ln|z| + i\mathrm{Arg}(z)$, where $\mathrm{Arg} : \mathbb{C} \setminus \mathbb{R}_- \to (-\pi, \pi)$ is the principal branch of the multifunction arg, the argument of a complex number. The function Log extends to a homeomorphism of $\overline{\mathbb{H}}_+ \setminus \{0\}$ onto Σ; here $\overline{\mathbb{H}}_+$ stands for the closure of $\mathbb{H}_+$.

We denote by $AP_{\mathcal{C}}(\Sigma)$ the algebra of uniformly continuous almost periodic functions on Σ (i.e., they are almost periodic on any line parallel to the x-axis). Clearly, we have $AP_{\mathcal{O}}(\Sigma) \subset AP_{\mathcal{C}}(\Sigma)$. Then, by Theorem 2.4 (the corona theorem for $AP_{\mathcal{O}}(\Sigma)$), the maximal ideal space $\mathcal{M}(AP_{\mathcal{C}}(\Sigma))$ of the algebra $AP_{\mathcal{C}}(\Sigma)$ is homeomorphic to $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$. In what follows we identify these spaces.

Definition 4.1. We say that $f : U_{z_0} \to \mathbb{C}$ is a (*continuous*) *almost periodic function* if there is a function $\widehat{f} \in AP_{\mathcal{C}}(\Sigma)$ such that

$$f(z) := \widehat{f}(\mathrm{Log}(\phi_{z_0}(z))) \ \text{ for all } \ z \in U_{z_0}.$$

If $\widehat{f} \in AP_{\mathcal{O}}(\Sigma)$, then f is called a *holomorphic almost periodic function.*

Suppose that $z_0 = e^{it_0}$. For $s \in (0, \pi)$ we set $\gamma_1(z_0, s) := \mathrm{Log}(\phi_{z_0}(\gamma_{t_0^1}(s))) \subset \mathbb{R}$ and $\gamma_{-1}(z_0, s) := \mathrm{Log}(\phi_{z_0}(\gamma_{t_0^{-1}}(s))) \subset \mathbb{R} + i\pi$.

Lemma 4.2. *Let $f \in \mathrm{SAP}(\{-z_0, z_0\})$. We set $f_k := f|_{\gamma_{t_0^k}(\pi)}$ and consider the functions $h_k = f_k \circ \varphi_{z_0}^{-1} \circ \mathrm{Log}^{-1}$ on $\gamma_k(z_0, s)$, $k \in \{-1, 1\}$. Then for any $\epsilon > 0$ there are points $s_\epsilon \in (0, s)$ and almost periodic functions h_1' on $\mathbb{R}$ and h_{-1}' on $\mathbb{R} + i\pi$ such that*

$$\sup_{z \in \gamma_k(z_0, s_\epsilon)} |h_k(z) - h_k'(z)| < \epsilon \ \text{ for each } \ k \in \{-1, 1\}. \tag{4.2}$$

Proof. We prove the result for f_1 only. The proof for f_{-1} is similar. By Theorem 1.5, it suffices to prove the lemma for $f_1 = g_{t_0}$ or $f_1 = e^{i\lambda \log_{t_0^1}}$, $\lambda \in \mathbb{R}$. In the first case we can choose a sufficiently small s_ϵ such that on $\gamma_{t_0^1}(s_\epsilon)$ the function g_{t_0} is uniformly approximated with an error less than ϵ by a constant function. Then for the role of h_1' we can choose the corresponding constant function on $\gamma_1(z_0, s_\epsilon)$. In the second case, by definition,

$$h_1(x) = e^{i\lambda \ln(\mathrm{Arg}(\frac{2i-e^x}{2i+e^x}))} e^{i\lambda \ln(\frac{4e^x}{4-e^{2x}}+o(e^{3x}))} = e^{i\lambda(x+o(e^x))} \quad \text{as} \quad x \to -\infty.$$

This implies that

$$|h_1(x) - e^{i\lambda x}| < \epsilon \quad \text{for all} \quad x \in \gamma_1(z_0, s_\epsilon)$$

if s_ϵ is sufficiently small. □

We also use the following well-known result.

Lemma 4.3. *Suppose that f_1 and f_2 are continuous almost periodic functions on $\mathbb{R}$ and $\mathbb{R} + i\pi$, respectively. Then there exists a function $F \in AP_{\mathcal{C}}(\Sigma)$ harmonic in Σ_0 whose boundary values are f_1 and f_2.*

Proof. Let F be a function harmonic in Σ_0 with boundary values f_1 and f_2. Since f_1 and f_2 are almost periodic, for any $\epsilon > 0$ there exists $l(\epsilon) > 0$ such that every interval $[t_0, t_0 + l(\epsilon)]$ contains a common ϵ-period of f_1 and f_2, say, τ_ϵ (see, e.g., [LZ]). Thus,

$$\sup_{x\in\mathbb{R}} |f_1(x+\tau_\epsilon) - f_1(x)| < \epsilon \quad \text{and} \quad \sup_{x\in\mathbb{R}} |f_2(x+i\pi+\tau_\epsilon) - f_2(x+i\pi)| < \epsilon.$$

Now, by the maximum principle for harmonic functions,

$$\sup_{x\in\mathbb{R}} |F(x+iy+\tau_\epsilon) - F(x+iy)| < \epsilon \quad \text{for each} \quad y \in [0,\pi],$$

that is, F is almost periodic on every line $\mathbb{R} + iy$, $y \in [0, \pi]$. □

Let $\mathcal{A}(U_{z_0})$ be the algebra of functions continuous on $\mathbb{D}$ and almost periodic on the circular neighborhood U_{z_0} of z_0. Let $\mathcal{A}_{z_0}$ denote the uniform closure of the algebra generated by all $\mathcal{A}(U_{z_0})$ and by the closure $\mathcal{M}_{z_0}$ of $\mathbb{D}$ in the maximal ideal space of $\mathcal{A}_{z_0}$. Since the algebra $\mathcal{A}_{z_0}$ is selfadjoint, the Stone-Weierstrass theorem shows that $\mathcal{M}_{z_0}$ coincides with the maximal ideal space of $\mathcal{A}_{z_0}$. Next, let $p_{z_0} : \mathcal{M}_{z_0} \to \overline{\mathbb{D}}$ be the continuous surjective map dual to the natural embedding $C(\overline{\mathbb{D}}) \hookrightarrow \mathcal{A}_{z_0}$.

