

NON-LOCAL SDES, CRITICAL DRIFTS AND LOCAL BLOW-UPS

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ABSTRACT. The paper is concerned with α -stable SDEs with singular time-inhomogeneous general drift. Our drifts satisfy a condition that is close to the minimal possible scaling-invariance and can introduce local blow-ups of the type arising in some particle systems with strong attracting interactions.

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1. INTRODUCTION

This paper is concerned with weak solution theory for SDEs with singular drift:

$$X_t = x - \int_0^t b(s, X_s) ds + Z_t, \quad t \geq 0, \quad (1.1)$$

where Z_t is an isotropic α -stable process in \mathbb{R}^d ($d \geq 2$) generated by the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$, $1 < \alpha < 2$ (the case $\alpha = 2$ is included as well, then Z_t is replaced by Brownian motion, but our focus is on $\alpha < 2$). The drift $b : \mathbb{R}^{1+d} \rightarrow \mathbb{R}^d$ is a Borel measurable vector field satisfying a close to the minimal possible condition of invariance with respect to the diffusive scaling, cf. (1.3). We consider drifts that can be singular enough to introduce local in space blow-up effects due to a strong attraction to a point or a submanifold, and can have critical singularities in time. Our drifts are general, i.e. they are not assumed to have any particular structure such as a gradient form or divergence of appropriate sign.

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SDEs with attracting singularities in the drift arise in a number of physical models, including the SDE behind the non-classical Feynman-Kac formula for the 2D δ -Bose gas [Ch] and the nonlinear SDE for the Keller-Segel model of chemotaxis, see [FJ] for $\alpha = 2$, and [E] for $\alpha < 2$. Another class of models where SDEs with singular (non-attracting) drift appear is related to the passive tracer model in hydrodynamics where the drift is the velocity field obtained by solving 3D Navier-Stokes equations [ZZ]. It is well known that the non-locality of the diffusion operator $(-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla$ behind SDE (1.1) makes the analysis of (1.1) and of the corresponding Kolmogorov equations easier in some aspects (e.g. the heat kernel of the fractional Laplacian vanishes at infinity only polynomially, which allows for some scaling arguments that do not exist when $\alpha = 2$, cf. [JW]) and considerably more difficult in some other aspects. We refer e.g. to [CaV, KSS, MM, S] for extensions of De Giorgi's and Nash's methods to non-local PDEs. The literature on non-local PDEs and stable-like diffusion processes [CF] is motivated, in particular, by physical models involving anomalous diffusions [KRS]. A part of it is the literature on non-local (= stable-driven) SDEs, which we will discuss in a moment.

Our main assumption on b is the following Rellich-type condition:

$$\begin{aligned} b \cdot \nabla \text{ is strongly bounded by } & \lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}} \\ \text{in } & L^p(\mathbb{R}^{1+d}, (1 + |b|)^{-p+1} dt dx), \quad p > d + 1, \\ \text{with sufficiently small bound, for some constant } & \lambda > 0, \end{aligned} \tag{1.2}$$

where the role of the weight $(1 + |b|)^{-p+1}$ is to desingularize the drift; see (1.16)-(1.18) for the complete form of (1.2). This condition seems to be rather well suited for handling particle systems with attracting interactions that are strong enough to introduce local blow-up effects: as the strength of attraction between the particles crosses a certain critical threshold, particles start to collide in finite time a.s. and the particle system ceases to have a global-in-time weak solution (hence the assumption of smallness in (1.2)).

An elementary sufficient condition for (1.2) (i.e. (1.16)-(1.18)) is the following parabolic Morrey class condition:

$$\| |b|^{\frac{1}{\alpha-1}} \|_{E_{1+\epsilon}} \text{ is sufficiently small.} \tag{1.3}$$

Definition. A function $v \in L_{loc}^{1+\epsilon}(\mathbb{R}^{1+d})$, $\epsilon > 0$, is said to be in the Morrey class $E_{1+\epsilon} = E_{1+\epsilon}^\alpha$ if it has finite Morrey norm

$$\|v\|_{E_{1+\epsilon}} := \sup_{\rho > 0, (t,x) \in \mathbb{R}^{1+d}} \rho \left(\frac{1}{\rho^{d+\alpha}} \int_{C_\rho^\alpha(t,x)} |v(s,y)|^{1+\epsilon} ds dy \right)^{\frac{1}{1+\epsilon}} < \infty,$$

where the supremum is taken over all α -parabolic cylinders $C_\rho^\alpha(t,x) := \{(s,y) \in \mathbb{R}^{1+d} \mid t \leq s \leq t + \rho^\alpha, |y - x| \leq \rho\}$.

See Proposition 1 for details. It refines some results of Adams and Krylov, and is of interest on its own.

Examples. 1. Condition (1.3) includes e.g.

$$b(x) = \pm \kappa \frac{x}{|x|^\alpha}, \quad \kappa > 0$$

in which case $\| |b|^{\frac{1}{\alpha-1}} \|_{E_{1+\epsilon}}^{\alpha-1} = c(d, \epsilon) \kappa$. This drift introduces strong attraction to the origin in (1.1) (+) or strong repulsion (-).

2. More generally, (1.3) includes $|b| \in L^{\frac{d}{\alpha-1}, \infty}$ (weak Lebesgue class), in particular, a drift with everywhere dense singular locus

$$b(t, x) = \sum_{z_i \in \mathbb{Q}^d} \kappa_i(t, x) \frac{x - z_i}{|x - z_i|^\alpha} \quad \text{provided } \sum_i \|\kappa_i\|_{L^\infty(\mathbb{R}^{1+d})} < \infty. \quad (1.4)$$

3. Another example is the following many-particle drift: $b : \mathbb{R}^{dN} \rightarrow \mathbb{R}^{dN}$, $b = (b^i)_{i=1}^N$,

$$b^i(x) := \frac{1}{N} \sum_{j=1, j \neq i}^N \frac{x^i - x^j}{|x^i - x^j|^\alpha}, \quad x = (x^1, \dots, x^N),$$

(see the proof of Theorem 3 for details). This drift has singularities along the collision hyperplanes $x^i = x^j$ and is not anywhere close to $L^{\frac{dN}{\alpha-1}}(\mathbb{R}^{dN})$.

4. Another example is given by a time-inhomogeneous Borel measurable vector field b satisfying, for some fixed $T > 0$,

$$|b(t, x)| \leq \mathbf{1}_{[-T, T]} |t - t_0|^{-\frac{\alpha-1}{2}} \quad \text{for all } (t, x) \in \mathbb{R}^{1+d}.$$

The previous examples can be combined, i.e. we can add them up, multiply by constants or by functions in L^∞ .

We can also verify (1.2) directly by means of various fractional Hardy inequalities [FLS, KPS], which provide control over what ‘‘sufficiently small’’ in (1.2), (1.3) is.

Our result, stated briefly, is as follows (for detailed statements see Theorems 1, 2). The part (i) also imposes a global L^1 integrability condition on the unbounded part of $|b|$.

Theorem. *If the unbounded part of the drift b satisfies (1.3), then the following are true:*

- (i) *For every $x \in \mathbb{R}^d$, SDE (1.1) has a conditionally unique weak solution in a large class. These weak solutions constitute a strong Markov family. In fact, they determine a strongly continuous Feller propagator.*
- (ii) *There is a detailed Sobolev regularity theory of the Kolmogorov backward and forward equations corresponding to SDE (1.1) that, in particular, yields the uniqueness of the law of X_t among laws having a sufficiently regular density satisfying the Kolmogorov forward equation in the weak sense.*

Example 1 (Counterexample to weak existence). The smallness condition in (1.3) is necessary. Namely, consider SDE (1.1) with the following drift that pushes the solution X_t towards the origin:

$$b(x) = \kappa \frac{x}{|x|^\alpha}, \quad \kappa > 0.$$

Fix some $\beta \in (1, \alpha)$ and put

$$c_{d, \alpha, \beta} := -2^\alpha \frac{\Gamma(\frac{\alpha-\beta}{2})\Gamma(\frac{d+\beta}{2})}{\Gamma(-\frac{\beta}{2})\Gamma(\frac{d+\beta-\alpha}{2})} > 0.$$

This constant is fixed by the identity

$$-(-\Delta)^{\frac{\alpha}{2}} |x|^\beta = c_{d, \alpha, \beta} |x|^{\beta-\alpha}$$

that follows e.g. from the Lévy representation of the fractional Laplacian (1.24) and the rotational invariance of the latter and of $|x|^\beta$. If the strength of attraction towards the origin satisfies

$$\kappa > \frac{c_{d, \alpha, \beta}}{\beta}, \quad (1.5)$$

then the SDE (1.1) has no weak solution started from $x = 0$. Indeed, supposing that there is a weak solution, and applying formally Itô's formula (1.27) to $|X_t|^\beta$, one obtains

$$\begin{aligned} \mathbb{E}|X_{t \wedge \tau_R \wedge \sigma_m}|^\beta &= \mathbb{E} \int_0^{t \wedge \tau_R \wedge \sigma_m} (-(-\Delta)^{\frac{\alpha}{2}} |X_s|^\beta - b(X_s) \cdot \nabla |X_s|^\beta) ds \\ &= \mathbb{E} \int_0^{t \wedge \tau_R \wedge \sigma_m} (c_{d,\alpha,\beta} - \kappa\beta) |X_s|^{\beta-\alpha} ds, \end{aligned} \quad (1.6)$$

where τ_R, σ_m are stopping times needed to address the growth of $|x|^\beta$ at infinity, see details in Appendix C. Now, if (1.5) holds, the left-hand side (1.6) is positive and the right-hand side is negative, which leads to a contradiction. This argument, as it is written, is formal since $|x|^\beta$ is continuous but is not smooth at the origin, so a justification of (1.6) is required (Appendix C).

SDE (1.1) can be viewed as a model (“two-particle”) case of the interacting system

$$X_t^i = X_0^i - \frac{1}{N} \sum_{j=1, j \neq i}^N \int_0^t K(X_s^i - X_s^j) ds + Z_t^i, \quad (1.7)$$

where Z_t^i are independent α -stable processes. We can reverse this relation and embed (1.7) in the class of our SDEs in the case $\alpha = 2$, and also, to some extent, in the case $1 < \alpha < 2$ (Section 3). In the Brownian case $\alpha = 2, d = 2, K(x) = \kappa x|x|^{-2} \mathbf{1}_{x \neq 0}$, particle system (1.7) becomes the finite-particle approximation of the celebrated Keller-Segel model of chemotaxis. Some sharp and detailed results are obtained in [FJ, FT] by exploiting the special form of the interaction kernel. Our condition on the drift provides us with flexible means for modifying the former, i.e. we can multiply the interaction kernel by a function having L^∞ norm less than or equal to one and still have (1.3) satisfied and, more generally, (1.2). Regarding time-inhomogeneous drifts, the dynamics of a typical particle in (1.7) as $N \rightarrow \infty$ is described by the McKean-Vlasov SDE

$$X_t = X_0 - \int_0^t (K \star \eta(s))(X_s) ds + Z_t, \quad (1.8)$$

where $\eta(t, \cdot)$ is the density of the law of X_t and \star is the convolution in the spatial variables [CJM]. Despite the regularizing effect of the convolution, this time-inhomogeneous drift is still highly singular near $t = 0$ if e.g. $\text{Law} X_0 = \delta_x$. However, if time-homogeneous interaction kernel K satisfies (1.3), then the convolution with η also satisfies (1.3) (Section 3).

To our knowledge, the present paper and the elliptic weakly form-bounded theory [KM] are the first to treat α -stable SDEs with *general*¹ drifts that can be singular enough to reach local blow-up phenomena.

In the Brownian case $\alpha = 2$, more is known; see discussion of the existing literature below.

1.1. Literature. The following is a brief survey of the literature on non-local (stable-driven) SDEs with irregular drift.

1. For bounded drifts $|b| \in L^\infty$, the weak existence and uniqueness in law for $1 < \alpha < 2$ are due to Komatsu [Ko].

¹That is, without necessarily profiting from the special structure of the drift, e.g. the regularizing effect of the convolution in distribution-dependent drifts and the regularity of the density of the initial distribution.

2. Portenko [P] and Podolynny-Portenko [PP] attained a sharp condition on the Lebesgue scale up to the strict inequality:

$$|b| \in L^q(\mathbb{R}^d) + L^\infty(\mathbb{R}^d), \quad q > \frac{d}{\alpha - 1}. \quad (1.9)$$

The time-inhomogeneous case,

$$|b| \in L^r(\mathbb{R}, L^q(\mathbb{R}^d)) + L^\infty(\mathbb{R}^{1+d}), \quad \frac{d}{q} + \frac{\alpha}{r} < \alpha - 1,$$

was handled in [J], see also the recent developments in [FJM].

3. Kim-Song [KSo] and Chen-Wang [CW] considered Kato class drifts, i.e.

$$\lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_{B_r(x)} \frac{|b(y)|}{|x - y|^{d+1-\alpha}} dy = 0 \quad (\text{in many results it suffices to require } \leq \delta \text{ for small } \delta),$$

proved the existence of a weak solution and, importantly, its uniqueness in law, and constructed a Feller semigroup, strengthening [P, PP]; [KSo] also consider measure-valued Kato class drifts. We emphasize that, for every $\varepsilon > 0$, there exist Kato class drifts b such that $|b|^{\frac{1}{\alpha-1}} \notin L_{\text{loc}}^{1+\varepsilon}(\mathbb{R}^d)$, i.e. situated far from (1.9). The Kato class condition can be restated in an equivalent form:

$$\|(\lambda + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{\alpha-1}{\alpha}} |b|\|_\infty \leq \delta \quad \left(\text{or, by duality, } \| |b| (\lambda + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{\alpha-1}{\alpha}} \|_{L^1(\mathbb{R}^d) \rightarrow L^1(\mathbb{R}^d)} \leq \delta \right) \quad (1.10)$$

(the proof can be found in [Si]). Kato class drifts arise in physical applications that are different from the ones that motivate the present work, such as diffusions on fractals; in particular, they do not introduce local blow-ups. The Kato class is essentially the largest class of general drifts for which one has the standard two-sided heat kernel bounds² of $\Lambda = (-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla$:

$$e^{-t\Lambda}(x, y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x - y), \quad (1.11)$$

see Bogdan-Jakubowski [BJ]; these two-sided bounds are used in the proofs in [KSo, CW]. In our setting, (1.11) is no longer valid, cf. (1.14). Still, we use in our proof of Theorem 1 some of the probabilistic arguments of [CW, P, PP].

In [KM], the authors considered SDE (1.1) with weakly form-bounded drift $b = b(x)$:

$$\| |b|^{\frac{1}{2}} (\lambda + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{1}{2}} (\lambda + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{\alpha-1}{\alpha}} \|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq \sqrt{\delta}. \quad (1.12)$$

This class contains the Kato class, the weak Lebesgue class $|b| \in L^{\frac{d}{\alpha-1}, \infty}(\mathbb{R}^d)$ and, more generally, $|b|^{\frac{1}{\alpha-1}}$ having sufficiently small elliptic Morrey norm

$$\| |b|^{\frac{1}{\alpha-1}} \|_{M_{1+\varepsilon}} := \sup_{x \in \mathbb{R}^d, r > 0} r \left(\frac{1}{r^d} \int_{B_r(x)} |b(y)|^{\frac{1+\varepsilon}{\alpha-1}} dy \right)^{\frac{1}{1+\varepsilon}}, \quad (1.13)$$

plus a bounded part of the drift. This class thus reaches drifts introducing local blow-up effects and destroying the standard heat kernel bounds (1.11). Namely, [KSS] extended some aspects of Nash's method to handle $\Lambda = (-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla$ with a critical attracting (weakly form-bounded) drift $b(x) = \kappa x |x|^{-\alpha}$, $\kappa \in]0, \kappa_*[$, and established the following two-sided heat kernel bounds:

$$e^{-t\Lambda}(x, y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x - y) (t^{-\frac{1}{\alpha}} |y| \wedge 1)^{-d+\gamma} \quad (1.14)$$

²Given $a(z), b(z) \geq 0$, we write $a(z) \approx b(z)$ if $c^{-1}b(z) \leq a(z) \leq cb(z)$ for some constant $c > 1$ and all admissible z .

for all $t > 0, x, y \in \mathbb{R}^d, y \neq 0$. The exponent $\gamma \in]\alpha, d]$ is uniquely determined by

$$\Lambda^* |\cdot|^{-d+\gamma} = 0 \quad (\Lambda^* \text{ is the formal adjoint of } \Lambda).$$

It is seen that already for this drift the heat kernel of Λ is singular at $y = 0$. For more general drifts (1.4) such explicit bounds are out of reach since the constants already in a bound of type (1.14) would depend on the distance between the singularities of the drift.

Concerning non-Morrey weakly form-bounded drifts, we refer to Chang-Wilson-Wolff [CWW] who improve the Morrey condition by some logarithmic-type factors (strictly speaking, here we would have to speak of a closely related class of form-bounded drifts for $\alpha = 2$ (Table 1), but it seems like the appropriate extension of [CWW] is possible with a few additional efforts). At the same time, the L^2 setting of (1.12) makes it convenient to e.g. apply Fourier transform to extend this class to other operators arising in some particle systems, see Theorem 3.

The approach of [KM] is, however, limited to the elliptic setting (or so it seems at the moment) and therefore to time-homogeneous drifts $b = b(x)$ since it uses the Stroock-Varopoulos inequalities for symmetric Markov generators. In the present paper we pursue an alternative to the argument in [KM], strengthen rather substantially the results in [KM] and also simplify some of the proofs there (Section 5).

4. There is a rich literature on the stable-driven SDE (1.1) with distributional drifts, in particular, divergence-free drifts in the Bessel or Besov scale. We refer to Athreya-Butkovsky-Mytnik [ABM], Chaudru de Raynal-Menozzi [CM], Fitoussi [F] and Kremp-Perkowski [KpP]. These results are somewhat orthogonal to what we do in the present paper, i.e. handling attraction phenomena. One interesting question that appears in relation to distributional drifts is how to define the product of two distributions $b \cdot \nabla u$ in the Kolmogorov PDE or how to interpret $\int_0^t b(s, X_s) ds$ in the SDE, see cited papers.

It is also interesting to note that one of the entry points to our analysis of the Kolmogorov PDE for (1.1), i.e. the fractional Hardy inequality

$$H_{d,\alpha} \int_{\mathbb{R}^d} |x|^{-\alpha} \varphi^2 dx \leq \int_{\mathbb{R}^d} |(-\Delta)^{\frac{\alpha}{4}} \varphi|^2 dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d), \quad (1.15)$$

can be obtained using the machinery of Besov spaces, see [BCG], moreover, they obtained a refined Hardy inequality having the additional feature of invariance under oscillations. However, the other Besov class distributions that satisfy the Hardy inequality obtained in this way must have sufficiently regular distributional derivatives, while in the present paper we are interested in conditions that involve only the drift itself (e.g. to be free to modify the interaction kernels). This observation is not a rigorous impossibility result for the SDE (1.1) since it is conditional on the use of our approach. That being said, to our knowledge, at the moment there are no well-posedness results for SDEs with Besov drifts that deal with the part of the Besov scale that allows one to handle the Hardy inequality or, more generally, deal with strong attracting singularities of the drift.

5. Strong/pathwise well-posedness for stable-driven SDEs with irregular (but more regular than (1.2) or (1.3)) drifts has been studied by Priola [Pr], who treated bounded Hölder drifts, and by X. Zhang [Z], who treated time-dependent Sobolev drifts. Related stable-like and supercritical results for Hölder drifts were obtained by Chen-Song-Zhang [CSZ], Chen-Zhang-Zhao [CZZ], and Knopova-Kulik [KnK], see also the recent result of Brzeźniak-Priola-Zhai-Zhu [BPZZ] regarding well-posedness of the stochastic transport equation with Lévy noise.

6. Surveying weak solutions to SDEs in the Brownian case $\alpha = 2$ would require another paper, so we mention only a few results dealing with general singular drifts that can introduce local blow-ups: by

Semënov and one of us [K1, KS], Röckner-Zhao [RZ], Krylov’s series of papers where he also handles VMO diffusion coefficients, including [Kr3, Kr4, Kr5], and [K3]. The last paper is a Brownian counterpart of the present paper, it was a continuation of [K1, KS]. However, the probabilistic arguments in [K3] are quite different from the present work. Also, a number of results below are new even in the case $\alpha = 2$, such as Theorem 2.

1.2. Proofs. The proof of Theorem 1 (SDE theory) has two components:

- The regularity theory of the Kolmogorov backward PDE. This is based on the following operator norm estimate (an equivalent form of (1.2))

$$\|b^{\frac{1}{p}} \cdot \nabla(\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-1} |b|^{\frac{1}{p}}\|_{L^p(\mathbb{R}^{1+d}) \rightarrow L^p(\mathbb{R}^{1+d})} < c_p \delta < 1, \quad (1.16)$$

where $b^{\frac{1}{p}} := b|b|^{-1+\frac{1}{p}}$. It allows us to construct the Duhamel series for the inhomogeneous equation

$$(\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla)u = f,$$

where (1.16) is the estimate on the denominator of the geometric series.

To extract useful information from the Duhamel series, i.e. to handle irregular f and to apply the parabolic Sobolev embedding to characterize the regularity of u , we will need two “almost-halves” of (1.16):

$$\|b^{\frac{1}{p}} \cdot \nabla(\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{(-1+\frac{1}{\alpha})\frac{1}{p}-\varepsilon_0}\|_{L^p(\mathbb{R}^{1+d}) \rightarrow L^p(\mathbb{R}^{1+d})} < \infty \quad (1.17)$$

$$\|(\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{(-1+\frac{1}{\alpha})\frac{1}{p}-\varepsilon_0} |b|^{\frac{1}{p}}\|_{L^p(\mathbb{R}^{1+d}) \rightarrow L^p(\mathbb{R}^{1+d})} < \infty, \quad (1.18)$$

where “almost” refers to $\varepsilon_0 > 0$; if $\varepsilon_0 = 0$, then the composition of these two estimates yields (1.16). The Sobolev embedding, when $p > d + 1$, provides us with the Feller propagator for SDE (1.1). It determines a family of probability measures \mathbb{P}_x on the canonical space of càdlàg paths parametrized by the initial point $x \in \mathbb{R}^d$ – these are the candidates for the weak solutions to SDE (1.1).