Lemma 4.4. (a) *For every neighborhood U of the compact set $F_{z_0} := p_{z_0}^{-1}(z_0)$, there is a circular neighborhood U_{z_0} of z_0 such that $U_{z_0} \cap \mathbb{D} \subset U \cap \mathbb{D}$.*
(b) *F_{z_0} is homeomorphic to $\mathcal{M}(\mathcal{AP}_{\mathcal{O}}(\Sigma))$.*
(c) *Each function $f \in \mathrm{SAP}(S) \cap H^\infty$ belongs to the algebra $\bigcap_{z\in\partial\mathbb{D}} \mathcal{A}_z$.*

Proof. (a), (b). Since the algebra $\mathcal{A}(U_{z_0})$ is selfadjoint, $\mathbb{D}$ is dense in the maximal ideal space $\mathcal{M}(U_{z_0})$. Then $\mathcal{M}_{z_0}$ is the inverse limit of the compact spaces $\mathcal{M}(U_{z_0})$ (because $\mathcal{A}_{z_0}$ is the uniform closure of the algebra generated by the algebras $\mathcal{A}(U_{z_0})$); see, e.g., [R]. For $U_{z_0} \subset V_{z_0}$, we denote by $p_{U_{z_0}V_{z_0}} : \mathcal{M}(U_{z_0}) \to \mathcal{M}(V_{z_0})$ the maps in this limit system and by $p_{U_{z_0}} : \mathcal{M}_{z_0} \to \mathcal{M}(U_{z_0})$ the corresponding (continuous and surjective) limit maps. Then, by the definition of the inverse limit, the base of topology on $\mathcal{M}_{z_0}$ consists of the sets $p_{U_{z_0}}^{-1}(U)$, where $U \subset \mathcal{M}(U_{z_0})$ is open and U_{z_0} is a circular neighborhood of z_0. In particular, since F_{z_0} is a compact set, for a neighborhood U of F_{z_0} there is a circular neighborhood $\widetilde{U}_{z_0}$ of z_0 and a neighborhood $\widetilde{U} \subset \mathcal{M}(\widetilde{U}_{z_0})$ of $F(\widetilde{U}_{z_0}) := p_{\widetilde{U}_{z_0}}(F_{z_0})$ such that $p_{\widetilde{U}_{z_0}}^{-1}(\widetilde{U}) \subset U$. Recall that $\mathbb{D}$ is a dense subset of $\mathcal{M}(\widetilde{U}_{z_0})$ and of $\mathcal{M}_{z_0}$. Also, $p_{\widetilde{U}_{z_0}}^{-1}(\mathbb{D})$ contains $\mathbb{D} \subset \mathcal{M}_{z_0}$. Thus, to prove (a) it suffices to show that there is $U_{z_0} \subset \widetilde{U}_{z_0}$ such that $U_{z_0} \cap \mathbb{D} \subset \widetilde{U} \cap \mathbb{D}$.

First, we study the structure of $\mathcal{M}(\widetilde{U}_{z_0})$. Let $\mathcal{A}^*(\widetilde{U}_{z_0})$ be the pullback to Σ via the map $(\mathrm{Log} \circ \phi_{z_0})^{-1}$ of the algebra $\mathcal{A}(\widetilde{U}_{z_0})$. Then $\mathcal{A}^*(\widetilde{U}_{z_0})$ consists of continuous functions on Σ_0 such that on $(\mathrm{Log}\circ\phi_{z_0})(\widetilde{U}_{z_0})$ they are restrictions of almost periodic functions on Σ. Since $\mathcal{A}^*(\widetilde{U}_{z_0})$ is isomorphic to $\mathcal{A}(\widetilde{U}_{z_0})$, we can naturally identify the maximal ideal spaces of these algebras.

Next, observe that there is $T<0$ such that $(\mathrm{Log}\circ\phi_{z_0})(\widetilde{U}_{z_0})$ contains the subset $\Sigma_T := \{z\in\Sigma \ :\ \mathrm{Re}\ z\le T\}$ of the strip Σ. By the definition of the topology on $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ (see §2), Σ_T is dense in $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$. Hence, the space $\mathcal{M}(\widetilde{U}_{z_0})$ contains $\mathcal{M}(AP_{\mathcal{C}}(\Sigma))(=\mathcal{M}(AP_{\mathcal{O}}(\Sigma)))$. Let K be the intersection of the closures of $\widetilde{U}_{z_0}$ and of $\mathbb{D}\setminus\widetilde{U}_{z_0}$ in $\mathbb{C}$. Then $K' := \mathrm{Log}\circ\phi_{z_0}(K)$ is a compact subset of Σ. In particular, $AP_{\mathcal{C}}(\Sigma)|_{K'} = C(K')$. This implies (by the Tietze extension theorem) that every bounded continuous function on $(\mathbb{D}\setminus\widetilde{U}_{z_0})\cup K$ can be extended to a function of class $\mathcal{A}(\widetilde{U}_{z_0})$ with the same supremum norm. Now, we can describe $\mathcal{M}(\widetilde{U}_{z_0})$ explicitly as follows (cf. the proof of Theorem 1.5 for a similar construction).

Let M be the maximal ideal space of the algebra of bounded continuous functions on $(\mathbb{D}\setminus\widetilde{U}_{z_0})\cup K$. We identify $K\subset M$ with $K'\subset\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ with the help of $\mathrm{Log}\circ\phi_{z_0}$. Under this identification, the quotient space of $M\sqcup\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ is homeomorphic to $\mathcal{M}(\widetilde{U}_{z_0})$.

By definition, the fiber F_{z_0} over z_0 consists of the limit points in $\mathcal{M}_{z_0}$ of all nets converging to $\{z_0\}$ inside $\mathbb{D}$. This and the above construction of $\mathcal{M}(\widetilde{U}_{z_0})$ show that $F(\widetilde{U}_{z_0})$ is homeomorphic to $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$. Moreover, F_{z_0} is the inverse limit of the compact sets $F(\widetilde{U}_{z_0})$, where the limit system is determined by the maps $p_{U_{z_0}V_{z_0}}|_{F(U_{z_0})}$ for $U_{z_0}\subset V_{z_0}$. We show that the maps $p_{U_{z_0}V_{z_0}}|_{F(U_{z_0})} : F(U_{z_0})\to F(V_{z_0})$ are homeomorphisms. Indeed, let $\{z_\alpha\}\subset\ U_{z_0}$ be a net converging to a point $\xi\in F(U_{z_0})$. Since $U_{z_0}\hookrightarrow V_{z_0}$, in our definitions of $\mathcal{M}(U_{z_0})$ and $\mathcal{M}(V_{z_0})$, the net $\{z_\alpha\}$ converges to the same point $\xi\in F(V_{z_0})$, which gives the required statement. Since all maps $p_{U_{z_0}V_{z_0}}|_{F(U_{z_0})}$ are homeomorphisms, the definition of the inverse limit implies that the map $p_{\widetilde{U}_{z_0}}|_{F_{z_0}} : F_{z_0}\to F(\widetilde{U}_{z_0})$ is also a homeomorphism. This completes the proof of (b).

Now, observe that in our model of $\mathcal{M}(\widetilde{U}_{z_0})$ the intersection of $\widetilde{U}$ with $\mathbb{D}$ contains $\widetilde{U}_{z_0}\cap\mathbb{D}$, which completes the proof of (a).

(c) Fix a point $z_*\in\partial\mathbb{D}$. We must show that every $f\in\mathrm{SAP}(S)\cap H^\infty$ belongs to $\mathcal{A}_{z_*}$. By (3.2) and Lemma 4.2, each $f\in\mathrm{SAP}(\partial\mathbb{D})\cap H^\infty$ can be approximated locally on open arcs of the form $\gamma_{t^k}(s)$, $k\in\{-1,1\}$, $s\in(0,\pi)$, $z:=e^{it}\in\partial\mathbb{D}$, by pullbacks of almost periodic functions on the boundary of Σ. Using the compactness of $\partial\mathbb{D}$, for any $\epsilon>0$ we can find finitely many points $z_1,\dots,z_n\in\partial\mathbb{D}$, arcs $\gamma_{t_l^k}(s_l)$, $k\in\{-1,1\}$, $s_l\in(0,\pi)$, $z_l := e^{it_l}$, and functions $f^l : \partial\mathbb{D}\setminus\{-z_l,z_l\}\to\mathbb{C}$ which are pullbacks of almost periodic functions on $\partial\Sigma$ by means of $\mathrm{Log}\circ\phi_{z_l}$, $1\le l\le n$, such that

$$\partial\mathbb{D}\setminus\{z_1,\dots,z_n\} = \bigcup_{1\le l\le n}(\gamma_{t_l^{-1}}\cup\gamma_{t_l^1}),\quad\text{and}$$

$$\operatorname*{ess\,sup}_{z\in\gamma_{t_l^1}(s_l)\cup\gamma_{t_l^{-1}}(s_l)}|f(z)-f^l(z)|<\epsilon$$

for all $1\le l\le n$. Without loss of generality we may assume that $z_*\in\{z_1,\dots,z_n\}$. Next, set $V_l := \gamma_{t_l^{-1}}\cup\gamma_{t_l^1}\cup\{z_l\}$. Then $(V_l)_{l=1}^n$ is a finite open cover of $\partial\mathbb{D}$. Let $\{\rho_l\}_{l=1}^n$ be a smooth partition of unity subordinate to this cover and such that $\rho_l(z_l)=1$. Consider the functions f_l on $\partial\mathbb{D}\setminus\{z_l\}$ defined by the formulas

$$f_l := \rho_l f^l,\quad 1\le l\le n.$$

Let F be the harmonic function on $\mathbb{D}$ such that $F|_{\partial\mathbb{D}} = \sum_{1\le l\le n} f_l$. Then

$$||f-F||_{L^\infty(\mathbb{D})}<\epsilon.$$

We prove that $F \in \mathcal{A}_{z_*}$. Since $\epsilon > 0$ is arbitrary, this will complete the proof of (c).

Let $F_{l,1}$ and $F_{l,2}$ be the harmonic functions on $\mathbb{D}$ with the boundary values f_l and $f^l - f_l$, respectively. Then $F_l := F_{l,1} + F_{l,2}$ is the harmonic function with the boundary values f^l. By Lemma 4.3, every F_l is almost periodic on $\overline{\mathbb{D}} \setminus \{\pm z_l\}$. Thus, if $z_l = z_*$, then $F_l \in \mathcal{A}_{z_*}$. If $z_l \neq z_*$, then F_l is continuous at z_*, and so $F_l \in \mathcal{A}_{z_*}$, by the definition of $\mathcal{A}_{z_*}$. Next, for a point z_l distinct from z_*, the function $F_{l,1}$ extends continuously to an open disk centered at z_* (because the support of f_l does not contain z_*). Hence, $F_{l,1} \in \mathcal{A}_{z_*}$. Assume now that $z_l = z_*$ for some l. Then the function $F_{l,2}$ extends continuously to an open disk centered at z_* (because the support of $f^l - f_l$ does not contain z_*). Thus, $F_{l,2} \in \mathcal{A}_{z_*}$, and in this case $F_{l,1} := F_l - F_{l,2} \in \mathcal{A}_{z_*}$. Since $F := \sum_{1 \leq l \leq n} F_{l,1}$, combining the cases considered above, we see that $F \in \mathcal{A}_{z_*}$.

This completes the proof of Lemma 4.4. $\square$

Theorem 4.5. *Let $f \in \mathrm{SAP}(S) \cap H^\infty$. Then for each $z_0 \in \partial\mathbb{D}$ and any $\epsilon > 0$ there is a circular neighborhood $U_{z_0} := U_{z_0}(f, \epsilon)$ of z_0 and a holomorphic almost periodic function f_{z_0} on U_{z_0} such that*

$$\sup_{z \in U_{z_0} \cap \mathbb{D}} |f(z) - f_{z_0}(z)| < \epsilon.$$

Proof. Fix a point $z_0 \in \partial\mathbb{D}$. By Lemma 4.4(c), the function $f \in \mathrm{SAP}(S) \cap H^\infty$ belongs to $\mathcal{A}_{z_0}$, so that it extends via the Gelfand transformation up to a continuous function $\widehat{f}$ on $\mathcal{M}_{z_0}$. We use the description of $\mathcal{M}_{z_0}$ presented in the proof of Lemma 4.4. Recall that in that construction the fiber $F_{z_0} \subset \mathcal{M}_{z_0}$ over z_0 is identified naturally with $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$.

Lemma 4.6. *The function $\widehat{f}|_{F_{z_0}}$ is holomorphic.*

Proof. In the proof we use the results of §2. Consider the map $i_\xi : \Sigma_0 \to \mathcal{M}(AP_{\mathcal{O}}(\Sigma))$, $\xi \in b\mathbb{Z}$. We must show that $\widehat{f} \circ i_\xi$ is holomorphic.