- Armed with these PDE regularity results and the probability measures $\{\mathbb{P}_x\}_{x \in \mathbb{R}^d}$, we apply the argument of Chen-Wang [CW], Portenko [P], Podolynny-Portenko [PP], i.e. we inspect the characteristic function and apply Banach’s contraction principle to show that

$$\mathbb{E}_{0,x} \left[e^{i\xi \cdot (\omega_t - x + \int_0^t b(s, \omega_s) ds)} \right] = e^{-t|\xi|^\alpha}, \quad \forall x, \xi \in \mathbb{R}^d, t \geq 0,$$

which is the characteristic function of the isotropic α -stable process.

The proof of Theorem 2 on the Kolmogorov forward (Fokker-Planck) equation is based on a dual variant of the PDE results used in the proof of Theorem 1. We emphasize that even if we take $\alpha = 2$ and $p = 2$, we do not end up with the standard triple of Hilbert spaces $W^{1,2} \rightarrow L^2 \rightarrow W^{-1,2}$; instead, we work in an asymmetric triple of Bessel potential spaces (cf. Theorem 2), which allows us to handle a more irregular right-hand side in the Fokker-Planck equation, and therefore more irregular initial data. By working in the asymmetric triple we lose the possibility to handle irregular diffusion coefficients, but this is not among our objectives in this work.

Condition (1.16) goes back to the work of Semënov on the regularity of eigenfunctions of Schrödinger operators, see [BS, LS], and one of us [K1] on the realization of elliptic Kolmogorov operator with singular weakly form-bounded drift $b = b(x)$ as a Feller generator. One important difference between these works is that a Schrödinger operator – having a singular form-bounded potential – is well defined

in L^2 via the KLMN theorem. One thus ventures into L^p to extract additional information about its domain. In contrast, in [K1] none of the classical methods of the perturbation theory would apply in L^p , including $p = 2$, so some rather deep semigroup theory, including Hille's theory of pseudoresolvents and Trotter's approximation theorem, had to be used. By working in the parabolic setting directly, we will use estimate (1.16) on *Semënov's denominator* to postulate the Duhamel series for the Kolmogorov equations, sacrificing a few spatial singularities compared to the weakly form-bounded drifts, but avoiding all the difficulties arising in the elliptic setting and also gaining critical singularities of the drift in time.

At the time of writing, we are aware of three ways to verify (1.16) or of its elliptic counterpart, see Table 1.

TABLE 1. Ways to verify estimates (1.16)-(1.18)

Condition on drift b	α	Features
1. Form-bounded $b = b(x)$ $\ b (\lambda - \Delta)^{-\frac{1}{2}} \ _{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq \delta$	$\alpha = 2$	Allows for explicit control of admissible δ for SDE [K4] Has a parabolic variant
2. Weakly form-bounded $b = b(x)$	$1 < \alpha \leq 2$	Largest class of spatial singularities δ is sufficiently small [KM, K1]
3. Morrey class condition (1.3) on $b = b(t, x)$	$1 < \alpha \leq 2$	Has parabolic variant Morrey norm ($\approx \delta$) is small Spatial singularities in-between 1 & 2 Can be split into "halves"

In this paper we pursue the third condition, i.e. the Morrey class (1.3). In fact, for the Morrey class it is possible to obtain (1.16), using the boundedness of the parabolic Riesz transform on $L^p(\mathbb{R}^{1+d})$, from the following two "halves": given $0 < \varepsilon (< d + \alpha - 1)$, for every $1 < p < \infty$,

$$\| |b|^{\frac{1}{p}} (\pm \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{(-1 + \frac{1}{\alpha})\frac{1}{p}} \|_{L^p(\mathbb{R}^{1+d}) \rightarrow L^p(\mathbb{R}^{1+d})} \leq c_{d,\alpha,p,\varepsilon} \| |b|^{\frac{1}{\alpha-1}} \|_{E_{1+\varepsilon}^{\frac{\alpha-1}{p}}}, \quad (1.19)$$

and, by duality,

$$\| (\pm \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{(-1 + \frac{1}{\alpha})\frac{1}{p'}} |b|^{\frac{1}{p'}} \|_{L^p(\mathbb{R}^{1+d}) \rightarrow L^p(\mathbb{R}^{1+d})} \leq c_{d,\alpha,p',\varepsilon} \| |b|^{\frac{1}{\alpha-1}} \|_{E_{1+\varepsilon}^{\frac{\alpha-1}{p'}}} \quad (1.20)$$

(so one actually can take $\varepsilon_0 = 0$ in (1.17), (1.18) – but this is not possible for weakly form-bounded drifts, for which (1.17), (1.18) require $\varepsilon_0 > 0$). See Section 4 for precise statement and the history of such estimates.

Let us also add that for the class of form-bounded drifts one can manage the gradient instead of getting rid of it via Riesz transform or even using pointwise heat kernel estimates, arriving at better conditions on δ , but this requires some rather delicate and strong gradient bounds [K4]; it is not clear at the moment how to obtain them for the other classes.

Our class of drifts is large enough to destroy L^p estimates, for p large, on the order α spatial derivatives of solutions of the Kolmogorov backward equation. Even in the case $\alpha = 2$, this excludes the possibility to also incorporate discontinuous diffusion coefficients, such as VMO; the reason is the size of our class of singular drifts.

Theorems 1, 2 allow us to describe quantitatively how the theory of SDE (1.1) and of the corresponding Kolmogorov equations changes if we change the drift b to cb for a constant $c > 0$ or, in other words, adjust the Morrey norm (form-bound) of the drift. Obtaining such a quantitative description is, essentially, our ultimate goal in the present work.

1.3. Notations.

(a) Put

$$\langle h \rangle := \int_{\mathbb{R}^{1+d}} h dz, \quad \langle h, g \rangle := \langle hg \rangle.$$

(b) Denote by $\mathcal{B}(X, Y)$ the space of bounded linear operators between Banach spaces $X \rightarrow Y$, endowed with the operator norm $\|\cdot\|_{X \rightarrow Y}$, and set $\mathcal{B}(X) := \mathcal{B}(X, X)$.

(c) We write $T = s\text{-}X\text{-}\lim_n T_n$ for $T, T_n \in \mathcal{B}(X)$ if $Tf = \lim_n T_n f$ in X for every $f \in X$.

(d) Put

$$\|h\|_p \equiv \|h\|_{L^p(\mathbb{R}^{1+d})} := \langle |h|^p \rangle^{\frac{1}{p}}.$$

Denote by $\|\cdot\|_{p \rightarrow q} := \|\cdot\|_{L^p(\mathbb{R}^{1+d}) \rightarrow L^q(\mathbb{R}^{1+d})}$ the corresponding operator norm.

(e) Let

$$C_\infty(\mathbb{R}^d) := \{f \in C(\mathbb{R}^d) \mid \lim_{|x| \rightarrow \infty} f(x) = 0\},$$

endowed with the sup-norm.

(f) We denote by $C_b(\mathbb{R}^d)$ the space of bounded continuous functions on \mathbb{R}^d , and by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of smooth functions rapidly decaying at infinity.

(g) We will need the polynomial weight

$$\rho(x) = (1 + \kappa|x|^2)^{-\frac{\theta}{2}}, \quad \theta > d, \quad \kappa > 0. \quad (1.21)$$

The derivatives of both ρ and ρ^{-1} are bounded from above by ρ and ρ^{-1} , respectively, times a constant that can be made as small as needed by selecting κ sufficiently small. Define the weighted Lebesgue space

$$L_\rho^p(\mathbb{R}^{1+d}) := L^p(\mathbb{R}^{1+d}, \rho(x) dt dx).$$

The dual space of $L_\rho^p(\mathbb{R}^{1+d})$ is $L_{\rho^{-1}}^{p'}(\mathbb{R}^{1+d})$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

(h) Recall that the integral kernel $q_\gamma(t, x)$, $0 < \gamma \leq 2$, of the operators ($\lambda \geq 0$)

$$(\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{\gamma}{2}} f(t, x) = \int_{\mathbb{R}^{1+d}} e^{-\lambda(t-s)} q_\gamma(t-s, x-y) f(s, y) dy ds$$

$$(\lambda - \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{\gamma}{2}} f(t, x) = \int_{\mathbb{R}^{1+d}} e^{-\lambda(s-t)} q_\gamma(s-t, x-y) f(s, y) dy ds$$

satisfies the pointwise bounds

$$c p_\gamma(t, x) \leq q_\gamma(t, x) \leq C p_\gamma(t, x) \quad (1.22)$$

for

$$p_\gamma(t, x) := \mathbf{1}_{\{t>0\}} t^{\frac{\gamma}{2}-1} \left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x|^{d+\alpha}} \right) = \mathbf{1}_{\{t>0\}} \begin{cases} t^{\frac{\gamma}{2}} |x|^{-d-\alpha}, & |x| \geq t^{\frac{1}{\alpha}}, \\ t^{\frac{\gamma}{2}-\frac{d+\alpha}{\alpha}}, & |x| < t^{\frac{1}{\alpha}}, \end{cases}$$

with constants $0 < c, C < \infty$ depending only on d, α , and γ . Furthermore,

$$|\nabla_x q_\gamma(t, x)| \leq C_1 p_{\gamma-\frac{\alpha}{2}}(t, x). \quad (1.23)$$

(i) Lévy representation for the fractional Laplacian:

$$-(-\Delta)^{\frac{\alpha}{2}} f(x) = c_{d,\alpha} \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \mathbf{1}_{\{|z| \leq 1\}} \nabla f(x) \cdot z \right) \frac{dz}{|z|^{d+\alpha}}. \quad (1.24)$$

(j) Let $\mathbb{W}_\alpha^{\gamma,p}(\mathbb{R}^{1+d})$ denote the fractional parabolic Bessel potential space:

$$\mathbb{W}_\alpha^{\gamma,p}(\mathbb{R}^{1+d}) := (\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{\gamma}{2}} L^p(\mathbb{R}^{1+d})$$

with norm

$$\|g\|_{\mathbb{W}_\alpha^{\gamma,p}} := \|(\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{\frac{\gamma}{2}} g\|_p.$$

(k) The parabolic Sobolev embedding theorem: for all $1 \leq p \leq q \leq \infty$ such that

$$\frac{\gamma}{2} > \left(1 + \frac{d}{\alpha}\right) \left(\frac{1}{p} - \frac{1}{q}\right), \quad (1.25)$$

one has

$$(\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{\gamma}{2}} \in \mathcal{B}(L^p, L^q), \quad \text{so } \mathbb{W}_\alpha^{\gamma,p} \subset L^q(\mathbb{R}^{1+d}). \quad (1.26)$$

See [AS].

We refer to the recent comprehensive treatment of the theory of elliptic and parabolic Morrey spaces by Krylov [Kr2]. See also Adams-Xiao [AX].

(l) Let I be either $\mathbb{R}_+ := [0, \infty[$ or a closed interval in \mathbb{R}_+ . We denote by $D = D(I, \mathbb{R}^d)$ the space of càdlàg paths $I \rightarrow \mathbb{R}^d$.

(m) A weak solution to the SDE (1.1) is a pair of càdlàg processes (X_t, Z_t) defined on a complete probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq s}, \mathcal{F}, \mathbb{P})$ with a right-continuous filtration, such that $X_0 = x$ \mathbb{P} -a.s.,

$$\int_0^T |b(s, X_s)| ds < \infty \quad \mathbb{P}\text{-a.s.},$$

Z_t is an isotropic α -stable process and the identity (1.1) holds \mathbb{P} -a.s.

In Theorem 1, we construct a weak solution to the SDE (1.1) by taking $\Omega := D([s, T], \mathbb{R}^d)$ (for $0 \leq s < T < \infty$ fixed), i.e. on the canonical space of càdlàg paths

$$\mathbf{D}_{s,T} := (D([s, T], \mathbb{R}^d), \{\mathcal{F}_t\}_{t \geq s}, \mathcal{F}),$$

where \mathcal{F}_t is the canonical filtration (complete, right-continuous). We will define $X_t(\omega) := \omega(t)$ to be the coordinate process and will define Z_t via identity (1.1). Our goal thus will be to find a probability measure on $\mathbf{D}_{s,T}$ that makes Z_t an isotropic α -stable process.

(n) The Itô formula for X_t , a weak solution of SDE (1.1), implies, in particular, that for every bounded $f \in C_t^1 C_x^2(\mathbb{R} \times \mathbb{R}^d)$ having bounded derivatives

$$f(t, X_t) = f(0, x) + \int_0^t (\partial_s f - b \cdot \nabla f - (-\Delta)^{\frac{\alpha}{2}} f)(s, X_s) ds + M_t, \quad (1.27)$$

where M_t is the compensated jump martingale part.

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2. MAIN RESULTS

We will state our results first in terms of the Morrey class norm – this is an easy-to-verify condition. Our Morrey class hypothesis will be of two types:

(H) Let $b : \mathbb{R}^{1+d} \rightarrow \mathbb{R}^d$ be a Borel measurable vector field of the form

$$b = b_s + b_b, \quad (2.1)$$

where $|b_b| \in L^\infty(\mathbb{R}^{1+d})$ (the bounded part) and the singular part has sufficiently small Morrey norm, i.e.

$$\| |b_s|^{\frac{1}{\alpha-1}} \|_{E_{1+\epsilon}} < c_{d,\alpha,\epsilon},$$

where $\epsilon > 0$ is fixed small³.

(H₁) Let (H) hold, and assume additionally that

$$|b_s| \in L^1(\mathbb{R}^{1+d}).$$

Fix some $T > 0$.

Theorem 1 (SDE theory). *Let $d \geq 2$. Assume that the Borel measurable drift $b : \mathbb{R}^{1+d} \rightarrow \mathbb{R}^d$ satisfies (H₁). (Or, in complete generality, assume that the operator norms in (5.6), (5.7) are sufficiently small.) Then the following are true:*

(i) (Weak existence) *For each $x \in \mathbb{R}^d$ and $0 \leq s < T$, there exists a probability measure $\mathbb{P}_{s,x}$ on $\mathbf{D}_{s,T}$ such that*

(a)

$$\mathbb{P}_{s,x}[\omega_s = x] = 1,$$

(b)

$$\mathbb{E}_{s,x} \int_s^t |b(r, \omega_r)| dr < \infty, \quad t \geq s,$$

(c)

$$\omega_t = x - \int_s^t b(r, \omega_r) dr + Z_t - Z_s,$$

for an α -stable process Z_t defined on the completion of $\mathbf{D}_{s,T}$ with respect to $\mathbb{P}_{s,x}$.

That is, the pair (coordinate process X_t , Z_t defined via (c)) is a weak solution to the SDE (1.1).

Also, assuming that the constant θ in the definition of weight ρ is fixed to satisfy $d < \theta < d + \alpha$, and κ is fixed sufficiently small, the following Krylov-type bound holds: for every $x \in \mathbb{R}^d$ and any Borel vector field $f : \mathbb{R}^{1+d} \rightarrow \mathbb{R}^d$ that admits decomposition (2.1) with finite Morrey norm of the singular part, for every $p > d + 1$ there exists a constant K such that

$$\mathbb{E}_{s,x} \int_s^T |f(r, \omega_r) h(r, \omega_r)| dr \leq K \| \mathbf{1}_{[0,T]} |f|^{\frac{1}{p}} h \|_{L_p^p}, \quad (2.2)$$

for all $h \in C_b(\mathbb{R}^{1+d})$. In particular, taking $f = b$, we obtain that the constructed weak solution to (1.1) does not spend too much time around the singular set of the drift. This Krylov-type bound also holds for $|f| = 1$ and $h \in (C_b \cap L^p)(\mathbb{R}^{1+d})$.

³The constant $c_{d,\alpha,\epsilon} > 0$ depends on the constants in some fundamental inequalities of harmonic analysis.

(ii) (Feller propagator) *The operators*

$$P^{t,r}f(x) := \mathbb{E}_{t,x}[f(X_r)], \quad 0 \leq t \leq r, \quad x \in \mathbb{R}^d \quad (2.3)$$

constitute a backward Feller propagator on C_∞ .

(iii) (Conditional weak uniqueness) *Let $x \in \mathbb{R}^d$. Any probability measure $\mathbb{Q}_{s,x}$ on $\mathbf{D}_{s,T}$ that satisfies (a)-(c) in (i) and, for some $p > d + 1$ from (i),*

$$\mathbb{E}_{\mathbb{Q}_{s,x}} \int_s^T |h(r, \omega_r)| dr \leq K \|\mathbf{1}_{[s,T]} h\|_p,$$

and

$$\mathbb{E}_{\mathbb{Q}_{s,x}} \int_s^T |b(r, \omega_r)h(r, \omega_r)| dr \leq K \|\mathbf{1}_{[s,T]} |b|^{\frac{1}{p}} h\|_p$$

for all $h \in (C_b \cap L^p)(\mathbb{R}^{1+d})$ also satisfies

$$\mathbb{Q}_{s,x} = \mathbb{P}_{s,x}$$

(it is seen that the weak solution constructed in (i) satisfies the previous two inequalities).

We consider the uniqueness class of (iii) – rather than the standard uniqueness class in terms of the Krylov bound – since this allows us to capture the following effect: the uniqueness class becomes larger as b becomes more regular. Namely, a nice b (that is, one having small Morrey norm) allows us to take a large p . So, in principle, in the limit, $p \rightarrow \infty$, e.g. bounded $|b|$, we recover the uniqueness in law.

Another form of conditional weak uniqueness of X_t is established in the next theorem.

We fix the following regularization of b :

$$b_n := b_{n,b} + b_{n,s}, \quad n = 1, 2, \dots, \quad (2.4)$$

where

$$b_{n,b} = \gamma_{\varepsilon_n} \star \mathbf{1}_n b_b, \quad b_{n,s} = \gamma_{\varepsilon_n} \star \mathbf{1}_n b_s, \quad (2.5)$$

and

$$\mathbf{1}_n := \mathbf{1}_{\{(t,x) \in \mathbb{R}^{1+d} \mid |t| \leq n, |x| \leq n, |b(t,x)| \leq n\}},$$

$\{\gamma_\varepsilon\}_{\varepsilon > 0}$ is the standard Friedrichs mollifier on \mathbb{R}^{1+d} with compact support. We can select $\varepsilon_n \downarrow 0$ sufficiently rapidly to have $\sup_n \| |b_{n,s}|^{\frac{1}{\alpha-1}} \|_{E_{1+\varepsilon}}$ as close to $\| |b_s|^{\frac{1}{\alpha-1}} \|_{E_{1+\varepsilon}}$ as needed, say,

$$\| |b_{n,s}|^{\frac{1}{\alpha-1}} \|_{E_{1+\varepsilon}} \leq (1 + 2^{-n}) \| |b_s|^{\frac{1}{\alpha-1}} \|_{E_{1+\varepsilon}}.$$

Alternatively, we can multiply b_n defined as above by $0 < c_n \uparrow 1$ (sufficiently slowly) to have inequality $\| |b_{n,s}|^{\frac{1}{\alpha-1}} \|_{E_{1+\varepsilon}} \leq \| |b_s|^{\frac{1}{\alpha-1}} \|_{E_{1+\varepsilon}}$ for all $n = 1, 2, \dots$.

Remark 1. By selecting $\varepsilon_n \downarrow 0$ sufficiently rapidly, we can make b_n as close to $\mathbf{1}_n b$ as needed in any $L^s(\mathbb{R}^{1+d})$, $1 \leq s < \infty$, so that we can treat $\{|b_n|\}_{n \geq 1}$ as essentially a monotone approximation of $|b|$, in the same way as $\{\mathbf{1}_n b\}_{n \geq 1}$ is.

Let us add that the role of $\lambda > 0$ below is to handle the bounded part of the drift, and also to let us work with nicer Sobolev embeddings. We also recall that λ enters the definition of the parabolic Bessel potential spaces $\mathbb{W}_\alpha^{\gamma,p}$.

Theorem 2 (PDE theory). *Assume that the Borel measurable drift $b : \mathbb{R}^{1+d} \rightarrow \mathbb{R}^d$ satisfies **(H)**. (Or, in complete generality, assume that the operator norms in (4.1), (4.2) are sufficiently small.) Then the following are true:*

(i) For every $1 < p < \infty$, there exist constants $c_{d,\alpha,p,\epsilon} > 0$ and $\lambda_{d,\alpha,p,\epsilon} \geq 0$ such that if

$$\| |b_s|^{\frac{1}{\alpha-1}} \|_{E_{1+\epsilon}} < c_{d,\alpha,p,\epsilon}$$

then, given a right-hand side

$$f \in \mathbb{W}_\alpha^{-\frac{2}{\alpha} + (-1 + \frac{1}{\alpha})\frac{2}{p'}, p}(\mathbb{R}^{1+d}) \quad (2.6)$$

and a sequence of functions $\{f_n\} \subset L^\infty(\mathbb{R}^{1+d})$ such that

$$f_n \rightarrow f \quad \text{in } \mathbb{W}_\alpha^{-\frac{2}{\alpha} + (-1 + \frac{1}{\alpha})\frac{2}{p'}, p},$$

the solutions η_n of the approximating inhomogeneous Kolmogorov-Fokker-Planck equations

$$\lambda \eta_n + \partial_t \eta_n + (-\Delta)^{\frac{\alpha}{2}} \eta_n - \operatorname{div}(b_n \eta_n) = f_n, \quad \lambda > \lambda_{d,\alpha,p,\epsilon},$$

where $\{b_n\}$ is defined in (2.4), converge as $n \rightarrow \infty$ in $\mathbb{W}_\alpha^{\frac{\alpha-1}{\alpha} \frac{2}{p'}, p}$ to the function (formal Duhamel series)

$$\eta = (\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-(-1 + \frac{1}{\alpha})\frac{1}{p}} (1 + G_p H_p)^{-1} (\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{1}{\alpha} + (-1 + \frac{1}{\alpha})\frac{1}{p'}} f \quad (2.7)$$

where

$$G_p := \nabla \cdot (\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{1}{\alpha} + (-1 + \frac{1}{\alpha})\frac{1}{p'}} b^{\frac{1}{p}},$$

$$H_p := |b|^{\frac{1}{p}} (\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-(-1 + \frac{1}{\alpha})\frac{1}{p}},$$

are bounded operators on $L^p(\mathbb{R}^{1+d})$, and the $L^p \rightarrow L^p$ operator norm of the denominator of the geometric series in (2.7) is dominated by

$$\|G_p\|_{p \rightarrow p} \|H_p\|_{p \rightarrow p} < 1,$$

so the series converges in $L^p(\mathbb{R}^{1+d})$. It follows that $\eta \in \mathbb{W}_\alpha^{(1 - \frac{1}{\alpha})\frac{2}{p'}, p}$.