For this, we transfer the function f with the help of the map $(\mathrm{Log} \circ \phi_{z_0})^{-1}$ to Σ_0 and denote by $\widetilde{f}$ the function pulled back. Fix a point $\eta \in i_\xi(\Sigma_0)$, say, $\eta := i_\xi(w)$, $w \in \Sigma_0$. Then there is a straight line $\mathbb{R} + iy \subset \Sigma_0$, $y \in (0, \pi)$, and a net $\{z_\alpha\} \subset \mathbb{R} + iy$ forming an infinite discrete subset of $\mathbb{R} + iy$ and such that $\{z_\alpha\}$ converges to η in the topology of $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$. Let $B \subset \mathbb{C}$ be an open disk such that $\{\mathbb{R} + iy\} + B \subset \Sigma_0$. By the definition of i_ξ, for each $z \in B$ the net $\{z_\alpha + z\}$ converges in $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ to $i_\xi(w + z)$. Also, by the definition of $\mathcal{M}_{z_0}$, we have

$$\lim_\alpha \widetilde{f}(z_\alpha + z) = (\widehat{f} \circ i_\xi)(w + z), \quad z \in B.$$

But the holomorphic functions $\widetilde{f}_\alpha(z) := \widetilde{f}(z_\alpha + z)$ form a normal family on B. Therefore, using an argument similar to that in the proof of Lemma 2.3 (1), we see that $\widehat{f} \circ i_\xi|_B$ is holomorphic. Since ξ and η are arbitrary, this implies that $\widehat{f}|_{F_{z_0}}$ is holomorphic. $\square$

Now, Lemma 2.3 shows the existence of a function $\widetilde{f}_{z_0} \in AP_{\mathcal{O}}(\Sigma)$ whose extension to $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ coincides with $\widehat{f}|_{F_{z_0}}$. Consider the function $f_{z_0} \in \mathcal{A}_{z_0}$ whose pullback to Σ via $(\mathrm{Log} \circ \phi_{z_0})^{-1}$ coincides with $\widetilde{f}_{z_0}$. Then, by the definition of the topology of $\mathcal{M}_{z_0}$, the extension $\widehat{f}_{z_0}$ of f_{z_0} to $\mathcal{M}_{z_0}$ satisfies $\widehat{f}_{z_0}|_{F_{z_0}} = \widehat{f}|_{F_{z_0}}$. Since F_{z_0} is a compact set, the latter implies that there is a neighborhood U of F_{z_0} in $\mathcal{M}_{z_0}$ such that $|\widehat{f}_{z_0}(x) - \widehat{f}(x)| < \epsilon$ for all $x \in U$. Finally, by Lemma 4.4(a), there is a circular neighborhood U_{z_0} such that $U_{z_0} \cap \mathbb{D} \subset U \cap \mathbb{D}$, whence $|f_{z_0}(z) - f(z)| < \epsilon$ for all $z \in U_{z_0} \cap \mathbb{D}$. $\square$

4.2. Proof of Theorem 1.8. We must show that $A_S = \mathrm{SAP}(S) \cap H^\infty$. We split the proof into several parts. First, we prove the following statement.

Lemma 4.7. $\mathrm{SAP}(S) \cap H^\infty$ *is the uniform closure of the algebra generated by all possible subalgebras* $\mathrm{SAP}(F) \cap H^\infty$ *with finite* $F \subset S$.

Then we shall prove that $A_F = \mathrm{SAP}(F) \cap H^\infty$ for every finite subset $F \subset \partial\mathbb{D}$. Combined with the above lemma and the fact that A_S is the uniform closure of the algebra generated by all possible subalgebras A_F with finite $F \subset S$, this will clearly complete the proof of the theorem.

Proof. By Theorem 4.5, for a given $f \in SAP(S) \cap H^\infty$ we can find finitely many points $z_1, \dots, z_n$, circular neighborhoods $U_{z_1}, \dots, U_{z_n}$, and holomorphic almost periodic functions $f_1, \dots, f_n$ defined on $U_{z_1}, \dots, U_{z_n}$, respectively, such that $(U_{z_i})_{1 \le i \le n}$ forms an open cover of $\partial\mathbb{D} \setminus \{z_1, \dots, z_n\}$ and

$$\max_i ||f|_{U_{z_i}} - f_i||_{L^\infty(U_{z_i})} < \epsilon.$$

Since the discontinuities of $f|_{\partial\mathbb{D}}$ belong to the closed set S, each function f_i with $z_i \notin S$ can be chosen also to be continuous on the closure $\overline{U}_{z_i}$.

Next, we form a cocycle $\{c_{ij}\}$ on the intersections of sets from the above cover by the formula

$$c_{ij}(z) := f_i(z) - f_j(z), \quad z \in U_{z_i} \cap U_{z_j}.$$

Reducing, if necessary, the sets of the above cover, we may assume without loss of generality that all $U_{z_i} \cap U_{z_j}$, $i \neq j$, do not contain the points $z_1, \dots, z_n$. Then each $U_{z_i} \cap U_{z_j}$, $i \neq j$, is a compact subset of $\overline{\mathbb{D}}$, and the corresponding c_{ij} are continuous and holomorphic in the interior of $U_{z_i} \cap U_{z_j}$.

Let $\{\rho_i\}$ be a smooth partition of unity subordinate to the cover $(U_{z_i})_{1 \le i \le n}$. We can require that every ρ_i be the restriction to $\overline{U}_{z_i}$ of a C^∞-function on $\mathbb{C}$ and that $\rho_i(z_i) = 1$. As usual, we employ this partition of unity to resolve the cocycle $\{c_{ij}\}$ by the formulas

$$\tilde{f}_j(z) = \sum_{k=1}^{n} \rho_k(z) c_{jk}(z), \quad z \in \overline{U}_{z_j}. \tag{4.3}$$

Hence,

$$c_{ij}(z) := \tilde{f}_i(z) - \tilde{f}_j(z), \quad z \in U_{z_i} \cap U_{z_j}.$$

In particular, since the c_{ij} are holomorphic in $\mathbb{D} \cap U_{z_i} \cap U_{z_j}$, the formula

$$h(z) := \frac{\partial \tilde{f}_i(z)}{\partial \overline{z}}, \quad z \in U_{z_i} \cap \mathbb{D},$$

determines a smooth bounded function in an open annulus $A \subset \cup_{i=1}^{n} \overline{U}_{z_i}$ with the outer boundary $\partial\mathbb{D}$. Also, by our choice of the partition of unity, h extends continuously to the closure $\overline{A}$ of A.

Consider the function

$$H(z) = \frac{1}{2\pi i} \int\int_{\zeta \in A} \frac{h(\zeta)}{\zeta - z}\, d\zeta \wedge d\overline{\zeta}, \quad z \in \overline{A}. \tag{4.4}$$

Passing in (4.4) to polar coordinates with origin at z, we easily obtain

$$\sup_{z \in A} |H(z)| \le Cw(A) \sup_{z \in A} |h(z)|, \tag{4.5}$$

where $w(A)$ is the width of A and $C > 0$ is an absolute constant. Moreover, $H \in C(\overline{A})$ and $\partial H/\partial\overline{z} = h$ in A (see, e.g., [G, Chapter VIII]). We can replace A by a similar annulus of a smaller width such that for this new A

$$\sup_{z\in A}|H(z)| < \epsilon.$$

Now, we set

$$c_i(z) := \widetilde{f}_i(z) - H(z), \quad z \in \overline{U}_{z_i} \cap \overline{A}.$$

Then each c_i is continuous on $U_{z_i} \cap \overline{A}$ and holomorphic at the interior points of this set, and

$$c_i(z) - c_j(z) = c_{ij}(z), \quad z \in U_{z_i} \cap U_{z_j}.$$

Since $|c_{ij}(z)| < 2\epsilon$ for all $z \in U_{z_i} \cap U_{z_j}$, we have

$$|c_i(z)| < 3\epsilon, \quad z \in U_{z_i} \cap U_{z_j}.$$

We define a global function f_ϵ on $\overline{A} \setminus \{z_1, \dots, z_n\}$ by the formulas

$$f_\epsilon(z) := f_i(z) - c_i(z), \quad z \in U_i \cap \overline{A}.$$

Since for $z_i \notin S$ the function f_i is continuous on $\overline{U}_{z_i}$, the above construction shows that $f_\epsilon \in H^\infty(A) \cap \mathrm{SAP}(F)$, where $F := \{z_1, \dots, z_n\} \cap S$. Also,

$$||f - f_\epsilon||_{L^\infty(A)} < 4\epsilon.$$

Let B be an open disk centered at 0 whose intersection with A is an annulus of width less than ϵ. Consider the cocycle c on $B \cap A$ defined by

$$c(z) = f(z) - f_\epsilon(z), \quad z \in B \cap A.$$

By definition, $|c(z)| \le 4\epsilon$ for all $z \in B \cap A$. Let A' be the open annulus with the interior boundary coinciding with the interior boundary of A and with the outer boundary $\{z \in \mathbb{C} \ : \ |z| = 2\}$. Then $A' \cap B = A \cap B$. Consider a smooth partition of unity subordinate to the cover $\{A', B\}$ of $\mathbb{D}$ and consisting of smooth radial functions ρ_1 and ρ_2 such that

$$\max_i ||\nabla\rho_i||_{L^\infty(\mathbb{C})} \le \widetilde{C} w(B \cap A) < \widetilde{C}\epsilon$$

for some absolute constant $\widetilde{C} > 0$. Then using arguments similar to the above and based on versions of (4.3), (4.4), and (4.5) for the cocycle c and the partition of unity $\{\rho_1, \rho_2\}$, we can find holomorphic functions $\overline{c}_1$ on B and $\overline{c}_2$ on A continuous on the corresponding boundaries and such that

$$\overline{c}_1(z) - \overline{c}_2(z) = c(z), \quad z \in B \cap A, \quad \text{and}$$

$$\max\{||\overline{c}_1||_{H^\infty(B)}, ||\overline{c}_2||_{H^\infty(A)}\} \le \overline{C}||c||_{H^\infty(A\cap B)},$$

where $\overline{C} > 0$ is an absolute constant. Finally, we define

$$F_\epsilon(z) := \begin{cases} f(z) - \overline{c}_1(z) & \text{if } z \in B, \\ f_\epsilon(z) - \overline{c}_2(z) & \text{if } z \in \mathbb{D} \setminus B. \end{cases}$$

Clearly,

$$||f - F_\epsilon||_{L^\infty(\mathbb{D})} < c\epsilon.$$

for some absolute constant $c > 0$, and $F_\epsilon \in \mathrm{SAP}(F) \cap H^\infty$, where $F := \{z_1, \dots, z_n\} \cap S$. Since ϵ is arbitrary, this completes the proof of the lemma. □

As the next step of the proof we establish Theorem 1.8 for $\mathrm{SAP}(F) \cap H^\infty$ with a finite set $F \subset \partial S$, say, $F = \{z_1, \dots, z_n\}$.

Lemma 4.8.

$$A_F = \mathrm{SAP}(F) \cap H^\infty.$$

Proof. We show that $\mathrm{SAP}(F)\cap H^\infty\subset A_F$. First, suppose that F contains at least two points.

Let $\psi_1:\partial\mathbb{D}\to\partial\mathbb{D}$ be the restriction to the boundary of a Möbius transformation of $\mathbb{D}$ that maps $-z_1$ to a point of F distinct from z_1 and preserves z_1. Then, by the definition of the Möbius transformations, ψ_1 is a C^1 diffeomorphism of $\partial\mathbb{D}$. In particular, by Corollary 1.6, for any $f\in\mathrm{SAP}(F)\cap H^\infty$ we have $f\circ\psi_1\in\mathrm{SAP}(F_1)\cap H^\infty$, where $F_1:=\psi_1^{-1}(F)$. Since $z_1\in F_1$, we can argue as in the proof of Theorem 4.5 to find an almost periodic holomorphic function g_1 on $\overline{\mathbb{D}}\setminus\{\pm z_1\}$ such that the function $f\circ\psi_1-g_1$ is continuous and equals 0 at z_1. We set

$$\widetilde{g}_1(z):=\frac{g(z)(z+z_1)}{2z_1},\quad z\in\overline{\mathbb{D}}\setminus\{z_1\}.$$