Another representation for η :

$$\eta = (\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-1} f - (\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-(-1 + \frac{1}{\alpha})\frac{1}{p}} G_p (1 + H_p G_p)^{-1} H_p (\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{1}{\alpha} + (-1 + \frac{1}{\alpha})\frac{1}{p'}} f. \quad (2.8)$$

(ii) For a fixed $1 < p < \infty$ from (i), given a right-hand side f satisfying (2.6), the function η given by (2.7) is the unique L^p weak solution (see the definition after the theorem) to the inhomogeneous Kolmogorov-Fokker-Planck equation

$$\lambda \eta + \partial_t \eta + (-\Delta)^{\frac{\alpha}{2}} \eta - \operatorname{div}(b \eta) = f. \quad (2.9)$$

(iii) In particular, whenever $1 < p < \frac{d+1}{d}$ in (ii), the following Cauchy problem for the Kolmogorov-Fokker-Planck equation

$$\begin{cases} \lambda \eta + \partial_t \eta + (-\Delta)^{\frac{\alpha}{2}} \eta - \operatorname{div}(b \eta) = 0, \\ \eta|_{t=0} = \delta_x, \end{cases} \quad (2.10)$$

has a unique weak solution $\eta(t, \cdot)$. This weak solution corresponds to selecting the right-hand side $f := \delta_{t=0} \delta_x$ in (ii) – it satisfies (2.6).

The following definition of the weak solution to PDE (2.9) is obtained from (2.9) by formally multiplying it by a test function φ and integrating by parts:

Definition. Let $1 < p < \infty$. An L^p weak solution to (2.9) is a Borel measurable function $\eta : \mathbb{R}^{1+d} \rightarrow \mathbb{R}$ having regularity

$$\eta \in \mathbb{W}_\alpha^{(1-\frac{1}{\alpha})\frac{2}{p}, p} \quad (2.11)$$

such that the following identity holds:

$$\begin{aligned} & \langle (\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{(1-\frac{1}{\alpha})\frac{1}{p}} \eta, (\lambda - \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{1-(1-\frac{1}{\alpha})\frac{1}{p}} \varphi \rangle \\ & + \langle H_p(\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{(1-\frac{1}{\alpha})\frac{1}{p}} \eta, -G_p^*(\lambda - \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{1-(1-\frac{1}{\alpha})\frac{1}{p}} \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^{1+d}), \end{aligned} \quad (2.12)$$

where the operators H_p, G_p are from Theorem 2.

By dealing with operators $\pm \partial_t + (-\Delta)^{\frac{\alpha}{2}}$, we consider time as another variable: in the analysis of parabolic equations one encounters other arguably non-physical constructions involving the time variable, such as complex time when dealing with holomorphic semigroups. We also consider Morrey norm $\|\cdot\|_{E_{1+\epsilon}}$ where t and x play equal roles in scaling. This, however, appears to be restrictive for further development of Theorems 1, 2, e.g. applications to McKean-Vlasov equations. It would be reasonable to consider more general anisotropic Morrey norms; such norms are discussed in [Kr2].

3. SOME APPLICATIONS TO PARTICLE SYSTEMS

Let us make a few comments, rather for illustration purposes, regarding applications of our results to the finite particle system (1.7) and to the McKean-Vlasov PDE behind (1.8). Regarding particle system (1.7), Theorems 1 and 2 require a modification – unless $\alpha = 2$ – namely, replacing the unperturbed operator by a quasi-isotropic α -stable generator

$$\sum_{i=1}^N (-\Delta)_{x^i}^{\frac{\alpha}{2}},$$

acting on $x = (x^1, \dots, x^N) \in \mathbb{R}^{dN}$. To this end, we can either use our proofs but employing a variant of Proposition 1 for the heat kernel

$$e^{-t \sum_{i=1}^N (-\Delta)_{x^i}^{\frac{\alpha}{2}}}(x - y) = \prod_{i=1}^N e^{-t (-\Delta)_{x^i}^{\frac{\alpha}{2}}}(x^i - y^i),$$

or follow the approach of [KM], i.e. the elliptic argument using estimates for weakly form-bounded drifts obtained via Stroock-Varopoulos inequalities. We will pursue the latter, taking advantage of its somewhat more abstract form.

Theorem 3. Let the interaction kernel $K : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be Borel measurable. The following are true:

- (i) (Finite particle system) Assume additionally that K is weakly form-bounded with respect to the fractional Laplacian on \mathbb{R}^d , i.e. $|K| \in L_{\text{loc}}^1(\mathbb{R}^d)$ and

$$\| |K|^{\frac{1}{2}} (\lambda + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{1}{2} \frac{\alpha-1}{\alpha}} \|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq \sqrt{\delta}$$

for some $\lambda > 0$. For instance, K satisfies the elliptic Morrey class condition

$$|K|^{\frac{1}{\alpha-1}} \in M_{1+\epsilon}$$

with sufficiently small elliptic Morrey norm (see (1.13)). Then there exists a constant $C_{d,N,\alpha}$ such that if $\delta < C_{d,N,\alpha}$, then there exist (unique, see below) probability measures $\{\mathbb{P}_x\}_{x \in \mathbb{R}^{dN}}$

on the canonical space of càdlàg paths $\mathbf{D} = \mathbf{D}_{[0, \infty[}$ that deliver a weak solution to the particle system

$$X_t^i = x^i - \frac{1}{N} \sum_{j=1, j \neq i}^N \int_0^t K(X_s^i - X_s^j) ds + Z_t^i, \quad (3.1)$$

where Z_t^i are independent α -stable processes, and determine a strongly continuous Feller semigroup via

$$e^{-t\Lambda} f(x) := \mathbb{E}_{\mathbb{P}_x} f(X_t^1, \dots, X_t^N), \quad f \in C_\infty(\mathbb{R}^{dN}),$$

where Λ is an appropriate operator realization of $\sum_{i=1}^N (-\Delta)_{x^i}^{\frac{\alpha}{2}} + \frac{1}{N} \sum_{i=1}^N \sum_{j=1, j \neq i}^N K(x^i - x^j) \cdot \nabla_{x^i}$ on C_∞ .

- (ii) (A priori Sobolev regularity for McKean-Vlasov equation) Assume that K satisfies the elliptic Morrey class condition

$$|K|^{\frac{1}{\alpha-1}} \in M_{1+\epsilon}$$

with sufficiently small Morrey norm. Then, for every $x \in \mathbb{R}^d$, the solution to the Cauchy problem for the McKean-Vlasov equation

$$\begin{cases} \lambda w + \partial_t w + (-\Delta)^{\frac{\alpha}{2}} w - \operatorname{div}((K \star w)w) = 0, \\ w|_{t=0} = \delta_x, \end{cases}$$

satisfies $w \in \mathbb{W}_\alpha^{\frac{\alpha-1}{\alpha} \frac{2}{p}, p} \cap L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^d))$ and

$$\begin{aligned} & (\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{\frac{\alpha-1}{\alpha} \frac{1}{p}} w \\ &= (\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-1 + \frac{\alpha-1}{\alpha} \frac{1}{p}} \delta_{t=0} \delta_x - \nabla (\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-1 + \frac{\alpha-1}{\alpha} \frac{1}{p}} [(K \star w)w]. \end{aligned} \quad (3.2)$$

Here we assume additionally that K is smooth to ensure the existence and uniqueness of w , but the Sobolev norm $\|w\|_{\mathbb{W}_\alpha^{\frac{\alpha-1}{\alpha} \frac{2}{p}, p}}$ is independent of the smoothness of K .

Theorem 3(i) expands the class of admissible singularities of the interaction kernel in (3.1), but it does not yet settle the question of the weak well-posedness of (3.1) since the condition on the interaction strength δ depends on N .

Theorem 3(ii) stays at the a priori level and exploits the fact that

$$K \star w \in E_{1+\epsilon} \quad (3.3)$$

(via, basically, Young's inequality for Morrey spaces), so Theorem 2 applies. In fact, this was done in [K2], but we include the details for the reader's convenience. Theorem 2 and (3.3) open up a way for launching the contraction principle to establish existence and uniqueness of the mild solution to the a posteriori (K no longer smooth) McKean-Vlasov equation, i.e. such that identity (3.2) holds. However, here we run into some limitations of our choice of Morrey class that treats time t and the spatial variable x in the same way, modulo the adjustment of scaling. We plan to address this elsewhere.

Remark 2 (On the critical strength of attraction). 1. In the Brownian case $\alpha = 2$, $d \geq 3$, one can prove weak existence, as well as some uniqueness results and the strong Markov property, for the particle system (3.1) with form-bounded interaction kernel (Table 1) for the strength of interaction δ going up to (but excluding) the first blow-up threshold, i.e. before the particles start to collide in finite time with positive probability and the PDE theory fails, see [K5, KS2]. This is done there using De Giorgi's method carried out in L^p where p is chosen as a function of the proximity of δ to the critical threshold.

2. When $d = 2$ and $K(x) = \kappa x|x|^{-2}$, some PDE results for the particle system (3.1), including heat kernel bounds, can still be proved, but, at the moment, the admissible strength of attraction degenerates, although very slowly, to zero as $N \rightarrow \infty$ [BK]. The probabilistic methods cover all strengths of attraction [FJ]; of course, they focus on well-posedness results of different character.

3. In the case $1 < \alpha \leq 2$ and $N = 2$, one has a PDE theory for the strength of attraction *equal* to the first blow-up threshold – this requires working in the hyperbolic Orlicz space [K6].

4. RELICH FROM MORREY

The proofs of Theorems 1, 2 use the following proposition as a means to verify the Rellich-type perturbation estimate (1.16) – it is needed to construct the Duhamel series in Theorem 2 and in the proof of Theorem 1.

Proposition 1. *Assume that the Borel measurable vector field $b : \mathbb{R}^{1+d} \rightarrow \mathbb{R}^d$ satisfies (H). Then, for every $1 < p < \infty$, for all $\lambda > 0$,*

$$\begin{aligned} & \| |b|^{\frac{1}{p}} (\lambda \pm \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{(-1 + \frac{1}{\alpha})\frac{1}{p}} \|_{L^p(\mathbb{R}^{1+d}) \rightarrow L^p(\mathbb{R}^{1+d})} \\ & \leq c_{d,\alpha,p,\epsilon} \| |b_s|^{\frac{1}{\alpha-1}} \|_{E_{1+\epsilon}}^{\frac{\alpha-1}{p}} + c\lambda^{(-1 + \frac{1}{\alpha})\frac{1}{p}} \| b_b \|_{\infty}^{\frac{1}{p}}, \end{aligned} \quad (4.1)$$

and, by duality,

$$\begin{aligned} & \| (\lambda \pm \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{(-1 + \frac{1}{\alpha})\frac{1}{p'}} |b|^{\frac{1}{p'}} \|_{L^p(\mathbb{R}^{1+d}) \rightarrow L^p(\mathbb{R}^{1+d})} \\ & \leq c_{d,\alpha,p',\epsilon} \| |b_s|^{\frac{1}{\alpha-1}} \|_{E_{1+\epsilon}}^{\frac{\alpha-1}{p'}} + c\lambda^{(-1 + \frac{1}{\alpha})\frac{1}{p'}} \| b_b \|_{\infty}^{\frac{1}{p'}}. \end{aligned} \quad (4.2)$$

The elliptic counterparts of estimates (1.19), (1.20) are due to Adams [A, Theorem 7.3] who, in fact, worked in the more general setting of A_p weights. Krylov obtained a parabolic analogue of Adams' result in [Kr1] for $\alpha = 2$ and for integer powers of $|b|$, as required for his approach to SDEs with singular drift and discontinuous diffusion coefficients. Estimates (1.19), (1.20) were obtained in [K3] by extending the Adams-Krylov argument to handle $|b|^{1/p}$ and also to include $\alpha < 2$. We use these estimates differently from Krylov. That said, the proof of Proposition 1 in [K3] is not far from the original arguments, so it is fair to refer to these estimates as *non-local Adams-Krylov estimates*.

For the reader's convenience, we include the proof of Proposition 1 in Appendix A.

5. LOCALIZATION

The proof of Theorem 1 requires us to control the convergence of the Duhamel series in the weighted Lebesgue space $L^p_\rho(\mathbb{R}^{1+d}) = L^p(\mathbb{R}^{1+d}, \rho(x)dxdt)$, where

$$\rho(x) = (1 + \kappa|x|^2)^{-\frac{\theta}{2}}, \quad \theta > d, \quad \kappa > 0. \quad (5.1)$$

This will be achieved via a localized (\equiv weighted) variant of Proposition 1.

1. In the case $\alpha = 2$, one can simply commute the weight ρ with $-\Delta$ and establish a weighted variant of Proposition 1 based on the fact that the derivatives of ρ are dominated by the weight itself times a constant that becomes smaller as $\kappa \downarrow 0$, see [K3]. In the case $1 < \alpha < 2$, such direct calculations are problematic.

2. In [KM], the authors considered weakly form-bounded drifts $b = b(x)$ (see the definition in the introduction), and, having at hand the estimate

$$\| |b|^{\frac{1}{p}} (\lambda + (-\Delta)^{\frac{\alpha}{2}})^{-1 + \frac{1}{\alpha}} |b|^{\frac{1}{p'}} \|_{L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)} \leq \delta, \quad (5.2)$$

obtained its localized counterpart

$$\| |b|^{\frac{1}{p}} (\lambda + (-\Delta)^{\frac{\alpha}{2}})^{-1 + \frac{1}{\alpha}} |b|^{\frac{1}{p'}} \|_{L^p(\mathbb{R}^d, \rho(x) dx) \rightarrow L^p(\mathbb{R}^d, \rho(x) dx)} \leq C\delta, \quad (5.3)$$

by showing that the fractional Laplacian in the weighted space is still the generator of a symmetric Markov semigroup, so that the Stroock-Varopoulos inequalities needed to deduce (5.2) from the weak form-boundedness of b still apply. This led to a rather lengthy proof. We first show that in the setting of [KM] there is a simple way to deduce (5.3) from (5.2) in an important special case $\lambda = 0$ (covered e.g. by the fractional Hardy inequality (1.15) and related Hardy-type inequalities). We will use Proposition 2 in the proof of Theorem 3(i).

Proposition 2. *Assume that (5.2) holds with $\lambda = 0$. Then (5.3) holds for $p \geq \frac{\theta}{2\alpha-1}$, for all $\lambda > 0$.*

Proof. Without loss of generality, we take $\lambda = 1$ in the following proof. We first note that for $\gamma > 0$

$$\begin{aligned} (1 + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{\gamma}{2}}(x - y) &= \int_0^\infty e^{-t} t^{\frac{\gamma}{2}-1} e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x - y) dt \\ &\quad (\text{use the standard upper bound on the heat kernel of the fractional Laplacian}) \\ &\leq C_1 \int_0^\infty e^{-t} t^{\frac{\gamma}{2}-1} (t^{-\frac{d}{\alpha}} \wedge t |x - y|^{-d-\alpha}) dt \\ &\leq C_2 \begin{cases} |x - y|^{-d-\alpha} & \text{if } |x - y| > 1 \\ |x - y|^{-d+\frac{\gamma\alpha}{2}} & \text{if } |x - y| \leq 1. \end{cases} \end{aligned} \quad (5.4)$$

Also, note that

$$(1 + |y|^2) \leq (1 + 2|x - y|^2 + 2|x|^2) \leq 2(1 + |x - y|^2)(1 + |x|^2), \quad x, y \in \mathbb{R}^d.$$

Therefore, we can estimate the integral kernel of “weighted” operator $\rho^{\frac{1}{p}}(1 + (-\Delta)^{\frac{\alpha}{2}})^{-1 + \frac{1}{\alpha}} \rho^{-\frac{1}{p}}$:

$$\begin{aligned} &[\rho^{\frac{1}{p}}(1 + (-\Delta)^{\frac{\alpha}{2}})^{-1 + \frac{1}{\alpha}} \rho^{-\frac{1}{p}}](x, y) \\ &\quad (\text{without loss of generality, } \kappa = 1 \text{ in the definition of } \rho) \\ &\leq (1 + |x|^2)^{-\frac{\theta}{2p}} (1 + (-\Delta)^{\frac{\alpha}{2}})^{-1 + \frac{1}{\alpha}}(x - y) (1 + |y|^2)^{\frac{\theta}{2p}} \\ &\leq 2^{\frac{\theta}{2p}} (1 + |x - y|^2)^{\frac{\theta}{2p}} (1 + (-\Delta)^{\frac{\alpha}{2}})^{-1 + \frac{1}{\alpha}}(x - y). \end{aligned} \quad (5.5)$$

We now apply the bound (5.4). That is, if $|x - y| \leq 1$, then the right-hand side of (5.5) is dominated by $|x - y|^{-d+\alpha-1}$ times a generic constant. If $|x - y| > 1$, then (5.5) is dominated by $|x - y|^{\frac{\theta}{p}-d-\alpha}$, in which case, if p is chosen so that $\frac{\theta}{p} - \alpha \leq \alpha - 1$, then (5.5) $\leq |x - y|^{-d+\alpha-1}$ (times a constant). To summarize, if $p \geq \frac{\theta}{2\alpha-1}$, then

$$[\rho^{\frac{1}{p}}(1 + (-\Delta)^{\frac{\alpha}{2}})^{-1 + \frac{1}{\alpha}} \rho^{-\frac{1}{p}}](x, y) \leq C(-\Delta)^{-\frac{\alpha-1}{2}}(x - y), \quad x, y \in \mathbb{R}^d,$$

which implies (5.3). \square

In the proof above we have exploited the fact that the kernel $(\lambda + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{\gamma}{2}}(x - y)$ tends to zero as $|x - y| \rightarrow \infty$ strictly faster than the corresponding Riesz kernel. This does not work for the parabolic kernel $(\lambda \pm \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{\gamma}{2}}(t, x - y)$. The Morrey class condition on $|b|$, however, provides some extra

flexibility: one can exploit the fact that to localize parabolic Adams' estimates (4.1), (4.2) one does not need to control the entire integral kernel $(\lambda \pm \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{1}{2}}(t, x - y)$. This is a localization argument due to Krylov, which we employ in the proof of Proposition 3. (To be more precise, we extend Krylov's argument from $\alpha = 2$, but in the process require an additional condition $\|b_s\|_{L^1(\mathbb{R}^{1+d})} < \infty$ due to worse decay properties of the heat kernel in the non-local case $1 < \alpha < 2$.)

Let $Q_r(x) \subset \mathbb{R}^d$ denote the cube centered at x with edge length r . Set

$$A_{\lambda,p} := \lambda^{-\frac{1}{p'} - (1 - \frac{1}{\alpha})\frac{1}{p}} \downarrow 0 \quad \text{as } \lambda \uparrow \infty.$$

The next Proposition 3 is used in the proof of Theorem 1. (The latter requires $p > d + 1$, and so the condition $p > \frac{\theta}{\alpha}$ in Proposition 3 is automatically satisfied.)

Proposition 3. *Assume that Borel measurable vector field b satisfies (\mathbf{H}_1) . Then, for every $p > \frac{\theta}{\alpha}$, provided that θ in the definition (5.1) of weight ρ is chosen so that $d < \theta < d + \alpha$ and κ is fixed sufficiently small (that is, depending only on d, α and the norms of b_b and b_s), the following are true for all $\lambda > 1$:*

(i)

$$\begin{aligned} & \| |b|^{\frac{1}{p}} (\lambda \pm \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{(-1 + \frac{1}{\alpha})\frac{1}{p}} \|_{L^p_\rho(\mathbb{R}^{1+d}) \rightarrow L^p_\rho(\mathbb{R}^{1+d})} \\ & \leq K_1 \sup_{x \in \mathbb{R}^d} \| \mathbf{1}_{Q_2(x)} |b_s|^{\frac{1}{\alpha-1}} \|_{E_{1+\epsilon}}^{\frac{\alpha-1}{p}} + K_2 A_{\lambda,p} \langle |b_s| \rangle^{\frac{1}{p}} + K_3 \lambda^{-\frac{1}{p}(1 - \frac{1}{\alpha})} \|b_b\|_{L^\infty(\mathbb{R}^{1+d})}^{\frac{1}{p}}, \end{aligned} \quad (5.6)$$

(ii)

$$\begin{aligned} & \| (\lambda \pm \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{(-1 + \frac{1}{\alpha})\frac{1}{p'}} |b|^{\frac{1}{p'}} \|_{L^p_\rho(\mathbb{R}^{1+d}) \rightarrow L^p_\rho(\mathbb{R}^{1+d})} \\ & \leq K_1 \sup_{x \in \mathbb{R}^d} \| \mathbf{1}_{Q_2(x)} |b_s|^{\frac{1}{\alpha-1}} \|_{E_{1+\epsilon}}^{\frac{\alpha-1}{p'}} + K_2 A_{\lambda,p'} \langle |b_s| \rangle^{\frac{1}{p'}} + K_3 \lambda^{-\frac{1}{p'}(1 - \frac{1}{\alpha})} \|b_b\|_{L^\infty(\mathbb{R}^{1+d})}^{\frac{1}{p'}}. \end{aligned} \quad (5.7)$$

for constants $K_i = K_i(d, \alpha, p, \epsilon, \theta, \kappa)$.