Then $\widetilde{g}_1$ has a discontinuity at z_1 only. We show that $\widetilde{g}_1\in A_{\{-z_1,z_1\}}$. Indeed, the definition of g_1 implies that the function $g_1\circ\phi_{z_1}^{-1}\circ\mathrm{Log}^{-1}$ belongs to $AP_{\mathcal{O}}(\Sigma)$. Therefore, it can be uniformly approximated on Σ by polynomials in variables $e^{i\lambda z}$, $\lambda\in\mathbb{R}$, see, e.g., [JT]. In its turn, g_1 can be uniformly approximated on $\overline{\mathbb{D}}\setminus\{\pm z_1\}$ by complex polynomials in variables $e^{i\lambda\mathrm{Log}\circ\phi_1}$. Now, for $z\in\partial\mathbb{D}$ we have

$$\mathrm{Im}\{(\mathrm{Log}\circ\phi_{z_1})(z)\}:=\begin{cases}0 & \text{if } 0\le\mathrm{Arg}(z/z_1)<\pi,\\ \pi & \text{if } 0\le\mathrm{Arg}(z_1/z)<\pi.\end{cases}$$

This implies that every function $e^{i\lambda\mathrm{Log}\circ\phi_1}$, $\lambda\in\mathbb{R}$, belongs to $A_{\{-z_1,z_1\}}$, whence $g_1\in A_{\{-z_1,z_1\}}$. Since $(z+z_1)/2z_1\in A_0$, we have $\widetilde{g}_1\in A_{\{-z_1,z_1\}}$ by definition. Thus, the function $h_1:=\widetilde{g}_1\circ\psi_1^{-1}$ belongs to A_F and is continuous outside z_1, and $f_1:=f-h_1\in\mathrm{SAP}(F^1)\cap H^\infty$, where $F^1:=F\setminus\{z_1\}$. By using similar arguments, we can find a function $h_2\in A_F$ continuous outside z_2 and such that $f_2:=f_1-h_2\in\mathrm{SAP}(F^2)\cap H^\infty$, where $F^2:=F^1\setminus\{z_2\}$ etc. After n steps we obtain functions $h_1,\dots,h_n\in A_F$ such that h_k is continuous outside z_k, $1\le k\le n$, and the function

$$h_{n+1}:=f-\sum_{k=1}^n h_k \tag{4.6}$$

has no discontinuities on $\partial\mathbb{D}$, that is, $h_{n+1}\in A_0$. Therefore, $f\in A_F$.

Next, if F consists of a single point, say z_0, then for any $f\in\mathrm{SAP}(F)\cap H^\infty$ the above argument yields a function $h\in A_{\{z_0,z_1\}}$ with a fixed $z_1\in\partial\mathbb{D}$ such that $f-h$ is continuous on F. Let $g\in A_0$ be a function equal to 1 on F and to 0 at z_1. Then $f-gh\in A_0$. This completes the first part of the proof.

Now we show that $A_F\subset\mathrm{SAP}(F)\cap H^\infty$. Again, assume first that F contains at least two points. Let $e^{\lambda f}\in A_F$, $\lambda\in\mathbb{R}$, where $\mathrm{Re}\, f$ is the characteristic function of an arc, say $[x,y]$, with $x,y\in F$. Let $\psi:\partial\mathbb{D}\to\partial\mathbb{D}$ be the restriction to $\partial\mathbb{D}$ of a Möbius transformation sending 1 to x and -1 to y. Then

$$(f\circ\psi\circ\phi_1^{-1})(z)=-\frac{i}{\pi}\mathrm{Log}\, z+C,\quad z\in\mathbb{H}_+,$$

for some constant C. Thus, we have

$$e^{(\lambda f\circ\psi\circ(\mathrm{Log}\circ\psi_1)^{-1})(z)}=e^{\lambda C}e^{-i\lambda z/\pi},\quad z\in\Sigma.$$

This means that $e^{\lambda f\circ\psi}\in\mathrm{SAP}(\{-1,1\})\cap H^\infty$. Then $e^{\lambda f}\in\mathrm{SAP}(F)\cap H^\infty$ by Corollary 1.6. Since A_F is generated by A_0 and such functions $e^{\lambda f}$, we arrive at the required implication.

If F is a single point, then we must show that $ge^{\lambda f}\in\mathrm{SAP}(F)\cap H^\infty$, $\lambda\in\mathbb{R}$, where $\mathrm{Re}\, f$ is the characteristic function of an arc with endpoint F, and $g\in A_0$ is such that

ge^f has discontinuity at F only. The result follows easily from the preceding part of the proof, because $e^{\lambda f}$ is almost periodic on $\partial\mathbb{D} \setminus \{F, y\}$ for some y and is continuous at y.

This completes the proof of the lemma. □

As was mentioned above, the required statement of the theorem follows from Lemmas 4.7 and 4.8. □

4.3. Proof of Example 1.10. Using the bilinear transformation $\phi_1 : \mathbb{D} \to \mathbb{H}_+$ (see (4.1)) that maps 1 to $0 \in \mathbb{R}$ and -1 to ∞, we can transform the problem to a similar one for functions on $\mathbb{H}_+$. Namely, let $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ be the sequence converging to 0 that is the image of the sequence $\{e^{it_k}\}_{k\in\mathbb{N}} \subset \partial\mathbb{D}$ in Example 1.10 under ϕ_1. Let $H : \mathbb{R} \to \{0, 1\}$ be the Heaviside function (i.e., the characteristic function of $[0, \infty)$). Then the pullback by ϕ_1^{-1} of the function u in Example 1.10 to the boundary $\mathbb{R}$ of $\mathbb{H}_+$ is the function

$$\widetilde{u}(x) := \sum_{k=1}^{\infty} \alpha_k H(x - x_k), \quad x \in \mathbb{R}.$$

We extend $\widetilde{u}$ up to a harmonic function on $\mathbb{H}_+$ by the Poisson integral. Let $\widetilde{v}$ be the harmonic conjugate to the extended function, determined on $\mathbb{R}$ by the formula

$$\widetilde{v}(x) = \sum_{k=1}^{\infty} \alpha_k \frac{\ln|x - x_k|}{\pi}, \quad x \in \mathbb{R}. \tag{4.7}$$

We set $\widetilde{h} := \widetilde{u} + i\widetilde{v}$. Then $\widetilde{h}$ is the pullback by ϕ_1^{-1} of a holomorphic function h on $\mathbb{D}$ such that $\operatorname{Re} h|_{\partial\mathbb{D}} = u$. Assume, to the contrary, that $e^h \in A_S$. Since $e^u \in R_S$, this assumption implies that $e^{iv} \in \mathrm{SAP}(S)$, where $v = \phi_1^*(\widetilde{v}|_{\mathbb{R}})$. Then, by the definition of the topology on $\mathcal{M}(\mathrm{SAP}(S))$, see Subsection 3.3, the functions $\cos(\widetilde{v}(e^t))$ and $\sin(\widetilde{v}(e^t))$, $t \in \mathbb{R}$, admit continuous extensions to $b\mathbb{R}$; these extensions are determined as follows.