We prove Proposition 3 in Section 7.

Proposition 3 imposes a more restrictive bound on p from below, due to the worse decay properties of the stable heat kernel at infinity compared to the stable Bessel potential.

6. PROOF OF THEOREM 2

Throughout this section, we write $A := (-\Delta)^{\frac{\alpha}{2}}$. Assertion (i) follows directly from Proposition 1 and

$$G_p(b_n) \rightarrow G_p(b), \quad H_p(b_n) \rightarrow H_p(b) \quad \text{strongly in } L^p(\mathbb{R}^{1+d}) \quad (6.1)$$

which, in turn, is a consequence of our choice of approximation $\{b_n\}$ (essentially monotone) and the Dominated Convergence Theorem.

Let us prove (ii). *Existence.* The fact that η constructed in (i) is a weak solution follows right away from the convergence

$$\eta_n \rightarrow \eta \quad \text{in } \mathbb{W}_\alpha^{(1 - \frac{1}{\alpha})\frac{2}{p}, p}(\mathbb{R}^{1+d}),$$

from (i), i.e.

$$(\lambda + \partial + A)^{(1 - \frac{1}{\alpha})\frac{1}{p}} \eta_n \rightarrow (\lambda + \partial + A)^{(1 - \frac{1}{\alpha})\frac{1}{p}} \eta \quad \text{in } L^p(\mathbb{R}^{1+d}), \quad (6.2)$$

and the convergence (6.1) which we supplement with $G_p^*(b_n) \rightarrow G_p^*(b)$ in $L^{p'}(\mathbb{R}^{1+d})$. (Having an explicit representation from (i) for the candidate for a weak solution, i.e. η , greatly simplifies the proof of weak existence.)

Uniqueness. First, we note that, given any weak solution η , in view of the inclusions $H_p \in \mathcal{B}(L^p)$, $G_p^* \in \mathcal{B}(L^{p'})$ established in Theorem 2, if the right-hand side f satisfies (2.6), then the identity (2.12) extends to

$$\begin{aligned} & \langle (\lambda + \partial_t + A)^{(1-\frac{1}{\alpha})\frac{1}{p}} \eta, (\lambda - \partial_t + A)^{1-(1-\frac{1}{\alpha})\frac{1}{p}} \varphi \rangle \\ & + \langle H_p(\lambda + \partial_t + A)^{(1-\frac{1}{\alpha})\frac{1}{p}} \eta, -G_p^*(\lambda - \partial_t + A)^{1-(1-\frac{1}{\alpha})\frac{1}{p}} \varphi \rangle \\ & = \langle (\lambda + \partial_t + A)^{-1+(1-\frac{1}{\alpha})\frac{1}{p}} f, (\lambda - \partial_t + A)^{1-(1-\frac{1}{\alpha})\frac{1}{p}} \varphi \rangle \end{aligned} \quad (6.3)$$

for all test functions $\varphi \in L^{p'}(\mathbb{R}^{1+d})$ such that

$$(\lambda - \partial_t + A)^{1-(1-\frac{1}{\alpha})\frac{1}{p}} \varphi \in L^{p'}(\mathbb{R}^{1+d}).$$

Now, let η be the weak solution constructed earlier, and let $\eta_\star \in \mathbb{W}_\alpha^{(1-\frac{1}{\alpha})\frac{2}{p}, p}(\mathbb{R}^{1+d})$ be another weak solution. Then, by the previous remark, identity (6.3) holds for η_\star as well. Therefore, the difference $\eta - \eta_\star$ satisfies

$$\begin{aligned} & \langle (\lambda + \partial + A)^{(1-\frac{1}{\alpha})\frac{1}{p}} (\eta - \eta_\star), (\lambda - \partial + A)^{1-(1-\frac{1}{\alpha})\frac{1}{p}} \varphi \rangle \\ & + \langle H_p(\lambda + \partial + A)^{(1-\frac{1}{\alpha})\frac{1}{p}} (\eta - \eta_\star), -G_p^*(\lambda - \partial + A)^{1-(1-\frac{1}{\alpha})\frac{1}{p}} \varphi \rangle = 0 \end{aligned} \quad (6.4)$$

for all test functions $\varphi \in L^{p'}(\mathbb{R}^{1+d})$ such that

$$(\lambda - \partial + A)^{1-(1-\frac{1}{\alpha})\frac{1}{p}} \varphi \in L^{p'}(\mathbb{R}^{1+d}).$$

Now, we choose in (6.4)

$$\varphi := (\lambda - \partial + A)^{-1+(1-\frac{1}{\alpha})\frac{1}{p}} [(\lambda + \partial + A)^{(1-\frac{1}{\alpha})\frac{1}{p}} (\eta - \eta_\star) |(\lambda + \partial + A)^{(1-\frac{1}{\alpha})\frac{1}{p}} (\eta - \eta_\star)|^{p-2}],$$

where the inclusion $(\lambda - \partial + A)^{1-(1-\frac{1}{\alpha})\frac{1}{p}} \varphi \in L^{p'}(\mathbb{R}^{1+d})$ follows from $\eta, \eta_\star \in \mathbb{W}_\alpha^{(1-\frac{1}{\alpha})\frac{2}{p}, p}(\mathbb{R}^{1+d})$, thus defined φ is an admissible test function. We arrive at

$$\begin{aligned} & \langle (\lambda + \partial + A)^{(1-\frac{1}{\alpha})\frac{1}{p}} (\eta - \eta_\star), (\lambda + \partial + A)^{(1-\frac{1}{\alpha})\frac{1}{p}} (\eta - \eta_\star) |(\lambda + \partial + A)^{(1-\frac{1}{\alpha})\frac{1}{p}} (\eta - \eta_\star)|^{p-2} \rangle \\ & + \langle H_p(\lambda + \partial + A)^{(1-\frac{1}{\alpha})\frac{1}{p}} (\eta - \eta_\star), -G_p^*(\lambda + \partial + A)^{(1-\frac{1}{\alpha})\frac{1}{p}} (\eta - \eta_\star) |(\lambda + \partial + A)^{(1-\frac{1}{\alpha})\frac{1}{p}} (\eta - \eta_\star)|^{p-2} \rangle \\ & = 0. \end{aligned}$$

Observe that the first term coincides with $\|\eta - \eta_\star\|_{\mathbb{W}_\alpha^{(1-\frac{1}{\alpha})\frac{2}{p}, p}(\mathbb{R}^{1+d})}^p$, while the absolute value of the second term is bounded from above by $\|H_p\|_{p \rightarrow p} \|G_p^*\|_{p' \rightarrow p'} \|\eta - \eta_\star\|_{\mathbb{W}_\alpha^{(1-\frac{1}{\alpha})\frac{2}{p}, p}}^p$. Hence

$$(1 - \|H_p\|_{p \rightarrow p} \|G_p^*\|_{p' \rightarrow p'}) \|\eta - \eta_\star\|_{\mathbb{W}_\alpha^{(1-\frac{1}{\alpha})\frac{2}{p}, p}}^p \leq 0.$$

By our assumption on the Morrey norm of b ,

$$\|H_p\|_{p \rightarrow p} \|G_p^*\|_{p' \rightarrow p'} \equiv \|H_p\|_{p \rightarrow p} \|G_p\|_{p \rightarrow p} < 1,$$

which implies $\eta = \eta_\star$. This ends the proof of assertion (ii).

(iii) is a consequence of (ii) and $\delta_{t=0}\delta_x \in \mathbb{W}_\alpha^{-\frac{2}{\alpha}+(-1+\frac{1}{\alpha})\frac{2}{p'},p}(\mathbb{R}^{1+d})$. In detail, by the parabolic Sobolev embedding property (1.25) (1.26), we have

$$\mathbb{W}_\alpha^{\frac{2}{\alpha}-(-1+\frac{1}{\alpha})\frac{2}{p'},p'}(\mathbb{R}^{1+d}) \hookrightarrow C_\infty(\mathbb{R}^{1+d}) \quad \text{provided } p < \frac{d+1}{d}.$$

Hence,

$$\delta_{t=0}\delta_x \in (C_\infty(\mathbb{R}^{1+d}))' \hookrightarrow (\mathbb{W}_\alpha^{\frac{2}{\alpha}-(-1+\frac{1}{\alpha})\frac{2}{p'},p'}(\mathbb{R}^{1+d}))' = \mathbb{W}_\alpha^{-\frac{2}{\alpha}+(-1+\frac{1}{\alpha})\frac{2}{p'},p}(\mathbb{R}^{1+d}).$$

□

7. PROOF OF PROPOSITION 3

We may establish (5.6), (5.7) separately for the bounded part b_b and the unbounded part b_s . The proof for the unbounded part b_s that we now give works for all $1 < p < \infty$. Put for brevity $A := (-\Delta)^{\frac{\alpha}{2}}$, $\partial := \partial_t$. Below C denotes a generic positive constant depending on d, α, p, ϵ whose value may change from line to line.

(i) Fix $x \in \mathbb{R}^d$. Let $\langle \cdot, \cdot \rangle$ denote, as before, the integration over \mathbb{R}^{1+d} . We have

$$\begin{aligned} \|\mathbf{1}_{Q_1(x)}|b_s|^{\frac{1}{p}}(\lambda + \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p}}f\|_p^p &= \langle \mathbf{1}_{Q_1(x)}|b_s|(\lambda + \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p}}f|^p \rangle \\ &\leq C \langle \mathbf{1}_{Q_1(x)}|b_s|(\lambda + \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p}}\mathbf{1}_{Q_2(x)}f|^p \rangle \\ &\quad + C \langle \mathbf{1}_{Q_1(x)}|b_s|(\lambda + \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p}}\mathbf{1}_{Q_2^c(x)}f|^p \rangle \\ &=: S(x) + R(x). \end{aligned}$$

Step 1. By Proposition 1,

$$S(x) \leq c_{d,\alpha,p,\epsilon}^p \|\mathbf{1}_{Q_1(x)}|b_s|^{\frac{1}{\alpha-1}}\|_{E_{1+\epsilon}}^{\alpha-1} \|\mathbf{1}_{Q_2(x)}f\|_p^p. \quad (7.1)$$

Step 2. On the complement $Q_2^c(x)$ we will use the separation property of the heat kernel of the fractional Laplacian:

$$R(x) \leq \sup_{(t,y) \in \mathbb{R}^{1+d}} |\mathbf{1}_{Q_1(x)}(y)|(\lambda + \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p}}\mathbf{1}_{Q_2^c(x)}f(t,y)|^p \langle \mathbf{1}_{Q_1(x)}|b_s| \rangle,$$

where, for every $(t,y) \in \mathbb{R}^{1+d}$, by the upper bound in (1.22),

$$\begin{aligned} &|\mathbf{1}_{Q_1(x)}(y)|(\lambda + \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p}}\mathbf{1}_{Q_2^c(x)}f(t,y)| \\ &\leq C \mathbf{1}_{Q_1(x)}(y) \int_{-\infty}^t \int_{\mathbb{R}^d} e^{-\lambda(t-s)}(t-s)^{(1-\frac{1}{\alpha})\frac{1}{p}}|y-z|^{-d-\alpha}\mathbf{1}_{Q_2^c(x)}(z)|f(s,z)|dzds \\ &\quad (\text{write } 1 = \rho^{-\frac{1}{p}}\rho^{\frac{1}{p}}) \\ &\leq C \mathbf{1}_{Q_1(x)}(y) \left(\int_{-\infty}^t \int_{\mathbb{R}^d} e^{-p'\lambda(t-s)}(t-s)^{(1-\frac{1}{\alpha})\frac{p'}{p}}|y-z|^{-p'(d+\alpha)}\mathbf{1}_{Q_2^c(x)}(z)\rho^{-\frac{p'}{p}}(z)dzds \right)^{\frac{1}{p'}} \\ &\quad \cdot \left(\int_{-\infty}^t \int_{\mathbb{R}^d} \mathbf{1}_{Q_2^c(x)}(z)\rho(z)|f(s,z)|^p dzds \right)^{\frac{1}{p}}. \end{aligned}$$

The second factor is bounded by $\|\rho^{\frac{1}{p}}f\|_p$. Hence, after factorizing the first integral as

$$\left(\int_{-\infty}^t e^{-p'\lambda(t-s)}(t-s)^{(1-\frac{1}{\alpha})\frac{p'}{p}} ds \right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^d} \mathbf{1}_{Q_1(x)}(y)|y-z|^{-p'(d+\alpha)}\mathbf{1}_{Q_2^c(x)}(z)\rho^{-\frac{p'}{p}}(z)dz \right)^{\frac{1}{p'}} =: a_\lambda(t)j(x,y),$$

we have

$$R(x) \leq CA_{\lambda,p}^p J^p(x) \|\rho^{\frac{1}{p}} f\|_p^p \langle \mathbf{1}_{Q_1(x)} | b_s \rangle$$

where

$$A_{\lambda,p} := \sup_{t \in \mathbb{R}} a_\lambda(t), \quad J(x) := \sup_{y \in \mathbb{R}^d} j(x, y) = \sup_{y \in Q_1(x)} j(x, y).$$

1) In fact, a_λ is independent of t : by the substitution $u = t - s$, we have

$$\begin{aligned} a_\lambda^{p'}(t) &= \int_{-\infty}^t e^{-p'\lambda(t-s)} (t-s)^{(1-\frac{1}{\alpha})\frac{p'}{p}} ds \\ &= \int_0^\infty e^{-p'\lambda u} u^{(1-\frac{1}{\alpha})\frac{p'}{p}} du < \infty \quad \text{since } \lambda > 0, \alpha > 1 \text{ and } 1 < p < \infty. \end{aligned}$$

Furthermore, $A_{\lambda,p} \equiv a_\lambda \downarrow 0$ as $\lambda \uparrow \infty$.

2) Let us estimate the rate of growth of $J(x)$ as $x \rightarrow \infty$. First, note that

$$z \mapsto \mathbf{1}_{Q_1(x)}(y) |y - z|^{-p'(d+\alpha)} \mathbf{1}_{Q_2^c(x)}(z) \rho^{-\frac{p'}{p}}(z)$$

is integrable since $p'(d+\alpha) - \theta \frac{1}{p-1} > d$ by our choice of θ ($< d + \alpha$), and therefore $j(x, y)$ is finite for all x, y . Since $y \in Q_1(x)$ and $z \in Q_2^c(x)$, there exists a constant $C > 0$ such that $|y - z| \geq C|x - z|$. It suffices to analyze the rate of growth of

$$\mathbb{R}^d \ni x \mapsto \int_{\mathbb{R}^d} |x - z|^{-p'(d+\alpha)} \mathbf{1}_{Q_2^c(x)}(z) \rho^{-\frac{p'}{p}}(z) dz.$$

We have, after the change of variable $w = x - z$,

$$\begin{aligned} x \mapsto & \int_{\mathbb{R}^d} |w|^{-p'(d+\alpha)} \mathbf{1}_{|w| \geq 1}(w) (1 + \kappa|x - w|^2)^{\frac{\theta}{2(p-1)}} dw \\ & \leq C \int_{\mathbb{R}^d} |w|^{-p'(d+\alpha)} \mathbf{1}_{|w| \geq 1}(w) (|w|^{\frac{\theta}{p-1}} + |x|^{\frac{\theta}{p-1}}) dw \\ & \leq C_1 + C_2 |x|^{\frac{\theta}{p-1}}. \end{aligned}$$

Therefore,

$$J(x) \leq (C_1 + C_2 |x|^{\frac{\theta}{p-1}})^{\frac{1}{p'}} \leq C_3 + C_4 |x|^{\frac{\theta}{p}}.$$

Hence

$$R(x) \leq A_{\lambda,p}^p (C_5 + C_6 |x|^\theta) \|\rho^{\frac{1}{p}} f\|_p^p \langle \mathbf{1}_{Q_1(x)} | b_s \rangle. \quad (7.2)$$

Step 3. Armed with estimates (7.1), (7.2), we now proceed to prove the sought weighted bound. Let us rewrite it as a bound on the operator norm $L^p(\mathbb{R}^{1+d}) \rightarrow L^p(\mathbb{R}^{1+d})$

$$\|\rho^{\frac{1}{p}} |b_s|^{\frac{1}{p}} (\lambda \pm \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{(-1+\frac{1}{\alpha})\frac{1}{p}} \rho^{-\frac{1}{p}}\|_{L^p(\mathbb{R}^{1+d}) \rightarrow L^p(\mathbb{R}^{1+d})}.$$

Selecting κ in the definition of ρ sufficiently small, i.e. so that

$$|\rho(x) - \sup_{y \in Q_2(x)} \rho(y)| \leq \frac{1}{10} \rho(x) \quad \forall x \in \mathbb{R}^d,$$

we have

$$\begin{aligned}
\|\rho^{\frac{1}{p}}|b_s|^{\frac{1}{p}}(\lambda + \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p}}f\|_p^p &= \sum_{x \in \mathbb{Z}^d} \|\rho^{\frac{1}{p}}\mathbf{1}_{Q_1(x)}|b_s|^{\frac{1}{p}}(\lambda + \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p}}f\|_p^p \\
&\leq \frac{11}{10} \sum_{x \in \mathbb{Z}^d} \rho(x)\|\mathbf{1}_{Q_1(x)}|b_s|^{\frac{1}{p}}(\lambda + \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p}}f\|_p^p \\
&\leq \frac{11}{10} \sum_{x \in \mathbb{Z}^d} \rho(x)(S(x) + R(x)) \\
&\leq \frac{11}{10} c_{d,p,q,\alpha}^p \sup_{x \in \mathbb{R}^d} \|\mathbf{1}_{Q_1(x)}|b_s|^{\frac{1}{\alpha-1}}\|_{E_{1+\epsilon}}^{\alpha-1} \sum_{x \in \mathbb{Z}^d} \rho(x)\|\mathbf{1}_{Q_2(x)}f\|_p^p \\
&\quad + \frac{11}{10} \sum_{x \in \mathbb{Z}^d} \rho(x)A_{\lambda,p}(C_5 + C_6|x|^\theta)\|\rho^{\frac{1}{p}}f\|_p^p \langle \mathbf{1}_{Q_1(x)}|b_s| \rangle \\
&\quad (\text{we are using } \sup_{x \in \mathbb{R}^d} \rho(x)(1 + |x|^\theta) < \infty) \\
&\leq K_1 c_{d,p,q,\alpha}^p \sup_{x \in \mathbb{R}^d} \|\mathbf{1}_{Q_1(x)}|b_s|^{\frac{1}{\alpha-1}}\|_{E_{1+\epsilon}}^{\alpha-1} \|\rho^{\frac{1}{p}}f\|_p^p + K_2 A_{\lambda,p}^p \langle |b_s| \rangle \|\rho^{\frac{1}{p}}f\|_p^p.
\end{aligned}$$

(ii) The result follows by a duality argument. For notational convenience, we replace p' by p in this part of the proof. The proof proceeds as in (i) – crucially, in (i) we allow any $1 < p < \infty$ – but now the estimate, rewritten as a bound on the usual operator norm $L^p(\mathbb{R}^{1+d}) \rightarrow L^p(\mathbb{R}^{1+d})$, has $\rho^{-\frac{1}{p}}$ outside and $\rho^{\frac{1}{p}}$ inside:

$$\|\rho^{-\frac{1}{p}}|b_s|^{\frac{1}{p}}(\lambda \pm \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{(-1+\frac{1}{\alpha})\frac{1}{p}}\rho^{\frac{1}{p}}\|_{L^p(\mathbb{R}^{1+d}) \rightarrow L^p(\mathbb{R}^{1+d})}.$$

The only modification is in the proof of 2). Now $j(x, y)$ is defined as

$$\left(\int_{\mathbb{R}^d} \mathbf{1}_{Q_1(x)}(y)|y - z|^{-p'(d+\alpha)}\mathbf{1}_{Q_2^c(x)}(z)\rho^{\frac{p'}{p}}(z)dz \right)^{\frac{1}{p'}},$$

i.e. the weight ρ^{-1} is transferred to the f -term. As before, $J(x) := \sup_{y \in Q_1(x)} j(x, y)$.

2') Let us estimate the rate of decay of $J(x)$ as $x \rightarrow \infty$. We only need to consider $|x| \geq 1$. Since $y \in Q_1(x)$, it suffices to analyze the rate of decay of

$$\mathbb{R}^d \ni x \mapsto \int_{\mathbb{R}^d} |x - z|^{-p'(d+\alpha)}\mathbf{1}_{Q_2^c(x)}(z)\rho^{\frac{p'}{p}}(z)dz.$$

After the change of variable $w = x - z$,

$$\begin{aligned}
x \mapsto &\int_{\mathbb{R}^d} |w|^{-p'(d+\alpha)}\mathbf{1}_{|w| \geq 1}(w)(1 + \kappa|x - w|^2)^{-\frac{\theta}{2(p-1)}}dw \\
&\leq C \int_{\{|w| \leq \frac{|x|}{2}\}} |w|^{-p'(d+\alpha)}\mathbf{1}_{|w| \geq 1}(w)|x|^{-\frac{\theta}{p-1}}dw + C \int_{\{|w| > \frac{|x|}{2}\}} |w|^{-p'(d+\alpha)}dw \\
&\leq C|x|^{-\frac{\theta}{p-1}} + C|x|^{-p'(d+\alpha)+d} \leq C_5|x|^{-\frac{\theta}{p-1}},
\end{aligned}$$

under our assumption $\theta < d + \alpha$. Therefore,

$$J(x) \leq C_6|x|^{-\frac{\theta}{p}}, \quad |x| \geq 1.$$

In the end, repeating the proof of (i), we multiply $J^p(x)$ by ρ^{-1} whose growth at infinity is cancelled by the decay of $|x|^{-\theta}$. This ends the proof of (ii) for b_s .

The proof for b_\flat is standard and is included in Appendix B for the reader's convenience. It gives the remaining terms in (i), (ii) with constant K_3 . \square

8. PROOF OF THEOREM 1

Put for brevity $A := (-\Delta)^{\frac{\alpha}{2}}$, $\partial := \partial_t$. The proof of Theorem 1 requires regularity theory of the inhomogeneous Kolmogorov backward equation (here, for the ease of stating the results, with time reversed)

$$\lambda u + \partial u + Au + b \cdot \nabla u = f \quad \text{on } \mathbb{R}^{1+d}. \quad (8.1)$$

The approximating bounded vector fields $\{b_n\}$ are defined by (2.4). We note that, with our hypothesis on b_s and the almost-monotone construction of $b_{n,s}$, we have $\sup_n \|b_{n,s}\|_{L^1(\mathbb{R}^{1+d})} < \infty$.

The results in 8.0.1 and 8.0.3 were proved in [K2].

8.0.1. *The dual variant of Theorem 2.* Let $1 < p < \infty$. Then, for the same constants $c_{d,\alpha,p,q}$ and $\lambda_{d,\alpha,p,q}$ as in Theorem 2, for a right-hand side

$$f \in \mathbb{W}_\alpha^{(-1+\frac{1}{\alpha})\frac{2}{p'},p}(\mathbb{R}^{1+d})$$

of (8.1) and a sequence of functions $\{f_n\} \subset L^\infty(\mathbb{R}^{1+d})$ such that

$$f_n \rightarrow f \quad \text{in } \mathbb{W}_\alpha^{(-1+\frac{1}{\alpha})\frac{2}{p'},p}$$

solutions u_n of the approximating Kolmogorov backward equations

$$\lambda u_n + \partial u_n + Au_n + b_n \cdot \nabla u_n = f_n, \quad \lambda > \lambda_{d,\alpha,p,q}$$

converge as $n \rightarrow \infty$ in $\mathbb{W}_\alpha^{\frac{2}{\alpha} + \frac{\alpha-1}{\alpha}\frac{2}{p},p}$ to the function

$$u = (\lambda + \partial + A)^{-\frac{1}{\alpha} + (-1+\frac{1}{\alpha})\frac{1}{p}} (1 + Q_p R_p)^{-1} (\lambda + \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p'}} f \quad (8.2)$$

where

$$R_p := b^{\frac{1}{p}} \cdot \nabla (\lambda + \partial + A)^{-\frac{1}{\alpha} + (-1+\frac{1}{\alpha})\frac{1}{p}}, \quad (8.3)$$

$$Q_p := (\lambda + \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p'}} |b|^{\frac{1}{p'}},$$

are bounded operators on $L^p(\mathbb{R}^{1+d})$, and

$$\|R_p\|_{p \rightarrow p} \|Q_p\|_{p \rightarrow p} < 1$$

so the geometric series in (8.2) converges in $L^p(\mathbb{R}^{1+d})$. In particular, it follows that

$$u \in \mathbb{W}_\alpha^{\frac{2}{\alpha} + \frac{\alpha-1}{\alpha}\frac{2}{p},p},$$

and if $p > d + 1$ then, by the Sobolev embedding (1.25), (1.26) u is bounded and continuous and the convergence of u_n to u is uniform on \mathbb{R}^{1+d} .

There is also the following alternative representation

$$\begin{aligned} u &= (\lambda + \partial + A)^{-1} f \\ &\quad - (\lambda + \partial + A)^{-\frac{1}{\alpha} + (-1+\frac{1}{\alpha})\frac{1}{p}} Q_p (1 + R_p Q_p)^{-1} R_p (\lambda + \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p'}} f. \end{aligned} \quad (8.4)$$

8.0.2. *Weighted counterpart of 8.0.1.* Using Proposition 3, we establish convergence of both fractional representations for the Duhamel series from 8.0.1 in $L^p_\rho(\mathbb{R}^{1+d})$. This, and the Sobolev Embedding Theorem, yield for $p > d + 1$ (note that, automatically, $p > \frac{d}{\theta}$): assuming additionally that λ is fixed sufficiently large, we have, for every $x \in \mathbb{R}^d$,

$$\sup_{t \in \mathbb{R}, y \in B_1(x)} |u(t, y)| \leq K \|f\|_{L^p_\rho(\mathbb{R}^{1+d})} \quad (8.5)$$

(the constant K does not depend on x , but does depend on the radius of the ball). This relaxes the condition on the behaviour of the right-hand side f at infinity compared to 8.0.1.

8.0.3. *Feller propagator.* Fix some $r \in \mathbb{R}$. Let v_n denote the solution of the initial-value problem

$$\begin{cases} (\lambda + \partial + A + b_n \cdot \nabla)v_n = 0, & (t, x) \in]r, \infty[\times \mathbb{R}^d, \\ v_n(r, \cdot) = g \in C_\infty(\mathbb{R}^d), \end{cases} \quad (8.6)$$

where b_n is defined by (2.4).

By a classical result, for each $n \geq 1$, the operators

$$U_n^{t,r} g := v_n(t), \quad -\infty < r \leq t < \infty$$

constitute a (forward) Feller propagator on $C_\infty(\mathbb{R}^d)$. We have conservation of probability:

$$\int_{\mathbb{R}^d} e^{\lambda(t-r)} U_n^{t,r}(x, y) dy = 1, \quad x \in \mathbb{R}^d.$$

Let $\delta_r(t)$ denote the delta-function (in time) concentrated at $t = r$. Put

$$(\lambda + \partial + A)^{-\frac{\gamma}{2}} \delta_r g(t, x) := \mathbf{1}_{t > r} e^{-\lambda(t-r)} \int_{\mathbb{R}^d} q_\gamma(t-r, x-y) g(y) dy, \quad (8.7)$$

i.e. by putting the delta-function in time in the right-hand side, the action of the parabolic operator becomes the action of the semigroup of the fractional Laplacian (times the exponential function in time) on $g = g(y)$.

Assume now that we are in the setting of the previous subsection 8.0.1 with $p > d + 1$. Then the limit

$$U^{t,r} := s\text{-}C_\infty(\mathbb{R}^d)\text{-}\lim_n U_n^{t,r} \quad \text{locally uniformly in } (r, t) \in \mathbb{R}^2, r \leq t$$

exists and determines a Feller propagator on $C_\infty(\mathbb{R}^d)$. Furthermore, for every initial function $g \in C_\infty(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$, function $v(t) := U^{t,r} g$ has representation

$$v = (\lambda + \partial + A)^{-1} \delta_r g - (\lambda + \partial + A)^{-\frac{1}{\alpha} + (-1 + \frac{1}{\alpha}) \frac{1}{p}} Q_p (1 + R_p Q_p)^{-1} N_p \cdot S_p g, \quad (8.8)$$

where

$$N_p := b^{\frac{1}{p}} (\lambda + \partial + A)^{(-1 + \frac{1}{\alpha}) \frac{1}{p}} \in \mathcal{B}(L^p(\mathbb{R}^{1+d}), [L^p(\mathbb{R}^{1+d})]^d)$$

and

$$S_p g := \nabla (\lambda + \partial + A)^{-\frac{1}{p} - \frac{1}{\alpha p}} \delta_r g, \quad \|S_p g\|_{L^p(\mathbb{R}^{1+d})} \leq C_{d,\alpha,p} \|\nabla g\|_{L^p(\mathbb{R}^d)}.$$

We are now in a position to begin the proof of Theorem 1.

Proof of (ii). The forward Feller propagator $\{U^{t,r}\}_{0 \leq r \leq t}$ of 8.0.3 is our point of departure. Define the backward Feller propagator $\{P^{t,r}\}_{0 \leq t \leq r \leq T}$ via

$$P^{t,r} := e^{\lambda(r-t)} U^{T-t, T-r},$$

where the exponential factor is needed to restore the preservation of probability (and so $P^{t,r}$ are independent of λ). Then, by a classical result, there exists a unique family of probability measures $\{\mathbb{P}_{s,x}\}_{x \in \mathbb{R}^d, s \geq 0}$ such that identity (2.3) holds.

Proof of (i). We first prove Krylov-type bound (2.2) for $\{\mathbb{P}_{s,x}\}_{x \in \mathbb{R}^d, s \geq 0}$. Recall that $\{b_n\}$ denotes the bounded smooth approximation of b defined by (2.4). We define $\{f_n\}$ in the same way. Let $\mathbb{E}_{s,x}^n$ denote the expectation with respect to the probability measure $\mathbb{P}_{s,x}^n := \mathbb{P}_{s,x}(b_n)$. We have

$$\mathbb{E}_{s,x}^n \int_s^T |f_n(r, \omega_r) h(r, \omega_r)| dr = u_n(s, x),$$

where $u_n = u_n(s, x)$ is the classical solution to the terminal-value problem in $s \leq T$

$$(-\partial + A + b_n \cdot \nabla)u_n = |f_n h|, \quad u_n|_{s=T} = 0,$$

which we put in the form

$$(-\partial + A + b_n \cdot \nabla)u_n = \mathbf{1}_{s \leq T} |f_n| h \quad \text{on } \mathbb{R}^{1+d}$$

and, furthermore, setting

$$e_\lambda(s) = e^{\lambda s},$$

we write

$$(\lambda - \partial + A + b_n \cdot \nabla)(e_\lambda u_n) = e_\lambda(s) \mathbf{1}_{s \leq T} |f_n| h \quad \text{on } \mathbb{R}^{1+d}.$$

Applying to the last equation (8.5) after reversing the direction of time, the Duhamel series representation of 8.0.1, we arrive at

$$\mathbb{E}_{s,x}^n \int_s^T |f_n(r, \omega_r) h(r, \omega_r)| dr \leq K \|\mathbf{1}_{[s,T]} |f_n|^{\frac{1}{p}} h\|_{L^p} \quad (8.9)$$

for all $h \in C_b(\mathbb{R}^{1+d})$. Our goal is to pass to the limit $n \rightarrow \infty$ in both sides. The passage to the limit on the right-hand side is straightforward. To pass to the limit on the left-hand side, we need the weak convergence of the probability measures:

$$\mathbb{P}_{s,x}^n \rightarrow \mathbb{P}_{s,x} \text{ weakly,} \quad (8.10)$$

which holds, by a classical result, due to the convergence of the Feller propagators in 8.0.3. We write

$$\mathbb{E}_{s,x}^n \int_s^T |f_n h| = \mathbb{E}_{s,x}^n \int_s^T (|f_n| - |f_m|) |h| + \mathbb{E}_{s,x}^n \int_s^T |f_m h| - \mathbb{E}_{s,x} \int_s^T |f_m h| + \mathbb{E}_{s,x} \int_s^T |f_m h|,$$

where m is fixed. Then, the difference between the second and third terms on the right-hand side vanishes as $n \rightarrow \infty$ by (8.10); the last term:

$$\begin{aligned} \mathbb{E}_{s,x} \int_s^T |f_m h| &= \lim_n \mathbb{E}_{s,x}^n \int_s^T |f_m h| \\ &\leq \liminf_n \mathbb{E}_{s,x}^n \int_s^T |f_n h| + \liminf_n \mathbb{E}_{s,x}^n \int_s^T |f_n - f_m| |h| \\ &\text{(apply (8.9) to } f_n \text{ and to } f_n - f_m, \text{ respectively)} \\ &\leq K \|\mathbf{1}_{[s,T]} |f|^{\frac{1}{p}} h\|_{L^p} + K \|\mathbf{1}_{[s,T]} |f - f_m|^{\frac{1}{p}} h\|_{L^p}. \end{aligned}$$

It remains to pass to the limit $m \rightarrow \infty$ on the left-hand side of this inequality. This is done by means of Fatou's lemma, once we note that if $S \subset \mathbb{R}^{1+d}$ has measure zero (i.e. the set on which $|fh|$ and $\liminf_n |f_n h|$ possibly differ), then

$$\mathbb{E}_{s,x} \int_s^T \mathbf{1}_S = 0.$$

Indeed, for every $\varepsilon > 0$ there exists a larger open set $U_\varepsilon \supset S$ such that its Lebesgue measure satisfies $|U_\varepsilon| < \varepsilon$. Since the set of paths that pass through U_ε is also open, we have the inequality

$$\mathbb{E}_{s,x} \int_s^T \mathbf{1}_{U_\varepsilon} \leq \liminf_n \mathbb{E}_{s,x}^n \int_s^T \mathbf{1}_{U_\varepsilon}.$$

In fact, we can further increase the right-hand side by replacing the indicator function with a continuous function that equals 1 on U_ε and 0 outside a slightly larger set of volume $< 2\varepsilon$. We can now show that the right-hand side goes to zero as $\varepsilon \downarrow 0$ using the bound (8.9).

Thus, after passing to the limit in m , we arrive at the sought Krylov-type bound

$$\mathbb{E}_{s,x} \int_s^T |f(r, \omega_r) h(r, \omega_r)| dr \leq K \|\mathbf{1}_{[s,T]} |f|^{\frac{1}{p}} h\|_{L^p}. \quad (8.11)$$

We will use (8.11) twice, in the proof of existence of weak solution and in the proof of conditional uniqueness.

We now proceed to the proof of the properties (a)-(c) in assertion (i). Without loss of generality, $s = 0$; (a) and (b) follow right away from the construction of $\mathbb{P}_{0,x}$ and from the Krylov-type bound (8.11) (take $f = b$, $h = 1$). It remains to verify (c), i.e. to show that

$$Z_t(\omega) := \omega_t - x + \int_0^t b(s, \omega_s) ds, \quad t \geq 0,$$

is an α -stable process under $\mathbb{P}_{0,x}$. Arguing as [P, PP, CW], we need to show that

$$\mathbb{E}_{0,x} \left[e^{i\xi \cdot (\omega_t - x + \int_0^t b(s, \omega_s) ds)} \right] = e^{-t|\xi|^\alpha}, \quad \forall x, \xi \in \mathbb{R}^d, t \geq 0, \quad (8.12)$$

where the right-hand side is the characteristic function of the isotropic α -stable process. We will use this below to identify Z as an isotropic α -stable process, and thus to prove (c). Specifically, the proof of (8.12) yields a more general result: for all $0 \leq r \leq t \leq T$,

$$\mathbb{E}_{r,x} \exp \left(i\xi \cdot \left(\omega_t - x + \int_r^t b(u, \omega_u) du \right) \right) = e^{-(t-r)|\xi|^\alpha}.$$

Therefore,

$$\mathbb{E}_{0,x} \left[e^{i\xi \cdot (Z_t - Z_r)} \middle| \mathcal{F}_r \right] = e^{-(t-r)|\xi|^\alpha}.$$

The right-hand side is deterministic, so $Z_t - Z_r$ is independent of \mathcal{F}_r , and has the isotropic α -stable law with characteristic function $e^{-(t-r)|\xi|^\alpha}$. Iterating this identity, we obtain

$$\mathbb{E}_{0,x} \exp \left(i \sum_{k=1}^n \xi_k \cdot (Z_{t_k} - Z_{t_{k-1}}) \right) = \prod_{k=1}^n e^{-(t_k - t_{k-1})|\xi_k|^\alpha}.$$

Thus the increments of $\{Z_t\}$ are independent and stationary. Since $\{Z_t\}$ has càdlàg paths and $Z_0 = 0$, it is an isotropic α -stable process.

We establish (8.12) by first showing that, for all $r \in [0, t]$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, both functions

$$\nu(r, t, x, \xi) := \mathbb{E}_{r,x} \left[e^{i\xi \cdot (\omega_t + \int_r^t b(s, \omega_s) ds)} \right]$$

and

$$\tilde{\nu}(r, t, x, \xi) := e^{i\xi \cdot x - (t-r)|\xi|^\alpha}$$

are bounded solutions of the same integral equation, and then showing that this integral equation can have only one bounded solution, so (8.12) follows.

Claim 1. $\nu(r, t, x, \xi)$ satisfies the integral equation

$$\nu(r, t, x, \xi) = \mathbb{E}_{r,x} [e^{i\xi \cdot \omega_t}] + i \int_r^t \mathbb{E}_{r,x} [\xi \cdot b(u, \omega_u) \nu(u, t, \omega_u, \xi)] du. \quad (8.13)$$

Proof of Claim 1. Let $r \leq s \leq t$. We write

$$\int_r^t \xi \cdot b(u, \omega_u) du = \int_r^s \xi \cdot b(u, \omega_u) du + \int_s^t \xi \cdot b(u, \omega_u) du$$

and obtain, using the Markov property,

$$\begin{aligned} \nu(r, t, x, \xi) &= \mathbb{E}_{r,x} \left[e^{i \int_r^s \xi \cdot b(u, \omega_u) du} e^{i \int_s^t \xi \cdot b(u, \omega_u) du} e^{i\xi \cdot \omega_t} \right] \\ &= \mathbb{E}_{r,x} \left[e^{i \int_r^s \xi \cdot b(u, \omega_u) du} \mathbb{E}_{r,x} \left[e^{i \int_s^t \xi \cdot b(u, \omega_u) du} e^{i\xi \cdot \omega_t} \mid \mathcal{F}_s \right] \right] \\ &= \mathbb{E}_{r,x} \left[e^{i \int_r^s \xi \cdot b(u, \omega_u) du} \mathbb{E}_{s, \omega_s} \left[e^{i \int_s^t \xi \cdot b(u, \omega_u) du} e^{i\xi \cdot \omega_t} \right] \right] \\ &= \mathbb{E}_{r,x} \left[e^{i \int_r^s \xi \cdot b(u, \omega_u) du} \nu(s, t, \omega_s, \xi) \right]. \end{aligned} \quad (8.14)$$

Note that

$$e^{i \int_r^s \xi \cdot b(u, \omega_u) du} = 1 + i \int_r^s \xi \cdot b(u, \omega_u) e^{i \int_u^s \xi \cdot b(l, \omega_l) dl} du$$

(integrate the derivative of the left-hand side). Therefore,

$$\begin{aligned} \nu(r, t, x, \xi) &= \mathbb{E}_{r,x} [\nu(s, t, \omega_s, \xi)] + i \int_r^s \mathbb{E}_{r,x} \left[\nu(s, t, \omega_s, \xi) \xi \cdot b(u, \omega_u) e^{i \int_u^s \xi \cdot b(l, \omega_l) dl} \right] du. \end{aligned}$$

Using the Markov property again and applying (8.14) in the other direction, we have

$$\begin{aligned} &\mathbb{E}_{r,x} \left[\nu(s, t, \omega_s, \xi) \xi \cdot b(u, \omega_u) e^{i \int_u^s \xi \cdot b(l, \omega_l) dl} \right] \\ &= \mathbb{E}_{r,x} \left[\mathbb{E}_{r,x} \left[\nu(s, t, \omega_s, \xi) \xi \cdot b(u, \omega_u) e^{i \int_u^s \xi \cdot b(l, \omega_l) dl} \mid \mathcal{F}_u \right] \right] \\ &= \mathbb{E}_{r,x} \left[\xi \cdot b(u, \omega_u) \mathbb{E}_{u, \omega_u} \left[\nu(s, t, \omega_s, \xi) e^{i \int_u^s \xi \cdot b(l, \omega_l) dl} \right] \right] \\ &= \mathbb{E}_{r,x} [\xi \cdot b(u, \omega_u) \nu(u, t, \omega_u, \xi)]. \end{aligned}$$

Thus,

$$\nu(r, t, x, \xi) = \mathbb{E}_{r,x} [\nu(s, t, \omega_s, \xi)] + i \int_r^s \mathbb{E}_{r,x} [\xi \cdot b(u, \omega_u) \nu(u, t, \omega_u, \xi)] du.$$

Note that

$$\nu(t, t, x, \xi) = \mathbb{E}_{t,x} [e^{i\xi \cdot \omega_t}] = e^{i\xi \cdot x}.$$

Taking $s = t$, we have

$$\nu(r, t, x, \xi) = \mathbb{E}_{r,x}[e^{i\xi \cdot \omega_t}] + i \int_r^t \mathbb{E}_{r,x}[\xi \cdot b(u, \omega_u) \nu(u, t, \omega_u, \xi)] du.$$

This completes the proof of Claim 1. \square

Claim 2. *The function $\tilde{\nu}(r, t, x, \xi) := e^{i\xi \cdot x - (t-r)|\xi|^\alpha}$ is a bounded solution of (8.13).*

Proof. In the proof we will be using the operators of 8.0.1. We represent

$$\tilde{\nu}(r, t, x, \xi) \left(= e^{i\xi \cdot x - (t-r)|\xi|^\alpha} \right) = \int_{\mathbb{R}^d} e^{-(t-r)A} (x - y) e^{i\xi \cdot y} dy.$$

We will show that (8.4) basically contains the Duhamel formula. We first give a formal calculation – it is formal because we apply (8.4) to f that does not vanish at infinity – and then make this calculation rigorous. We will use

$$e^{-(t-u)(\lambda+A)} h(x) = (\lambda - \partial + A)^{-1} (\delta_t h)(u, x)$$

(cf. notation (8.7)). We have

$$ie^{\lambda(r-t)} \mathbb{E}_{r,x} \int_r^t [\xi \cdot b(u, \omega_u) \tilde{\nu}(u, t, \omega_u, \xi)] du \equiv e^{\lambda(r-t)} \mathbb{E}_{r,x} \int_r^t b(u, \omega_u) \cdot \nabla e^{-(t-u)A} e^{i\xi \cdot \omega_u} du \quad (8.15)$$

(put $h(x) := e^{i\xi \cdot x}$ and formally apply (8.4) with $f(u, x) := b(u, x) \cdot \nabla e^{-(t-u)(\lambda+A)} h(x)$, but after reversing the direction of time)

$$\begin{aligned} &= (\lambda - \partial + A)^{-\frac{1}{\alpha} + (-1 + \frac{1}{\alpha}) \frac{1}{p}} Q_p R_p (\lambda - \partial + A)^{(-1 + \frac{1}{\alpha}) \frac{1}{p'}} \delta_t h(x) \\ &\quad - (\lambda - \partial + A)^{-\frac{1}{\alpha} + (-1 + \frac{1}{\alpha}) \frac{1}{p}} Q_p (1 + R_p Q_p)^{-1} R_p Q_p R_p (\lambda - \partial + A)^{(-1 + \frac{1}{\alpha}) \frac{1}{p'}} \delta_t h(x) \\ &=: \Theta(b) \delta_t h(x), \end{aligned} \quad (8.16)$$

where, with some abuse of notation, Q_p and R_p are defined in the same way as in 8.0.1, but with time derivative $-\partial$ instead of ∂ . On the other hand, referring again to (8.4) and also to (8.8), one sees that the last expression coincides with

$$- (\lambda - \partial + A + b \cdot \nabla)^{-1} \delta_t h(x) + (\lambda - \partial + A)^{-1} \delta_t h(x) \quad (8.17)$$

$$= -e^{\lambda(r-t)} \mathbb{E}_{r,x}[h(\omega_t)] + e^{-(t-r)(\lambda+A)} h(x) \left(= -e^{\lambda(r-t)} \mathbb{E}_{r,x}[h(\omega_t)] + e^{\lambda(r-t)} \tilde{\nu}(r, t, x, \xi) \right). \quad (8.18)$$

This completes the proof of Claim 2, once we justify that, indeed, (8.15) = (8.16) and (8.17) = (8.18).