If $\{s_\alpha\} \subset \mathbb{R}$ is a net converging in $b\mathbb{R}$ to a point $\eta \in b\mathbb{R}$, then the values at η of the extended functions are

$$\lim_\alpha \cos(\widetilde{v}(e^{s_\alpha})) \quad \text{and} \quad \lim_\alpha \sin(\widetilde{v}(e^{s_\alpha})),$$

respectively. In particular, this definition requires the existence of these limits.

Now, by (4.7), there are sequences of points $\{x_k'\}_{k\in\mathbb{N}}$ and $\{x_k''\}_{k\in\mathbb{N}}$ in $\mathbb{R}_+$ such that x_k' and x_k'' are sufficiently close to x_k and

$$\lim_{k\to\infty} \left| \frac{x_k'}{x_k''} \right| = 1, \quad \cos(\widetilde{v}(x_k')) = 0, \quad \cos(\widetilde{v}(x_k'')) = 1, \ \ k \in \mathbb{N}.$$

We set $t_k' := \ln x_k'$ and $t_k'' := \ln x_k''$, $k \in \mathbb{N}$. Without loss of generality, we assume that $\{t_k'\}$ forms a net converging in the topology of $b\mathbb{R}$ to a point $\xi \in b\mathbb{R}$ (otherwise we replace $\{t_k\}$ by a proper subset satisfying this property). Since $\lim_{k\to\infty} |t_k' - t_k''| = 0$ by definition, and almost periodic functions on $\mathbb{R}$ are uniformly continuous, the family $\{t_k''\}$ forms a net with the same indices as for the net formed by $\{t_k'\}$ whose limit in $b\mathbb{R}$ is ξ. Hence, we have

$$\lim_k \cos(\widetilde{v}(e^{t_k'})) = \lim_k \cos(\widetilde{v}(e^{t_k''})),$$

a contradiction. Therefore, $e^h \notin A_S$.

Now, let $f \in A_0$ be such that $f(1) = 0$. Then the function $(fe^h)|_{\partial\mathbb{D}}$ is continuous at 1. Thus, $(fe^h)|_{\partial\mathbb{D}}$ can be uniformly approximated by constant functions on open arcs containing 1. The same is true for each point e^{it_k}. This implies immediately that $fe^h \in A_S$. □

§5. Proofs of Theorems 1.12 and 1.14

5.1. Proof of Theorem 1.12. We recall that A_S is the uniform closure of the algebra generated by all possible A_F with finite $F \subset S$. Therefore, the maximal ideal space $\mathcal{M}(A_S)$ of A_S is the inverse limit of the maximal ideal spaces $\mathcal{M}(A_F)$ of A_F. In particular, if we prove that $\mathbb{D}$ is dense in each $\mathcal{M}(A_F)$, then, by the definition of the inverse limit, this will imply that $\mathbb{D}$ is dense in $\mathcal{M}(A_S)$, as required. Thus, it suffices to prove the theorem for A_F with $F = \{z_1, \dots, z_n\} \subset \partial\mathbb{D}$.

Theorem 5.1. *$\mathbb{D}$ is dense in $\mathcal{M}(A_F)$.*

Proof. Let $\mathcal{I}_k \subset A_F$ be the closed ideal consisting of all functions that are continuous and equal to 0 at z_k. Let A_k denote the quotient Banach algebra $A_F/\mathcal{I}_k$ equipped with the quotient norm. We recall that $\mathcal{M}_{z_k}$ is the maximal ideal space of the algebra $\mathcal{A}_{z_k}$, which is the uniform closure of the algebra of functions continuous on $\mathbb{D}$ and almost periodic in circular neighborhoods of z_k. Also, by Lemma 4.4 (b), there is a natural continuous projection $p_{z_k} : \mathcal{M}_{z_k} \to \overline{\mathbb{D}}$, and $p_{z_k}^{-1}(z_k)$ is homeomorphic to $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$. Moreover, by Lemma 4.4 (c), each $f \in A_F$ extends to a function continuous on $\mathcal{M}_{z_k}$ and holomorphic on $p_{z_k}^{-1}(z_k)$. Hence, there is a continuous map $H_k : \mathcal{M}_{z_k} \to \mathcal{M}(A_F)$ whose image coincides with the closure of $\mathbb{D}$. Moreover, in accordance with the decomposition obtained in the proof of Lemma 4.8, see (4.6), H_k maps $p_{z_k}^{-1}(z_k)$ homeomorphically onto its image.

Lemma 5.2. *Let $\phi_k : A_F \to AP_{\mathcal{O}}(\Sigma)$ be the composition of the homomorphism of extension of functions in A_F to $\mathcal{M}_{z_k}$ and the homomorphism of restriction of functions on $\mathcal{M}_{z_k}$ to $p_{z_k}^{-1}(z_k)$. Then* Ker $\phi_k = \mathcal{I}_k$ *and A_k is isomorphic to $AP_{\mathcal{O}}(\Sigma)$.*

Proof. Clearly, $\mathcal{I}_k \subset \operatorname{Ker}\phi_k$. We check the reverse inclusion. Let $f \in \operatorname{Ker}\phi_k$. The proof of Lemma 4.8, see (4.6), shows that there are continuous linear operators $T_k : AP_{\mathcal{O}}(\Sigma) \to A_{\{z_k\}} \subset A_F$ such that $\phi_k \circ T_k = \mathrm{id}$. Moreover, $T_0 := I - \sum_{k=1}^n T_k \circ \phi_k$, where I is the identity map, maps A_F onto A_0. In particular, we have $T_k(\phi_k(f)) = 0$. Thus, $f = -T_0(f) + \sum_{s\neq k} T_s(\phi_s(f))$. Since $\phi_k(f) = 0$, this implies that f is continuous and equal to 0 at z_k. Now from the formula $\phi_k \circ T_k = \mathrm{id}$, it follows that A_k is isomorphic to $AP_{\mathcal{O}}(\Sigma)$. □

Let $i : A_0 \hookrightarrow A_F$ be the natural inclusion. Its dual determines a continuous surjective map $a_F : \mathcal{M}(A_F) \to \overline{\mathbb{D}}$. Next, taking the dual map to ϕ_k, we see that each $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ is embedded in $\mathcal{M}(A_F)$, its image coincides with $H_k(p_{z_k}^{-1}(z_k))$, and a_F maps $H_k(p_{z_k}^{-1}(z_k))$ to z_k.