1) “(8.15) = (8.16)”. By the classical theory, for every $n = 1, 2, \dots$, we have

$$ie^{\lambda(r-t)} \mathbb{E}_{r,x}^n \int_r^t [\xi \cdot b_n(u, \omega_u) \tilde{\nu}(u, t, \omega_u, \xi)] du = \Theta(b_n) \delta_t h(x), \quad (8.19)$$

where, recall, $\{b_n\}$ is a bounded smooth approximation of b defined by (2.4), and $\mathbb{E}_{r,x}^n$ is the expectation with respect to the probability measure $\mathbb{P}_{r,x}^n$. This identity is simply the relationship between taking the expectation of an integral in time and solving the corresponding inhomogeneous parabolic equation. We need to pass to the limit $n \rightarrow \infty$ on both sides of (8.19).

We pass to the limit on the right-hand side of (8.19) using

$$\Theta(b_n) \delta_t h \rightarrow \Theta(b) \delta_t h \quad \text{in } C_\infty(\mathbb{R} \times B_1(x)), \quad (8.20)$$

cf. 8.0.2, 8.0.3. Namely, everywhere except the right-most occurrence of R_p in the definition (8.16) of Θ we use the convergence

$$Q_p(b_n) \rightarrow Q_p(b), R_p(b_n) \rightarrow R_p(b) \quad \text{strongly in } L_\rho^p(\mathbb{R}^{1+d})$$

in [K2] – this is straightforward since $\{b_n\}$ can be treated as essentially monotone approximation of b . For the right-most R_p in (8.16), we need to prove

$$R_p(b_n)(\lambda - \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p'}} \delta_t h \rightarrow R_p(b)(\lambda - \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p'}} \delta_t h \quad \text{in } L_\rho^p(\mathbb{R}^{1+d}).$$

Here we write R_p with some abuse of notation: $h(x) = e^{i\xi \cdot x}$ is only bounded, so $(\lambda - \partial + A)^{(-1+\frac{1}{\alpha})\frac{1}{p'}} \delta_t h$ is also only bounded in the spatial variables. However, expanding the definition of R_p , one sees that there we are multiplying a bounded function by $|b|^{\frac{1}{p}}$ which is, by our assumption, in $L_\rho^p(\mathbb{R}^{1+d})$. (Let us add that having the gradient in R_p does not cause any problem here: after commuting it with the parabolic operator, we put it on $h(x) := e^{i\xi \cdot x}$, which still gives us a bounded function.) The convergence now follows easily since, once again, b_n is, essentially, a monotone approximation of b .

In order to pass to the limit $n \rightarrow \infty$ on the left-hand side of (8.19), we will argue as in the beginning of the present proof, i.e. as in the proof of the Krylov-type bound (8.11). That is, we will use once again weak convergence of probability measures (8.10), i.e. $\mathbb{P}_{r,x}^n \rightarrow \mathbb{P}_{r,x}$ weakly as $n \rightarrow \infty$. We write, for m fixed,

$$\begin{aligned} \mathbb{E}_{r,x}^n \int_r^t \xi \cdot b_n \tilde{\nu} &= \mathbb{E}_{r,x}^n \int_r^t \xi \cdot (b_n - b_m) \tilde{\nu} + \mathbb{E}_{r,x}^n \int_r^t \xi \cdot b_m \tilde{\nu} - \mathbb{E}_{r,x} \int_r^t \xi \cdot b_m \tilde{\nu} + \mathbb{E}_{r,x} \int_r^t \xi \cdot b_m \tilde{\nu} \\ &\quad \text{(we expand the last term)} \\ &= \mathbb{E}_{r,x}^n \int_r^t \xi \cdot (b_n - b_m) \tilde{\nu} + \mathbb{E}_{r,x}^n \int_r^t \xi \cdot b_m \tilde{\nu} - \mathbb{E}_{r,x} \int_r^t \xi \cdot b_m \tilde{\nu} \\ &\quad + \mathbb{E}_{r,x} \int_r^t \xi \cdot (b_m - b) \tilde{\nu} + \mathbb{E}_{r,x} \int_r^t \xi \cdot b \tilde{\nu}. \end{aligned}$$

By (8.10), the difference between the second and the third terms on the right-hand side goes to zero as $n \rightarrow \infty$. We apply the Krylov-type bound to the first and the fourth terms, and hence conclude that they tend to zero as $n, m \rightarrow \infty$. The desired convergence follows.

2) “(8.17) = (8.18)”. We only need to show that $(-\partial + A + b \cdot \nabla)^{-1} \delta_t h(x) = \mathbb{E}_{r,x}[h(\omega_t)]$. By the classical theory, for each finite n ,

$$(-\partial + A + b_n \cdot \nabla)^{-1} \delta_t h(x) = \mathbb{E}_{r,x}^n[h(\omega_t)],$$

so we need to pass to the limit $n \rightarrow \infty$. On the left-hand side the convergence is due to 8.0.3; we only need to take into account that h is only bounded, but this is done in the same way as in the proof of convergence (8.20). On the right-hand side we use weak convergence (8.10).

The proof of Claim 2 is completed. \square

Let

$$\mu(r, t, x, \xi) := \nu(r, t, x, \xi) - \tilde{\nu}(r, t, x, \xi).$$

Then, comparing the definitions of ν and $\tilde{\nu}$, we have

$$\mu(r, t, x, \xi) = i \int_r^t \mathbb{E}_{r,x}[\xi \cdot b(u, \omega_u) \mu(u, t, \omega_u, \xi)] du, \quad 0 \leq r \leq t, \quad x \in \mathbb{R}^d, \quad (8.21)$$

To conclude our proof that $Z_t(\omega)$ is an isotropic α -stable process under $\mathbb{P}_{0,x}$ it remains to establish the following claim.

Claim 3. μ is identically zero.

Proof. We first show that, for fixed $\xi \in \mathbb{R}^d$, there exists $a > 0$ sufficiently small such that on the interval $r \in [t - a, t] \subset [0, t]$, the only bounded solution to the homogeneous equation (8.21) is $\mu \equiv 0$.

In what follows, by the L^∞ norm we mean the sup-norm (not esssup, i.e. we do not ignore sets of measure zero). Define the operator

$$\mu \mapsto \Gamma\mu(r, t, x, \xi) := i \int_r^t \mathbb{E}_{r,x}[\xi \cdot b(u, \omega_u) \mu(u, t, \omega_u, \xi)] du, \quad r \in [t - a, t], \quad x \in \mathbb{R}^d,$$

acting on functions $\mu(\cdot, t, \cdot, \xi) \in L^\infty([t - a, t] \times \mathbb{R}^d)$. The constant a will be chosen sufficiently small so that Γ is a contraction on $L^\infty([t - a, t] \times \mathbb{R}^d)$. We have

$$\begin{aligned} & \sup_{x \in \mathbb{R}^d} \sup_{r \in [t-a, t]} |\Gamma\mu(r, t, x, \xi)| \\ & \leq \sup_{x \in \mathbb{R}^d} \sup_{r \in [t-a, t]} |\xi| \int_r^t \mathbb{E}_{r,x}[|b(u, \omega_u)| |\mu(u, t, \omega_u, \xi)|] du \\ & \leq \sup_{x \in \mathbb{R}^d} \sup_{r \in [t-a, t]} |\xi| \int_r^t \mathbb{E}_{r,x}[|b(u, \omega_u)|] du \|\mu(\cdot, t, \cdot, \xi)\|_{L^\infty([t-a, t] \times \mathbb{R}^d)}. \end{aligned}$$

By the Krylov-type bound (8.11),

$$\sup_{x \in \mathbb{R}^d} \sup_{r \in [t-a, t]} \mathbb{E}_{r,x} \int_r^t |b(u, \omega_u)| du \leq K \|b\|_{L^1_\rho([t-a, t] \times \mathbb{R}^d)}.$$

Now, for given ξ , we select $a > 0$ sufficiently small so that $|\xi|K\|b\|_{L^1_\rho([t-a, t] \times \mathbb{R}^d)} < 1$. Furthermore, a simple argument based on the Dominated convergence theorem shows that we can and will select one $a > 0$ such that $|\xi|K\|b\|_{L^1(I \times \mathbb{R}^d)} < 1$ for every sub-interval $I \subset [0, t]$ of length a . We will use this below.

Then Γ is a contraction on $L^\infty([t - a, t] \times \mathbb{R}^d)$. We conclude that μ , as a solution of (8.21), must be identically equal to zero, for all $r \in [t - a, t]$ and $x \in \mathbb{R}^d$.

We now extend this conclusion to the whole interval $[0, t]$. Indeed, for $r \in [t - 2a, t - a]$, we have

$$\begin{aligned} \mu(r, t, x, \xi) &= i \int_r^t \mathbb{E}_{r,x}[\xi \cdot b(u, \omega_u) \mu(u, t, \omega_u, \xi)] du \\ &= i \int_r^{t-a} \mathbb{E}_{r,x}[\xi \cdot b(u, \omega_u) \mu(u, t, \omega_u, \xi)] du + i \int_{t-a}^t \mathbb{E}_{r,x}[\xi \cdot b(u, \omega_u) \mu(u, t, \omega_u, \xi)] du \\ &= i \int_r^{t-a} \mathbb{E}_{r,x}[\xi \cdot b(u, \omega_u) \mu(u, t, \omega_u, \xi)] du, \end{aligned}$$

where we have used that $\mu = 0$ on $[t - a, t] \times \mathbb{R}^d$ established in the last step. Applying the bound (8.11) once again, we obtain

$$\begin{aligned} & \|\mu(\cdot, t, \cdot, \xi)\|_{L^\infty([t-2a, t-a] \times \mathbb{R}^d)} \\ & \leq \sup_{x \in \mathbb{R}^d} \sup_{r \in [t-2a, t-a]} |\xi| \int_r^{t-a} \mathbb{E}_{r,x}[|b(u, \omega_u)|] du \|\mu(\cdot, t, \cdot, \xi)\|_{L^\infty([t-2a, t-a] \times \mathbb{R}^d)} \\ & \leq |\xi|K\|b\|_{L^1_\rho([t-2a, t-a] \times \mathbb{R}^d)} \|\mu(\cdot, t, \cdot, \xi)\|_{L^\infty([t-2a, t-a] \times \mathbb{R}^d)}, \end{aligned}$$

where $|\xi|K\|b\|_{L^1_p([t-2a, t-a] \times \mathbb{R}^d)} < 1$. It follows immediately that $\mu = 0$ for all $t-2a \leq r \leq t-a$, $x \in \mathbb{R}^d$. Repeating this finitely many times, we obtain that μ is identically equal to zero for all $r \in [0, t]$, $x \in \mathbb{R}^d$, as claimed. \square

Claim 3 gives us

$$\nu(r, t, x, \xi) = \tilde{\nu}(r, t, x, \xi),$$

which concludes the proof of (c) \Rightarrow (i).

Proof of (iii). Without loss of generality, we assume that the initial time $s = 0$. Suppose that there exist two weak solutions $\mathbb{P}_x^1, \mathbb{P}_x^2$ to (1.1) that satisfy, for $p > d + \alpha - 1$ from (i),

$$\mathbb{E}_x^i \int_0^T |h(t, \omega_t)| dt \leq K \|\mathbf{1}_{[0, T]} h\|_{L^p_p} \quad (8.22)$$

and

$$\mathbb{E}_x^i \int_0^T |b(t, \omega_t) h(t, \omega_t)| dt \leq K \|\mathbf{1}_{[0, T]} |b|^{\frac{1}{p}} h\|_{L^p_p} \quad (8.23)$$

for all $h \in (C_b \cap L^p)(\mathbb{R}^{1+d})$, with a constant K independent of h ($i = 1, 2$). It suffices to show that, for every $F \in C_c^\infty(\mathbb{R}^{1+d})$,

$$\mathbb{E}_x^1 \left[\int_0^T F(t, \omega_t) dt \right] = \mathbb{E}_x^2 \left[\int_0^T F(t, \omega_t) dt \right], \quad (8.24)$$

to conclude $\mathbb{P}_x^1 = \mathbb{P}_x^2$.

Proof of (8.24). Let $u_n \in C([0, T], C_\infty(\mathbb{R}^d))$ be the classical solution to the terminal-value problem

$$(-\partial + A + b_n \cdot \nabla) u_n = F, \quad u_n(T, \cdot) = 0, \quad (8.25)$$

where, recall, the bounded smooth regularization b_n , $n = 1, 2, \dots$, of b is defined via (2.4). We have, by the properties of mollifiers,

$$b_n \rightarrow b \quad \text{in } L_{\text{loc}}^{1+\varepsilon}(\mathbb{R}^{1+d}, \mathbb{R}^d).$$

Set $\tau_R := \inf\{t \geq 0 \mid |\omega_t| \geq R\}$, $R > 0$, i.e. the first exit time from the ball $B_R = B_R(0)$. By Itô's formula (Section 1.3),

$$\begin{aligned} \mathbb{E}_x^i u_n(T \wedge \tau_R, \omega_{T \wedge \tau_R}) &= u_n(0, x) + \mathbb{E}_x^i \int_0^{T \wedge \tau_R} F(t, \omega_t) dt \\ &\quad + \mathbb{E}_x^i \int_0^{T \wedge \tau_R} [(b - b_n) \cdot \nabla u_n](t, \omega_t) dt \end{aligned} \quad (8.26)$$

($i = 1, 2$). Our goal is to show that the last term tends to zero (see below). Applying (8.23) and (8.22), we obtain

$$\begin{aligned} &\left| \mathbb{E}_x^i \int_0^{T \wedge \tau_R} [(b - b_n) \cdot \nabla u_n](t, \omega_t) dt \right| \\ &\leq \mathbb{E}_x^i \int_0^{T \wedge \tau_R} |b(1 - \mathbf{1}_n) \cdot \nabla u_n|(t, \omega_t) dt + \mathbb{E}_x^i \int_0^{T \wedge \tau_R} |(\mathbf{1}_n b - b_n) \cdot \nabla u_n|(t, \omega_t) dt \\ &\leq K \|\mathbf{1}_{[0, T] \times B_R} b^{\frac{1}{p}} (1 - \mathbf{1}_n) \cdot \nabla u_n\|_p + K \|\mathbf{1}_{[0, T] \times B_R} (\mathbf{1}_n b - b_n) \cdot \nabla u_n\|_p \\ &=: I_n + J_n. \end{aligned}$$

1) Let us deal with I_n . We fix $p_1 > p$ close to p so that it is admissible in Theorem 1(i), and estimate

$$\|\mathbf{1}_{[0,T] \times B_R} b^{\frac{1}{p}}(1 - \mathbf{1}_n) \cdot \nabla u_n\|_p \leq \|\mathbf{1}_{[0,T] \times B_R} b^{\frac{1}{q}}(1 - \mathbf{1}_n)\|_q \|b^{\frac{1}{p_1}} \cdot \nabla u_n\|_{p_1},$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q}$. Up to this moment we have not used the theory of equation (8.25), but will need it now. Note that $\tilde{u}_n(t) := e_\lambda(t)u_n(t)$ (set here $e_\lambda(t) := e^{\lambda(t-T)}$) satisfies

$$(\lambda - \partial + A + b_n \cdot \nabla)\tilde{u}_n = \mathbf{1}_{[0,T]}e_\lambda F.$$

Hence, we can apply to $b^{\frac{1}{p_1}} \cdot \nabla u_n$ the solution representation of 8.0.1 with $f := \mathbf{1}_{[0,T]}e_\lambda F$ after reversing the direction of time. Upon noting that

$$b^{\frac{1}{p_1}} \cdot \nabla u_n = e_{-\lambda} R_{p_1} (1 + Q_{p_1} R_{p_1})^{-1} (\lambda - \partial + (-\Delta)^{\frac{\alpha}{2}})^{(-1 + \frac{1}{\alpha})\frac{1}{p_1}} \mathbf{1}_{[0,T]}e_\lambda F,$$

we thus obtain

$$I_n \leq K_1 e^{\lambda T} \|\mathbf{1}_{[0,T] \times B_R} b^{\frac{1}{q}}(1 - \mathbf{1}_n)\|_q \|F\|_{p_1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

2) Next, we deal with J_n . This term is easier since $\mathbf{1}_n b - b_n$ is simply the difference between the cutoff of b and the mollification of this cutoff. Therefore, given any sequence $\alpha_n \downarrow 0$ and any $p_1 > p$ close to p , by selecting $\varepsilon_n \downarrow 0$ in the definition of b_n sufficiently rapidly and using the fact that we are working on the ball B_R , we have

$$J_n = K \|\mathbf{1}_{[0,T] \times B_R} |\mathbf{1}_n b - b_n| |\nabla u_n|\|_p \leq K \alpha_n \|\nabla u_n\|_{p_1}.$$

Using again the results of 8.0.1, we obtain that $\|\nabla u_n\|_{p_1} \leq K_2 e^{\lambda T} \|F\|_{p_1}$, and so

$$J_n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence

$$\mathbb{E}_x^i \int_0^{T \wedge \tau_R} [(b - b_n) \cdot \nabla u_n](t, \omega_t) dt \rightarrow 0 \quad (n \rightarrow \infty).$$

It remains to note, using again 8.0.1, that solutions u_n converge to a function $u \in C([0, T], C_\infty(\mathbb{R}^d))$. Therefore, we can pass to the limit in (8.26), first in n and then in $R \rightarrow \infty$, to obtain

$$0 = u(0, x) + \mathbb{E}_x^i \int_0^T F(t, \omega_t) dt \quad i = 1, 2,$$

which gives (8.24).

This ends the proof of Theorem 1. □

9. PROOF OF THEOREM 3

(i) Set $A := \sum_{i=1}^N (-\Delta)_{x^i}^{\frac{\alpha}{2}}$. In order to apply the argument of [KM], we need to establish the following:

a) Pointwise bound

$$|\nabla_x (\lambda + A)^{-1}(x - y)| \leq c_{d,N,\alpha} (\lambda + A)^{-1 + \frac{1}{\alpha}}(x - y)$$

for all $x, y \in \mathbb{R}^{dN}$, $x \neq y$. Indeed, recall that, for $z \in \mathbb{R}^d$,

$$e^{-t(-\Delta)_{\mathbb{R}^d}^{\frac{\alpha}{2}}}(z) \approx t(|z|^{-d-\alpha} \wedge t^{-\frac{d+\alpha}{\alpha}})$$

and, using [BJ, proof of Lemma 5],

$$\begin{aligned} |\nabla_z e^{-t(-\Delta)_{\mathbb{R}^d}^{\frac{\alpha}{2}}}(z)| &\leq K_1 t (|z|^{-d-\alpha-1} \wedge t^{-\frac{d+\alpha+1}{\alpha}}) \\ &\quad \text{(use the previous estimate)} \\ &\leq K_2 t^{-\frac{1}{\alpha}} e^{-t(-\Delta)_{\mathbb{R}^d}^{\frac{\alpha}{2}}}(z). \end{aligned}$$

Therefore, taking into account the factorization of e^{-tA} , for all $x \in \mathbb{R}^{dN}$,

$$\begin{aligned} |\nabla_{x^i} e^{-tA}(x)| &\leq |\nabla_{x^i} e^{-t(-\Delta)_{x^i}^{\frac{\alpha}{2}}}(x^i)| \prod_{j \neq i} e^{-t(-\Delta)_{x^j}^{\frac{\alpha}{2}}}(x^j) \\ &\leq K_2 t^{-\frac{1}{\alpha}} e^{-tA}(x). \end{aligned}$$

Hence

$$|\nabla_x e^{-tA}(x)| \leq C t^{-\frac{1}{\alpha}} e^{-tA}(x).$$

Multiplying the previous inequality by $e^{-\lambda t}$ and integrating in t from 0 to ∞ , we arrive at the sought bound using $(\lambda + A)^{-\frac{\gamma}{2}}(x) = \frac{1}{\Gamma(\frac{\gamma}{2})} \int_0^\infty e^{-\lambda t} t^{\frac{\gamma}{2}-1} e^{-tA}(x) dt$ ($0 < \gamma \leq 2$).

b) The many-particle drift $b = (b^1, \dots, b^N)$,

$$b_i(x) = \frac{1}{N} \sum_{j=1, j \neq i}^N K(x^i - x^j),$$

is weakly form-bounded on \mathbb{R}^{dN} with respect to A . This follows by first noting that $K_{ij}(x) := K(x^i - x^j)$ satisfies

$$\| |K_{ij}|^{\frac{1}{2}} (\lambda + (-\Delta)_{x^i}^{\frac{\alpha}{2}})^{-\frac{1}{2} \frac{\alpha-1}{\alpha}} \|_{L^2(\mathbb{R}^{dN}) \rightarrow L^2(\mathbb{R}^{dN})} \leq \sqrt{\delta}$$

and therefore $\| |K_{ij}|^{\frac{1}{2}} (\lambda + A)^{-\frac{1}{2} \frac{\alpha-1}{\alpha}} \|_{L^2(\mathbb{R}^{dN}) \rightarrow L^2(\mathbb{R}^{dN})} \leq \sqrt{\delta}$, so it remains to add up these inequalities, taking into account the normalization by $\frac{1}{N}$, in order to arrive at

$$\| |b|^{\frac{1}{2}} (\lambda + A)^{-\frac{1}{2} \frac{\alpha-1}{\alpha}} \|_{L^2(\mathbb{R}^{dN}) \rightarrow L^2(\mathbb{R}^{dN})} \leq \sqrt{N-1} \sqrt{\delta}.$$

Armed with a), b) and using the fact that A is a symmetric Markov generator, we obtain the estimate

$$\| |b|^{\frac{1}{p}} \cdot \nabla (\lambda + A)^{-1} |b|^{\frac{1}{p'}} \|_{L^p(\mathbb{R}^{dN}) \rightarrow L^p(\mathbb{R}^{dN})} \leq c_{d,\alpha,p} (N-1) \delta$$

and repeat the argument of [KM], i.e. postulate the Neumann series for $A + b \cdot \nabla$, show that it determines the resolvent of a Feller generator, and then use the argument of Podolynny-Portenko and Chen-Wang – i.e. the one that we also employ in the present paper – to show that the Feller semigroup indeed delivers weak solutions to the particle system.

(ii) It suffices for us to show that the drift $K \star w$ satisfies $\| |K \star w|^{\frac{1}{\alpha-1}} \|_{E_{1+\epsilon}} \leq \| |K|^{\frac{1}{\alpha-1}} \|_{M_{1+\epsilon}}$ ($\leq c_{d,\alpha,p,\epsilon}$) and then apply the regularity result of Theorem 2. Indeed,

$$\begin{aligned}
\| |K \star w|^{\frac{1}{\alpha-1}} \|_{E_{1+\epsilon}} &= \sup_{(t,x) \in \mathbb{R}^{1+d}, r>0} r \left(\frac{1}{|C_r^\alpha|} \int_{C_r^\alpha(t,x)} |K \star w|^{\frac{1+\epsilon}{\alpha-1}} \right)^{\frac{1}{1+\epsilon}} \\
&= \sup_{(t,x) \in \mathbb{R}^{1+d}, r>0} r \left(\frac{1}{|C_r^\alpha|} \int_t^{t+r^\alpha} \| |K \star w|^{\frac{1+\epsilon}{\alpha-1}} \|_{L^{\frac{1+\epsilon}{\alpha-1}}(B_r(x))} ds \right)^{\frac{1}{1+\epsilon}} \\
&= \sup_{(t,x) \in \mathbb{R}^{1+d}, r>0} r \left(\frac{1}{|C_r^\alpha|} \int_t^{t+r^\alpha} \left\| \int_{\mathbb{R}^d} K(\cdot - y) w(s, y) dy \right\|_{L^{\frac{1+\epsilon}{\alpha-1}}(B_r(x))}^{\frac{1+\epsilon}{\alpha-1}} ds \right)^{\frac{1}{1+\epsilon}} \\
&\leq \sup_{(t,x) \in \mathbb{R}^{1+d}, r>0} r \left(\frac{1}{|C_r^\alpha|} \int_t^{t+r^\alpha} \left(\int_{\mathbb{R}^d} \|K(\cdot - y)\|_{L^{\frac{1+\epsilon}{\alpha-1}}(B_r(x))} w(s, y) dy \right)^{\frac{1+\epsilon}{\alpha-1}} ds \right)^{\frac{1}{1+\epsilon}} \\
&\leq \sup_{(t,z) \in \mathbb{R}^{1+d}, r>0} r \left(\frac{1}{|C_r^\alpha|} \|K\|_{L^{\frac{1+\epsilon}{\alpha-1}}(B_r(z))}^{\frac{1+\epsilon}{\alpha-1}} \int_t^{t+r^\alpha} \left(\int_{\mathbb{R}^d} w(s, y) dy \right)^{\frac{1+\epsilon}{\alpha-1}} ds \right)^{\frac{1}{1+\epsilon}} \\
&\leq \sup_{z \in \mathbb{R}^d, r>0} r \left(\frac{1}{|B_r|} \|K\|_{L^{\frac{1+\epsilon}{\alpha-1}}(B_r(z))}^{\frac{1+\epsilon}{\alpha-1}} \right)^{\frac{1}{1+\epsilon}} = \| |K|^{\frac{1}{\alpha-1}} \|_{M_{1+\epsilon}},
\end{aligned}$$

where at the last step we have used $\int_{\mathbb{R}^d} w(s, y) dy \leq 1$. (Above we have, in fact, reproved for the reader's convenience a special case of Young's inequality for Morrey spaces [BT].) \square

APPENDIX A. PROOF OF PROPOSITION 1

The following proof is taken from [K2]. It suffices to show:

Let $|b|^{\frac{1}{\alpha-1}} \in E_q$ for some $1 < q < d + \alpha$. Then for every $1 < p < \infty$ there exists constant $C_{d,p,q,\alpha}$ such that

$$\begin{aligned}
|\langle |b|(P_{(1-\frac{1}{\alpha})\frac{2}{p}} f)^p \rangle| &\leq C_{d,p,q,\alpha}^p \| |b|^{\frac{1}{\alpha-1}} \|_{E_q}^{\alpha-1} \|f\|_p^p, \\
|\langle |b|(P_{(1-\frac{1}{\alpha})\frac{2}{p}}^* f)^p \rangle| &\leq C_{d,p,q,\alpha}^p \| |b|^{\frac{1}{\alpha-1}} \|_{E_q}^{\alpha-1} \|f\|_p^p,
\end{aligned} \tag{A.1}$$

where

$$P_\gamma f(t, x) := \int_{\mathbb{R}^{1+d}} p_\gamma(s, y) f(t + s, x + y) dy ds, \tag{A.2}$$

$$P_\gamma^* f(t, x) := \int_{\mathbb{R}^{1+d}} p_\gamma(s, y) f(t - s, x + y) dy ds \tag{A.3}$$

on $f \in C_c(\mathbb{R}^{1+d})$.

Indeed, by (1.22) we have

$$\begin{aligned}
P_\gamma f &\geq C^{-1} (-\partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{\gamma}{2}} f, \\
P_\gamma^* f &\geq C^{-1} (\partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{\gamma}{2}} f
\end{aligned}$$

on positive f , and so the assertion of Proposition 1 follows.

We will only prove (A.1). Since our regularization $\{b_n\}$ of b can be treated as essentially monotone (i.e. at the level of Euclidean norms), we will consider $b_n := \mathbf{1}_n b$, where $\mathbf{1}_n$ is the indicator of $\{|t| \leq n, |x| \leq n, |b(t, x)| \leq n\}$ and then use the Dominated convergence theorem.

We will need the following notations and estimates. For brevity, below we write b instead of b_n . Set

$$M_\beta f(t, x) := \sup_{\rho > 0} \rho^\beta \frac{1}{|C_\rho(t, x)|} \int_{C_\rho(t, x)} |f(s, y)| dy ds, \quad 0 \leq \beta \leq d + \gamma,$$

where we put for brevity $C_\rho := C_\rho^\alpha$, and define the maximal function $M := M_0$. Also, define

$$\hat{M}f(t, x) := \sup_{(t, x) \in C} \frac{1}{|C|} \int_C |f| dy ds,$$

where the supremum is taken over all parabolic α -cylinders C containing (t, x) . This function is required for harmonic-analytic arguments that are transferred from the elliptic setting (i.e. from the cubes). The fact that our ‘‘cubes’’ C are squeezed in one of the coordinates presents no problem.

Lemma A.1. *The following are true for every $\beta \in]\frac{\alpha\gamma}{2}, d + \gamma]$, for all $(t, x) \in \mathbb{R}^{1+d}$:*

(i) *For all $\rho > 0$, we have*

$$\begin{aligned} P_\gamma(\mathbf{1}_{C_\rho^c} f)(t, x) &\leq K \rho^{\frac{\alpha\gamma}{2} - \beta} M_\beta f(t, x), & K &:= \frac{1}{\frac{\alpha\gamma}{2} - \beta} \left(-\frac{\alpha\gamma}{2} + d + \alpha\right), \\ P_\gamma(\mathbf{1}_{C_\rho} f)(t, x) &\leq N \rho^{\frac{\alpha\gamma}{2}} M f(t, x), & N &:= \frac{2}{\alpha\gamma} \left(-\frac{\alpha\gamma}{2} + d + \alpha\right) \end{aligned}$$

for all $0 \leq f \in C_c(\mathbb{R}^{1+d})$.

(ii)

$$P_\gamma f(t, x) \leq C (M_\beta f(t, x))^{\frac{\alpha\gamma}{2} \frac{1}{\beta}} (M f(t, x))^{1 - \frac{\alpha\gamma}{2} \frac{1}{\beta}}, \quad C := K^{\frac{\alpha\gamma}{2} \frac{1}{\beta}} N^{1 - \frac{\alpha\gamma}{2} \frac{1}{\beta}}.$$

Proof of Lemma. We repeat Krylov [Krl, proof of Lemma 2.2] with some straightforward modifications needed to accommodate integral kernel $p_\gamma(s, y) := \mathbf{1}_{\{s > 0\}} s^{\frac{\gamma}{2}} (|y|^{-d-\alpha} \wedge s^{-\frac{d+\alpha}{\alpha}})$ instead of the Gaussian density $(4\pi s)^{-\frac{d}{2}} e^{-|y|^2/s}$ times $s^{\frac{\gamma}{2}-1}$.

(i) It suffices to carry out the proof for $(t, x) = (0, 0)$. Below $s \geq 0$, $y \in \mathbb{R}^d$. Set

$$Q := \{(s, y) \mid |y| \geq s^{\frac{1}{\alpha}}\}, \quad Q^c := \{(s, y) \mid |y| < s^{\frac{1}{\alpha}}\}.$$

With a slight abuse of notation, we write $p_\gamma(s, r)$ instead of $p_\gamma(s, y)$, where $r = |y|$.

The following result is obtained using integrating by parts. Let $\xi > 0$ and $g : [0, \infty[\rightarrow [0, \infty[$ be such that $t^{-\xi} \int_0^t g(s) ds \rightarrow 0$ as $t \uparrow \infty$. Then, for every $\tau \geq 0$,

$$\int_\tau^\infty t^{-\xi} g(t) dt \leq \xi \int_\tau^\infty t^{-\xi-1} \left(\int_\tau^t g(s) ds \right) dt. \quad (\text{A.4})$$

1. If $r \geq s^{\frac{1}{\alpha}}$, then $p_\gamma(s, r) \leq r^{\frac{\alpha\gamma}{2}-d-\alpha}$. Therefore,

$$\begin{aligned}
P_\gamma(f\mathbf{1}_{Q \cap C_\rho^c})(0) &\leq \int_0^\infty \int_{\mathbb{R}^d} |y|^{\frac{\alpha\gamma}{2}-d-\alpha} f(s, y) \mathbf{1}_{Q \cap C_\rho^c} dy ds \\
&= \int_{\{0 \leq s \leq |y|^\alpha, |y| \geq \rho\}} |y|^{\frac{\alpha\gamma}{2}-d-\alpha} f(s, y) dy ds \\
&= \int_\rho^\infty r^{\frac{\alpha\gamma}{2}-d-\alpha} \int_0^{r^\alpha} \int_{\{|y|=r\}} f(s, y) d\sigma_r ds dr \\
&\text{(we apply (A.4) with } \xi \equiv C_1 = -\frac{\alpha\gamma}{2} + d + \alpha) \\
&\leq C_1 \int_\rho^\infty r^{\frac{\alpha\gamma}{2}-d-\alpha-1} \int_\rho^r \int_0^{\theta^\alpha} \int_{\{|y|=\theta\}} f(s, y) d\sigma_\theta ds d\theta dr \\
&\leq C_1 \int_\rho^\infty r^{\frac{\alpha\gamma}{2}-d-\alpha-1} \int_0^r \int_0^{\theta^\alpha} \int_{\{|y|=\theta\}} f(s, y) d\sigma_\theta ds d\theta dr \\
&= C_1 \int_\rho^\infty r^{\frac{\alpha\gamma}{2}-d-\alpha-1} I(r) dr,
\end{aligned}$$

where

$$I(r) := \int_{C_r} f(s, y) ds dy.$$

Clearly, $I(r) \leq r^{d+\alpha-\beta} M_\beta f(0)$, so

$$\begin{aligned}
P_\gamma(f\mathbf{1}_{Q \cap C_\rho^c})(0) &\leq C_1 \int_\rho^\infty r^{\frac{\alpha\gamma}{2}-d-\alpha-1} r^{d+\alpha-\beta} M_\beta f(0) \\
&\leq K \rho^{\frac{\alpha\gamma}{2}-\beta} M_\beta f(0), \quad K = \frac{1}{\frac{\alpha\gamma}{2}-\beta} C_1 \equiv \frac{1}{\frac{\alpha\gamma}{2}-\beta} \left(-\frac{\alpha\gamma}{2} + d + \alpha\right),
\end{aligned}$$

where we have used the hypothesis $\beta > \frac{\alpha\gamma}{2}$.

2. If $r \leq s^{\frac{1}{\alpha}}$, then $p_\gamma(s, r) = s^{\frac{\gamma}{2}-\frac{d}{\alpha}-1}$. Therefore,

$$\begin{aligned}
P_\gamma(f\mathbf{1}_{Q^c \cap C_\rho^c})(0) &= \int_{\rho^\alpha}^\infty s^{\frac{\gamma}{2}-\frac{d}{\alpha}-1} \int_{\{|y| \leq s^{\frac{1}{\alpha}}\}} f(s, y) dy ds \\
&\text{(we apply (A.4) with } \xi \equiv C_2 := -\frac{\gamma}{2} + \frac{d}{\alpha} + 1) \\
&\leq C_2 \int_{\rho^\alpha}^\infty s^{\frac{\gamma}{2}-\frac{d}{\alpha}-2} \int_{\rho^\alpha}^s \int_{|y| \leq \theta^{\frac{1}{\alpha}}} f(\vartheta, y) dy d\vartheta ds \\
&\leq C_2 \int_{\rho^\alpha}^\infty s^{\frac{\gamma}{2}-\frac{d}{\alpha}-2} \int_0^s \int_{|y| \leq \theta^{\frac{1}{\alpha}}} f(\vartheta, y) dy d\vartheta ds \\
&= C_2 \int_{\rho^\alpha}^\infty s^{\frac{\gamma}{2}-\frac{d}{\alpha}-2} I(s^{\frac{1}{\alpha}}) ds = \alpha C_2 \int_\rho^\infty r^{\frac{\alpha\gamma}{2}-d-\alpha-1} I(r) dr \\
&\leq \alpha C_2 \int_\rho^\infty r^{\frac{\alpha\gamma}{2}-d-\alpha-1} r^{d+\alpha-\beta} dr M_\beta f(0) \\
&= K \rho^{\frac{\alpha\gamma}{2}-\beta} M_\beta f(0)
\end{aligned}$$

(we have used $\beta > \frac{\alpha\gamma}{2}$). Combined with Step 1, this yields the first inequality of Lemma A.1(i).

Let us prove the second inequality in Lemma A.1(i):

3. We have, using integration by parts,

$$\begin{aligned}
P_\gamma(f\mathbf{1}_{Q \cap C_\rho})(0) &\leq \int_0^\rho r^{\frac{\alpha\gamma}{2}-d-\alpha} \int_0^{r^\alpha} \int_{\{|y|=r\}} f(s, y) d\sigma_r ds dr \\
&\leq \int_0^\rho r^{\frac{\alpha\gamma}{2}-d-\alpha} \frac{\partial}{\partial r} \int_0^r \int_0^{\theta^\alpha} \int_{\{|y|=\theta\}} f(s, y) d\sigma_\theta ds d\theta dr \\
&= J_1 + C_3 \int_0^\rho r^{\frac{\alpha\gamma}{2}-d-\alpha-1} \int_0^r \int_0^{\theta^\alpha} \int_{\{|y|=\theta\}} f(s, y) d\sigma_\theta ds d\theta dr \\
&\leq J_1 + C_3 \int_0^\rho r^{\frac{\alpha\gamma}{2}-d-\alpha-1} I(r) dr,
\end{aligned}$$

where $C_3 := -\frac{\alpha\gamma}{2} + d + \alpha$ and

$$\begin{aligned}
J_1 &:= \rho^{\frac{\alpha\gamma}{2}-d-\alpha} \int_0^\rho \int_0^{\theta^\alpha} \int_{\{|y|=\theta\}} f(s, y) d\sigma_\theta ds d\theta \\
&\leq \rho^{\frac{\alpha\gamma}{2}-d-\alpha} I(\rho).
\end{aligned}$$

Since $I(\rho) \leq \rho^{d+\alpha} Mf(0)$, we have

$$\begin{aligned}
P_\gamma(f\mathbf{1}_{Q \cap C_\rho})(0) &\leq \rho^{\frac{\alpha\gamma}{2}-d-\alpha} \rho^{d+\alpha} Mf(0) + C_3 \int_0^\rho r^{\frac{\alpha\gamma}{2}-d-\alpha-1} r^{d+\alpha} dr Mf(0) \\
&= \rho^{\frac{\alpha\gamma}{2}} Mf(0) + \frac{2C_3}{\alpha\gamma} \rho^{\frac{\alpha\gamma}{2}} Mf(0) \\
&= N \rho^{\frac{\alpha\gamma}{2}} Mf(0), \quad N = 1 + \frac{2C_3}{\alpha\gamma} \equiv \frac{2}{\alpha\gamma} \left(-\frac{\alpha\gamma}{2} + d + \alpha\right).
\end{aligned}$$

4. Next,

$$\begin{aligned}
P_\gamma(f\mathbf{1}_{Q^c \cap C_\rho})(0) &= \int_0^{\rho^\alpha} s^{\frac{\gamma}{2}-\frac{d}{\alpha}-1} \int_{\{|y| \leq s^{\frac{1}{\alpha}}\}} f(s, y) dy ds \\
&\leq J_2 + C_4 \int_0^{\rho^\alpha} s^{\frac{\gamma}{2}-\frac{d}{\alpha}-2} I(s^{\frac{1}{\alpha}}) ds \\
&= J_2 + \alpha C_4 \int_0^\rho r^{\frac{\alpha\gamma}{2}-d-\alpha-1} I(r) dr,
\end{aligned}$$

where $C_4 = -\frac{\gamma}{2} + \frac{d}{\alpha} + 1$ and

$$\begin{aligned}
J_2 &:= \rho^{\frac{\alpha\gamma}{2}-d-\alpha} \int_0^{\rho^\alpha} \int_{\{|y| \leq \vartheta^{\frac{1}{\alpha}}\}} f(\vartheta, y) dy d\vartheta \\
&\leq \rho^{\frac{\alpha\gamma}{2}-d-\alpha} I(\rho).
\end{aligned}$$

Thus, applying again inequality $I(\rho) \leq \rho^{d+\alpha} Mf(0)$, we obtain $P_\gamma(f\mathbf{1}_{Q^c \cap C_\rho})(0) \leq N \rho^{\frac{\alpha\gamma}{2}} Mf(0)$. Together with the result of step 1, this completes the proof of the second inequality in Lemma A.1(i).