Let $\xi \in \mathcal{M}(A_F)$, and let $\mathfrak{m} :=$ Ker $\xi \subset A_F$ be the corresponding maximal ideal. First, assume that there exists k such that $\mathcal{I}_k \subset \mathfrak{m}$. Then $\mathfrak{m}_k = \phi_k(\mathfrak{m})$ is a maximal ideal of A_k. Let $\xi_k \in \mathcal{M}(AP_{\mathcal{O}}(\Sigma))$ denote the character corresponding to $\mathfrak{m}_k$. Then $\xi = \phi^*(\xi_k) \in H_k(p_{z_k}^{-1}(z_k))$. Now, by the definition of H_k, the point ξ belongs to the closure of $\mathbb{D}$ in $\mathcal{M}(A_F)$. We continue with the following lemma.

Lemma 5.3. *Assume that a maximal ideal $\mathfrak{m}$ of A_F contains none of $\mathcal{I}_k$. Then $\mathfrak{m}$ does not contain $\cap_{1\le k\le n}\mathcal{I}_k$.*

Proof. Suppose, to the contrary, that $\cap_{1\le k\le n}\mathcal{I}_k \subset \mathfrak{m}$. Let $x_k \in \mathcal{I}_k$, $1 \le k \le n$, be such that $x_k \notin \mathfrak{m}$. Since $\mathcal{I}_k$ are ideals, $x_1 \cdots x_n \in \cap_{1\le k\le n}\mathcal{I}_k$. Thus, $x_1 \cdots x_n \in \mathfrak{m}$. Since $\mathfrak{m}$ is a prime ideal, there is some k such that $x_k \in \mathfrak{m}$, a contradiction. □

This lemma shows that for the proof of the theorem it remains to consider the case where $\mathfrak{m} \not\subset \cap_{1\le k\le n}\mathcal{I}_k$. Observe that $\cap_{1\le k\le n}\mathcal{I}_k$ consists of all functions in A_0 that vanish on F. Thus, there is $f \in \cap_{1\le k\le n}\mathcal{I}_k$ such that $f(\xi)\neq 0$. This implies that $a_F(\xi)\notin F$. For every $g\in A_F$, consider the function gf. By definition, $gf\in A_0$. Thus, we have

$$g(a_F(\xi))f(a_F(\xi)) = (gf)(a_F(\xi)) = (gf)(\xi) = g(\xi)f(\xi) = g(\xi)f(a_F(\xi)).$$

Equivalently,

$$g(\xi) = g(a_F(\xi)) \quad \text{for all} \quad g\in A_F.$$

This implies that $a_F^{-1}(a_F(\xi)) = \{\xi\}$. Therefore, $a_F : \mathcal{M}(A_F)\setminus a_F^{-1}(F)\to \overline{\mathbb{D}}\setminus F$ is a homeomorphism. In particular, ξ belongs to the closure of $\mathbb{D}$.

The proof of Theorem 1.12 is complete. $\square$

5.2. Proof of Theorem 1.14. Statements (1) and (3) of the theorem follow easily from similar statements for a_F with a finite subset $F\subset S$, proved in Subsection 5.1, and the properties of the inverse limit. To prove (3), we start with the case of a finite subset $F\subset S$.

Since $A_0\subset A_F$ and the modulus of each function in A_0 attains its maximum on $\partial\mathbb{D}$, we have $K_F\subset a_F^{-1}(\partial\mathbb{D})$. Theorem 1.8 shows that $A_F\hookrightarrow \mathrm{SAP}(F)$. Also, the extensions of functions in A_F to $\mathcal{M}(\mathrm{SAP}(F))$ separate points there. Therefore, $\mathcal{M}(\mathrm{SAP}(F))$ embeds in $\mathcal{M}(A_F)$. Identifying $\mathcal{M}(\mathrm{SAP}(F))$ with its image under this embedding, we have $\mathcal{M}(\mathrm{SAP}(F))\subset a_F^{-1}(\partial\mathbb{D})$. Since $a_F^{-1}(\partial\mathbb{D})\setminus a_F^{-1}(F)\to\partial\mathbb{D}\setminus F$ is a homeomorphism and each $z\in\partial\mathbb{D}$ is a peak point for A_0, the set K_F contains the closure of $\mathbb{D}\setminus F$, which, by definition, coincides with $\mathcal{M}(\mathrm{SAP}(F))$. Assume that there exists $\xi\in K_F\setminus\mathcal{M}(\mathrm{SAP}(F))$. Then $a_F(\xi) := z^*\in F$. Next, identifying $a_S^{-1}(z^*)$ with $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$, from our assumption we deduce that $\xi\in i_\eta(\Sigma_0)$ for some $\eta\in b\mathbb{Z}$, see §2; here Σ_0 is the interior of Σ. Then, since $i_\eta(\Sigma_0)$ is dense in $\mathcal{M}(AP_{\mathcal{O}}(\Sigma))$, each function $f\in A_F$ with $\max_{\mathbb{D}}|f| = |f(\xi)|$ is constant on $a_S^{-1}(z^*)$ by the maximum modulus principle. Thus, such f attains the maximum of the modulus also on $\mathcal{M}(\mathrm{SAP}(F))$. This contradicts the minimality of K_F. Therefore, $K_F = \mathcal{M}(\mathrm{SAP}(F))$.

Furthermore, $\mathcal{M}(\mathrm{SAP}(S))$ is the inverse limit of compact sets $\mathcal{M}(\mathrm{SAP}(F))$ for all finite $F\subset S$. As before, we naturally identify $\mathcal{M}(\mathrm{SAP}(S))$ with a subset of $\mathcal{M}(A_S)$. Then, since A_S is the uniform closure of the algebra generated by all possible A_F with finite $F\subset S$, we have $K_S\subseteq\mathcal{M}(\mathrm{SAP}(S))$ by the definition of the inverse limit. But in fact the set K_S coincides with $\mathcal{M}(\mathrm{SAP}(S))$, because otherwise its projection to some of $\mathcal{M}(A_F)$ is a boundary of A_F and a proper subset of $\mathcal{M}(\mathrm{SAP}(F))$, a contradiction. $\square$

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