(ii) is obtained from (i) by adding up both inequalities and minimizing in ρ . \square

Armed with Lemma A.1(ii), we now prove (A.1). Put $u := P_{(1-\frac{1}{\alpha})\frac{2}{p}} f$. Then

$$\begin{aligned} |\langle |b|(P_{(1-\frac{1}{\alpha})\frac{2}{p}} f)^p \rangle| &= |\langle |b||u|^{p-1}, P_{(1-\frac{1}{\alpha})\frac{2}{p}} f \rangle| \\ &\leq |\langle P_{(1-\frac{1}{\alpha})\frac{2}{p}}^* (|b||u|^{p-1}), f \rangle| \leq \|P_{(1-\frac{1}{\alpha})\frac{2}{p}}^* (|b||u|^{p-1})\|_{p'} \|f\|_p. \end{aligned} \quad (\text{A.5})$$

Thus, to obtain (A.1), we need to bound the coefficient $\|P_{(1-\frac{1}{\alpha})\frac{1}{p}}^* (|b||u|^{p-1})\|_{p'}$. To this end, we apply pointwise estimate

$$P_{(1-\frac{1}{\alpha})\frac{2}{p}}^* (|b||u|^{p-1}) = P_{(1-\frac{1}{\alpha})\frac{2}{p}}^* (|b|^{\frac{1}{p}+\nu} |b|^{\frac{1}{p'}-\nu} |u|^{p-1}) \leq P_{(1-\frac{1}{\alpha})\frac{2}{p}}^* (|b|^{1+\nu p})^{\frac{1}{p}} (P_{(1-\frac{1}{\alpha})\frac{2}{p}}^* (|b|^{1-\nu p'} |u|^p))^{\frac{1}{p'}}$$

for a small $\nu > 0$ such that $1 + \nu p < q_0$ for some fixed $q_0 < q$. Hence

$$\|P_{(1-\frac{1}{\alpha})\frac{1}{p}}^* (|b||u|^{p-1})\|_{p'} \leq \langle |b|^{1-\nu p'} |u|^p, P_{(1-\frac{1}{\alpha})\frac{2}{p}} [P_{(1-\frac{1}{\alpha})\frac{2}{p}}^* (|b|^{1+\nu p})]^{\frac{1}{p-1}} \rangle. \quad (\text{A.6})$$

Step 1. Applying Lemma A.1(ii) with $\gamma := (1 - \frac{1}{\alpha})\frac{2}{p}$, $\beta := (\alpha - 1)(1 + \nu p)$ (clearly, $\beta > \frac{\alpha\gamma}{2}$) or, rather, its straightforward variant for P_α^* and noting that in this case

$$\frac{\alpha\gamma}{2} \frac{1}{\beta} = \frac{1}{p} \frac{1}{1 + \nu p},$$

and (this will be needed a few lines later)

$$\begin{aligned} M_{(\alpha-1)(1+\nu p)} |b|^{1+\nu p}(t, x) &= \sup_{\rho>0} \rho^{(\alpha-1)(1+\nu p)} \frac{1}{|C_\rho(t, x)|} \int_{C_\rho(t, x)} |b(s, y)|^{1+\nu p} dy ds \\ &\leq \sup_{\rho>0} \rho^{(\alpha-1)(1+\nu p)} \left(\frac{1}{|C_\rho(t, x)|} \int_{C_\rho(t, x)} |b(s, y)|^{\frac{1+\nu p}{\alpha-1}} dy ds \right)^{\alpha-1} \\ &= \left[\sup_{\rho>0} \rho \left(\frac{1}{|C_\rho(t, x)|} \int_{C_\rho(t, x)} |b(s, y)|^{\frac{1+\nu p}{\alpha-1}} dy ds \right)^{\frac{1}{1+\nu p}} \right]^{(\alpha-1)(1+\nu p)} \\ &\leq \| |b|^{\frac{1}{\alpha-1}} \|_{E_{1+\nu p}}^{(\alpha-1)(1+\nu p)}. \end{aligned}$$

we obtain

$$\begin{aligned} P_{(1-\frac{1}{\alpha})\frac{2}{p}}^* (|b|^{1+\nu p}) &\leq N (M_{(\alpha-1)(1+\nu p)} |b|^{1+\nu p})^{\frac{1}{p} \frac{1}{1+\nu p}} (M |b|^{1+\nu p})^{1-\frac{1}{p} \frac{1}{1+\nu p}} \\ &\leq C \| |b|^{\frac{1}{\alpha-1}} \|_{E_{1+\nu p}}^{\frac{\alpha-1}{p}} (\hat{M} |b|^{1+\nu p})^{1-\frac{1}{p} \frac{1}{1+\nu p}} \\ &\leq C \| |b|^{\frac{1}{\alpha-1}} \|_{E_{q_0}}^{\frac{\alpha-1}{p}} (\hat{M} |b|^{1+\nu p})^{1-\frac{1}{p} \frac{1}{1+\nu p}}. \end{aligned}$$

At this point, let us assume that $|b|^{1+\nu p}$ is an A_1 -weight, i.e.

$$\hat{M} |b|^{1+\nu p} \leq C_0 |b|^{1+\nu p} \quad (\text{A.7})$$

for some constant C_0 (we will get rid of this assumption later).

Remark 3. Up to this point all constants were explicit. Now we picked up constant C_0 which, as will be explained below (Step 4), depends on the constants in some classical inequalities of harmonic analysis.

Then

$$P_{(1-\frac{1}{\alpha})\frac{2}{p}}^* (|b|^{1+\nu p}) \leq C_2 \| |b|^{\frac{1}{p}} \|_{E_{q_0}} |b|^{1+\nu p - \frac{1}{p}}.$$

Step 2. After applying the previous estimate in (A.6), one sees that now we need to estimate

$$P_{(1-\frac{1}{\alpha})\frac{2}{p}}(|b|^{(1+\nu p-\frac{1}{p})\frac{1}{p-1}}) = P_{(1-\frac{1}{\alpha})\frac{2}{p}}(|b|^{\frac{1}{p}+\nu p'}).$$

Selecting ν even smaller, if needed, one may assume that $\frac{1}{p} + \nu p' < q_0$. By Lemma A.1(ii) with $\gamma := (1 - \frac{1}{\alpha})\frac{2}{p}$, $\beta := (\alpha - 1)(\frac{1}{p} + \nu p')$ (in this case, again, $\beta > \frac{\alpha\gamma}{2}$),

$$\begin{aligned} P_{(1-\frac{1}{\alpha})\frac{2}{p}}(|b|^{\frac{1}{p}+\nu p'}) &\leq C(M_{(\alpha-1)(\frac{1}{p}+\nu p')}|b|^{\frac{1}{p}+\nu p'})^{\frac{1}{p}\frac{1}{\frac{1}{p}+\nu p'}} (M|b|^{\frac{1}{p}+\nu p'})^{1-\frac{1}{p}\frac{1}{\frac{1}{p}+\nu p'}} \\ &\leq C\| |b|^{\frac{1}{\alpha-1}} \|_{E_{\frac{1}{p}+\nu p'}}^{\frac{\alpha-1}{p}} (\hat{M}|b|^{\frac{1}{p}+\nu p'})^{1-\frac{1}{p}\frac{1}{\frac{1}{p}+\nu p'}} \\ &\leq C\| |b|^{\frac{1}{\alpha-1}} \|_{E_{q_0}}^{\frac{\alpha-1}{p}} (\hat{M}|b|^{\frac{1}{p}+\nu p'})^{1-\frac{1}{p}\frac{1}{\frac{1}{p}+\nu p'}}. \end{aligned}$$

In addition to (A.7), let us temporarily assume that

$$\hat{M}|b|^{\frac{1}{p}+\nu p'} \leq C_0|b|^{\frac{1}{p}+\nu p'}. \quad (\text{A.8})$$

Then

$$P_{(1-\frac{1}{\alpha})\frac{2}{p}}(|b|^{\frac{1}{p}+\nu p'}) \leq C_2\| |b|^{\frac{1}{\alpha-1}} \|_{E_{q_0}}^{\frac{\alpha-1}{p}} |b|^{\nu p'}.$$

Step 3. Applying the results of Steps 1 and 2 in (A.6), we obtain

$$\|P_{(1-\frac{1}{\alpha})\frac{2}{p}}^*(|b||u|^{p-1})\|_{p'} \leq C_3\| |b|^{\frac{1}{\alpha-1}} \|_{E_{q_0}}^{\frac{\alpha-1}{p}} \langle |b||u|^p \rangle_{p'}^{\frac{1}{p'}}.$$

Therefore, (A.5) yields $\langle |b||u|^p \rangle \leq C_3\| |b|^{\frac{1}{\alpha-1}} \|_{E_{q_0}}^{\frac{\alpha-1}{p}} \langle |b||u|^p \rangle_{p'}^{\frac{1}{p'}}$, hence

$$\langle |b||u|^p \rangle_{p'}^{\frac{1}{p}} \leq C_3\| |b|^{\frac{1}{\alpha-1}} \|_{E_{q_0}}^{\frac{\alpha-1}{p}} \|f\|_p. \quad (\text{A.9})$$

Step 4. Now, we get rid of the assumptions (A.7) and (A.8) at expense of replacing $\|b\|_{E_{q_0}}$ in (A.9) by $\|b\|_{E_q}$, where, recall, $q_0 < q$. This, in turn, will give us (A.1). Fix $q_0 < q_1 < q$ and define

$$\tilde{b} := (\hat{M}|b|^{q_1})^{\frac{1}{q_1}}.$$

Then $\tilde{b} \geq |b|$ and \tilde{b} satisfies

$$\hat{M}\tilde{b}^{q_0} \leq C_0\tilde{b}^{q_0}, \quad (\text{A.10})$$

see [GR, p.158]. Since $1 + \nu p < q_0$, $\frac{1}{p} + \nu p' < q_0$, both inequalities (A.7) and (A.8) for \tilde{b} follow from (A.10), and so we have

$$\langle |b||u|^p \rangle_{p'}^{\frac{1}{p}} \leq C_3\| \tilde{b}^{\frac{1}{\alpha-1}} \|_{E_{q_0}}^{\frac{\alpha-1}{p}} \|f\|_p.$$

It remains to apply inequality $\| \tilde{b}^{\frac{1}{\alpha-1}} \|_{E_{q_0}} \leq C\| |b|^{\frac{1}{\alpha-1}} \|_{E_q}$, which was established in [Kr1, proof of Theorem 4.1] for $\alpha = 2$. In fact, [Kr1] gave two proofs of this fact, one is based on the Fefferman-Stein inequality, and the other one is more direct. In both cases, as is mentioned in [Kr1], transferring the corresponding harmonic-analytic results from cubes to \hat{M} defined on the parabolic cylinders (with $\alpha = 2$) presents no problem. (Note that our parabolic cylinders with $1 < \alpha < 2$ get closer to the cubes as $\alpha \downarrow 1$. So, at this last step of the proof we find ourselves between [Kr1] and the classical setting of cubes in \mathbb{R}^{1+d} .) \square

APPENDIX B. ESTIMATES USED IN THE PROOF OF PROPOSITION 3

Let us prove: provided $\theta > d$, for every $\frac{\theta}{\alpha} < p < \infty$, for all $\lambda > 1$,

$$\|\rho^{\frac{1}{p}}(\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{1}{p}(1-\frac{1}{\alpha})}\rho^{-\frac{1}{p}}g\|_{L^p(\mathbb{R}^{1+d})} \leq \lambda^{-\frac{1}{p}(1-\frac{1}{\alpha})}\|g\|_{L^p(\mathbb{R}^{1+d})}, \quad (\text{B.1})$$

$$\|\rho^{-\frac{1}{p}}(\lambda + \partial_t + (-\Delta)^{\frac{\alpha}{2}})^{-\frac{1}{p}(1-\frac{1}{\alpha})}\rho^{\frac{1}{p}}g\|_{L^p(\mathbb{R}^{1+d})} \leq \lambda^{-\frac{1}{p}(1-\frac{1}{\alpha})}\|g\|_{L^p(\mathbb{R}^{1+d})}. \quad (\text{B.2})$$

Proof. Let us prove (B.1). Write for brevity $\beta := \frac{1}{p}(1 - \frac{1}{\alpha})$, $A := (-\Delta)^{\frac{\alpha}{2}}$, $\partial := \partial_t$. We have

$$\begin{aligned} \|\rho^{\frac{1}{p}}(\lambda + \partial + A)^{-\beta}\rho^{-\frac{1}{p}}g\|_{L^p(\mathbb{R}^{1+d})} &= \|\rho^{\frac{1}{p}} \int_0^\infty e^{-\lambda s} s^{\beta-1} e^{-sA} \rho^{-\frac{1}{p}} g(t-s, \cdot) ds\|_{L^p(\mathbb{R}^{1+d})} \\ &\leq \left\| \int_0^\infty e^{-\lambda s} s^{\beta-1} \|\rho^{\frac{1}{p}} e^{-sA} \rho^{-\frac{1}{p}} g(t-s, \cdot)\|_{L^p(\mathbb{R}^d)} ds \right\|_{L^p(\mathbb{R}, dt)}. \end{aligned}$$

For every $s \geq 0$ (denoting by $p_s(z)$ the integral kernel of e^{-sA}),

$$\begin{aligned} \|\rho^{\frac{1}{p}} e^{-sA} \rho^{-\frac{1}{p}} g(t-s, \cdot)\|_{L^p(\mathbb{R}^d)} &= \langle \rho(x) | \langle p_s(z) \rho^{-\frac{1}{p}}(x-z) g(t-s, x-z) \rangle_z \rangle_x^{\frac{1}{p}} \\ &\quad (\text{use } \rho^{-\frac{1}{p}}(x-z) \leq c \rho^{-\frac{1}{p}}(x) \rho^{-\frac{1}{p}}(z)) \\ &\leq c \langle | \langle p_s(z) \rho^{-\frac{1}{p}}(z) g(t-s, x-z) \rangle_z |^p \rangle_x^{\frac{1}{p}} \\ &\quad (\text{apply Young's inequality}) \\ &\leq c \|p_s(\cdot)(1 + \kappa|\cdot|^2)^{\frac{\theta}{2p}}\|_{L^1(\mathbb{R}^d)} \|g(t-s, \cdot)\|_{L^p(\mathbb{R}^d)} \quad (\text{we use } \frac{\theta}{p} < \alpha) \\ &\leq C(1 + s^{\frac{\theta}{\alpha p}}) \|g(t-s, \cdot)\|_{L^p(\mathbb{R}^d)}. \end{aligned} \quad (\text{B.3})$$

We now return to the first chain of inequalities, substitute there the last estimate, and apply Young's inequality in the time variable ($1 + \frac{1}{p} = \frac{1}{1} + \frac{1}{p}$):

$$\begin{aligned} \|\rho^{\frac{1}{p}}(\lambda + \partial_t + A)^{-\frac{1}{p}(1-\frac{1}{\alpha})}\rho^{-\frac{1}{p}}g\|_{L^p(\mathbb{R}^{1+d})} &\leq C \int_0^\infty e^{-\lambda s} s^{\beta-1} (1 + s^{\frac{\theta}{\alpha p}}) ds \|g\|_{L^p(\mathbb{R}^{1+d})} \\ &\leq C_1 \lambda^{-\beta} \|g\|_{L^p(\mathbb{R}^{1+d})}, \quad \lambda > 1. \end{aligned}$$

This ends the proof of (B.1).

The proof of (B.2) is the same, except that in the counterpart of line (B.3) we need to use inequality $(1 + |x-z|^2)^{-\frac{\theta}{2p}} \leq c(1 + |x|^2)^{-\frac{\theta}{2p}}(1 + |z|^2)^{\frac{\theta}{2p}}$, i.e. $\rho^{\frac{1}{p}}(x-z) \leq c\rho^{\frac{1}{p}}(x)\rho^{-\frac{1}{p}}(z)$. \square

APPENDIX C. COUNTEREXAMPLE TO WEAK EXISTENCE

Below we provide a detailed justification of the counterexample to weak existence (Example 1). In fact, it only remains for us to prove identity (1.6) assuming that there exists a weak solution (X_t, Z_t) to SDE (1.1) with drift $b(x) = \kappa x|x|^{-\alpha}$, $\kappa > 0$, departing from the origin at time $t = 0$.

Lemma C.1. (1.6) holds.

Proof. Set

$$f(x) = |x|^\beta, \quad \beta \in (1, \alpha).$$

Since f is not C^2 at the origin, we regularize it by

$$f_\varepsilon(x) := (|x|^2 + \varepsilon^2)^{\frac{\beta}{2}}, \quad \varepsilon > 0.$$

Then $f_\varepsilon \in C^2(\mathbb{R}^d)$. Let

$$\tau_R := \inf\{t \geq 0 : |X_t| \geq R\}, \quad R > 0$$

and

$$\sigma_m := \inf\left\{u \geq 0 : \int_0^u |X_s|^{1-\alpha} ds \geq m\right\}, \quad m \geq 1.$$

Here, $\sigma_m \uparrow \infty$ a.s., because for every $t > 0$,

$$\int_0^t |X_s|^{1-\alpha} ds < \infty \quad \text{a.s.}$$

Applying Itô's formula (1.27) to $\chi_n f_\varepsilon$, where $\chi_n \in C_c^\infty(\mathbb{R}^d)$ satisfies $0 \leq \chi_n \leq 1$, $\chi_n = 1$ on $B_n(0)$, (so that $\chi_n f_\varepsilon$ is bounded and has bounded derivatives), and then letting $n \rightarrow \infty$, we obtain

$$\mathbb{E}f_\varepsilon(X_{t \wedge \tau_R \wedge \sigma_m}) = f_\varepsilon(0) + \mathbb{E} \int_0^{t \wedge \tau_R \wedge \sigma_m} \left(Af_\varepsilon(X_s) - b(X_s) \cdot \nabla f_\varepsilon(X_s) \right) ds, \quad (\text{C.1})$$

where, throughout this appendix,

$$A := -(-\Delta)^{\frac{\alpha}{2}}.$$

We now justify the passage to the limit $\varepsilon \downarrow 0$ in (C.1), in order to obtain the sought identity (1.6).

Step 1. Convergence of the left-hand side of (C.1). Using the SDE, we find

$$X_{t \wedge \tau_R \wedge \sigma_m} = - \int_0^{t \wedge \tau_R \wedge \sigma_m} b(X_s) ds + Z_{t \wedge \tau_R \wedge \sigma_m}.$$

Moreover, by the definition of σ_m ,

$$\int_0^{t \wedge \tau_R \wedge \sigma_m} |b(X_s)| ds = \kappa \int_0^{t \wedge \tau_R \wedge \sigma_m} |X_s|^{1-\alpha} ds \leq \kappa m.$$

Hence

$$|X_{t \wedge \tau_R \wedge \sigma_m}| \leq \kappa m + \sup_{0 \leq s \leq t} |Z_s|.$$

Since $\beta < \alpha$, we have

$$\mathbb{E} \sup_{0 \leq s \leq t} |Z_s|^\beta < \infty.$$

Thus $f_\varepsilon(X_{t \wedge \tau_R \wedge \sigma_m})$ is dominated by an integrable random variable independent of ε , and therefore

$$\mathbb{E}f_\varepsilon(X_{t \wedge \tau_R \wedge \sigma_m}) \rightarrow \mathbb{E}|X_{t \wedge \tau_R \wedge \sigma_m}|^\beta.$$

Step 2. Convergence of the right-hand side of (C.1). Clearly, $f_\varepsilon(0) \rightarrow 0$. We next pass to the limit in the drift term and the fractional Laplacian term:

1) For the drift term, we have

$$b(x) \cdot \nabla f_\varepsilon(x) = \kappa \beta (|x|^2 + \varepsilon^2)^{\frac{\beta}{2}-1} |x|^{2-\alpha},$$

and therefore

$$|b(x) \cdot \nabla f_\varepsilon(x)| \leq \kappa \beta |x|^{\beta-\alpha}, \quad x \neq 0.$$

On $\{s \leq \tau_R\}$,

$$|X_s|^{\beta-\alpha} = |X_s|^{1-\alpha} |X_s|^{\beta-1} \leq R^{\beta-1} |X_s|^{1-\alpha},$$

while

$$\int_0^{t \wedge \tau_R \wedge \sigma_m} |X_s|^{\beta-\alpha} ds \leq R^{\beta-1} m \quad \text{a.s..}$$

Therefore, by the Dominated convergence theorem,

$$\mathbb{E} \int_0^{t \wedge \tau_R \wedge \sigma_m} b(X_s) \cdot \nabla f_\varepsilon(X_s) ds \rightarrow \kappa \beta \mathbb{E} \int_0^{t \wedge \tau_R \wedge \sigma_m} |X_s|^{\beta-\alpha} ds.$$

2) For the fractional Laplacian term, we first show that

$$Af_\varepsilon(x) \rightarrow A|x|^\beta = c_{d,\alpha,\beta}|x|^{\beta-\alpha}, \quad x \neq 0. \quad (\text{C.2})$$

Indeed, recall the Lévy representation for the fractional Laplacian,

$$Af(x) = \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - \mathbf{1}_{\{|z| \leq 1\}} \nabla f(x) \cdot z \right) \nu(dz) \quad \text{with } \nu(dz) = c_{d,\alpha} |z|^{-d-\alpha} dz.$$

Define

$$F_\varepsilon(x, z) := f_\varepsilon(x+z) - f_\varepsilon(x) - \mathbf{1}_{\{|z| \leq 1\}} \nabla f_\varepsilon(x) \cdot z.$$

Fix $x \neq 0$. We estimate $F_\varepsilon(x, z)$ separately on three regions:

For $|z| \leq |x|/2$, we have $|x+z| \geq \frac{|x|}{2}$, and so

$$|F_\varepsilon(x, z)| \leq C|z|^2 \sup_{0 \leq \theta \leq 1} |D^2 f_\varepsilon(x + \theta z)| \leq C|x|^{\beta-2}|z|^2. \quad (\text{C.3})$$

For $|x|/2 < |z| \leq 1$, using $|\nabla f_\varepsilon(x)| \leq C|x|^{\beta-1}$ and the Mean value theorem, we have

$$|f_\varepsilon(x+z) - f_\varepsilon(x)| \leq C(|z|^\beta + |x|^{\beta-1}|z|),$$

and then

$$|F_\varepsilon(x, z)| \leq C(|z|^\beta + |x|^{\beta-1}|z|). \quad (\text{C.4})$$

For $|z| > 1$, since

$$|f_\varepsilon(x)| \leq C(1 + |x|^\beta), \quad (\text{C.5})$$

we have

$$|F_\varepsilon(x, z)| \leq C(|x|^\beta + |z|^\beta). \quad (\text{C.6})$$

Since $\alpha < 2$ and $\beta < \alpha$, these bounds are ν -integrable on their corresponding regions. Therefore, by the Dominated convergence theorem, we conclude (C.2).

Next, we show that

$$\mathbb{E} \int_0^{t \wedge \tau_R \wedge \sigma_m} Af_\varepsilon(X_s) ds \rightarrow c_{d,\alpha,\beta} \mathbb{E} \int_0^{t \wedge \tau_R \wedge \sigma_m} |X_s|^{\beta-\alpha} ds. \quad (\text{C.7})$$

Using (C.3)-(C.6), we have

$$\begin{aligned} |Af_\varepsilon(x)| &\leq \left(\int_{|z| \leq |x|/2} + \int_{|x|/2 < |z| \leq 1} + \int_{|z| > 1} \right) |F_\varepsilon(x, z)| \nu(dz) \\ &\leq C(|x|^{\beta-\alpha} + |x|^\beta + 1), \end{aligned}$$

while on $\{0 < |x| \leq R\}$ we have

$$|Af_\varepsilon(x)| \leq C_R(1 + |x|^{\beta-\alpha}).$$

Therefore, on $\{s \leq \tau_R\}$,

$$|Af_\varepsilon(X_s)| \leq C_R(1 + |X_s|^{\beta-\alpha}) \leq C_R(1 + R^{\beta-1}|X_s|^{1-\alpha}),$$

hence,

$$\int_0^{t \wedge \tau_R \wedge \sigma_m} |Af_\varepsilon(X_s)| ds \leq C_R(t + R^{\beta-1}m) \quad \text{a.s..}$$

Therefore, by the Dominated convergence theorem, we conclude (C.7).

Step 3. Armed with the convergence results of Steps 1 and 2, we now pass to the limit $\varepsilon \downarrow 0$ in (C.1) to obtain (1.6). \square

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