ON PARTICLE SYSTEMS AND CRITICAL STRENGTHS OF GENERAL SINGULAR INTERACTIONS

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ABSTRACT. For finite interacting particle systems with strong repulsing-attracting or general interactions, we prove global weak well-posedness almost up to the critical threshold of the strengths of attracting interactions (independent of the number of particles), and establish other regularity results, such as a heat kernel bound in the regions where strongly attracting particles are close to each other. Our main analytic instruments are a variant of De Giorgi's method in L^p with appropriately chosen large p, and an abstract desingularization theorem.

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1. Introduction

The paper is concerned with well-posedness and other properties of N-particle system

$$dX_{i} = -\frac{1}{N} \sum_{j=1, j \neq i}^{N} K_{ij}(X_{i} - X_{j})dt + M_{i}(X_{i})dt + \sqrt{2}dB_{i}, \qquad X_{i}(0) = x_{i} \in \mathbb{R}^{d}$$
(1.1)

 $\{B_i(t)\}_{t>0}$ are independent d-dimensional Brownian motions,

under broad assumptions on singular (i.e. locally unbounded) interaction kernels and drifts K_{ij} , M_i : $\mathbb{R}^d \to \mathbb{R}^d$ (i = 1, ..., N) that can have repulsion/attraction structure or can be of general form. Our primary goal is to obtain conditions on M_i and K_{ij} that

- 1) reach blow up effects, and
- 2) withstand the passage to the limit $N \to \infty$.

Interacting particle systems of type (1.1) arise in many physical and biological models [3, 9, 10, 13, 15, 16, 19, 23, 45, 52, 53]. Many of these models require one to deal with the interactions that are not only singular but are so singular that they reach blow up effects: replacing K_{ij} in (1.1) by $(1+\varepsilon)K_{ij}$, i.e. increasing the strength of interactions by factor $1+\varepsilon$, can lead to a collapse in the well-posedness of (1.1) even if $\varepsilon > 0$ is small. That is, the particles start to collide in finite time with positive probability, and (1.1) ceases to have a weak solution.

One of the main questions studied in the present paper is what is the critical threshold value of the strength of general singular interactions that separates the well-posedness of (1.1) from a blow up.

Throughout the paper, dimension $d \geq 3$. Important case d = 2 requires a separate study which we plan to carry out elsewhere.

To illustrate the blow up effect in particle systems, and to formalize the notion of the "strength of interactions", consider a particle system with the model singular attracting kernel (1.10):

$$dX_{i} = -\frac{1}{N} \sum_{j=1, j \neq i}^{N} \sqrt{\kappa} \frac{d-2}{2} \frac{X_{i} - X_{j}}{|X_{i} - X_{j}|^{2}} dt + \sqrt{2} dB_{i},$$
(1.2)

where κ measures the strength of attraction between the particles. (It is convenient for us to include factor $\frac{d-2}{2}$ in the coefficient in (1.10) because we are going to use Hardy's inequalities, see Example 1(2). The kind of Hardy inequalities that we need are not valid in two dimensions.)

(a) In the two-particle case N=2, a simple argument shows that for

$$\kappa > 16 \left(\frac{d}{d-2} \right)^2$$

and $X_1(0) = X_2(0)$ the particle system (1.2) does not have a weak solution. Informally, in the struggle between the drift and the diffusion the former starts to have an upper hand. If fact, already if $\kappa > 16$ the particles collide in finite time with positive probability even

if $X_1(0)$, $X_2(0)$ are uniformly distributed e.g. in a cube, see [7] for detailed proof. On the other hand, if

$$\kappa < 16$$
,

then (1.2) has a global in time weak solution for any initial configuration $X_1(0)$, $X_2(0) \in \mathbb{R}^d$. The latter can be seen from the results of the present paper.

(b) The weak well-posedness and the blow-up effects for the two-dimensional counterpart of (1.2)

$$dX_{i} = -\frac{1}{N} \sum_{j=1, j \neq i}^{N} \sqrt{\kappa} \frac{X_{i} - X_{j}}{|X_{i} - X_{j}|^{2}} dt + \sqrt{2} dB_{i}, \quad N \ge 2,$$
(1.3)

were studied in detail, among other problems connected to the Keller-Segel model of chemotaxis, in [10, 15].

(c) The density of the formal invariant measure of (1.2)

$$\psi(x) = \prod_{1 \le i \le j \le N} |x_i - x_j|^{-\sqrt{\kappa} \frac{d-2}{2} \frac{1}{N}}$$

is locally summable if and only if $\kappa < 16(\frac{d}{d-2})^2$. Also, as κ reaches and surpasses $\kappa = 16$, ψ ceases to be in $W_{\rm loc}^{2,1}$.

Another analytic fact that suggests that the singularities of the drift in (1.2) are critical for any $N \geq 2$, i.e. κ in general cannot be too large, is the estimates on the constant in the many-particle Hardy inequality (1.20) due to [22]. We employ their result in the proof of Theorem 2(iii).

(d) The blow up effects are observed for the Keller-Segel model (here in the parabolic-elliptic form):

$$\begin{cases} \partial_t \rho - \Delta \rho + \sqrt{\kappa} \operatorname{div} (\rho \nabla v) = 0, & \rho(0, \cdot) = \rho_0(\cdot), \\ -\Delta v = \rho, \end{cases}$$

where ρ is the population density and v is the chemical density [10, 13, 15, 16, 23, 52], see also references therein. Of course, $\int_{\mathbb{R}^d} \rho_0(x) dx = 1$ propagates to $\int_{\mathbb{R}^d} \rho(t, x) dx = 1$ for all t > 0. Solving the elliptic equation, one obtains the expression for the drift:

$$\nabla v = -K_1 * \rho, \quad K_1(y) = c_d \frac{y}{|y|^d}, \quad c_d > 0$$
 (1.4)

(we can further redefine κ to have $c_d = 1$). The resulting McKean-Vlasov PDE

$$\partial_t \rho - \Delta \rho - \sqrt{\kappa} \operatorname{div} \left(\rho(K_1 * \rho) \right) = 0 \tag{1.5}$$

is comparable to (1.2) only in dimension d=2, where it does indeed arise at the mean field limit of particle system (1.3) as $N \to \infty$ [10, 15, 16, 52]. The fact that one needs d=2 is at the first regard somewhat disappointing for us, see, however, the end of remark (iv) below. It should be added that there is a significant recent progress in understanding

the behaviour of (1.5), (1.3) around and at the critical threshold $\kappa = 16$ both at the level of PDEs and at the SDE level, see [16, 52]. At the PDE level there are other important results on admissible strengths of critical singular interactions and the McKean-Vlasov equation, including recent result in [8], see Section 1.2.

The proofs of the results described in (b) and in (d) depend on the special form of the Riesz interaction kernels in (1.2) and (1.4).

Despite the prominent role played in applications by the Riesz interaction kernels, there are many other situations where one needs to handle more general critical-order singular interactions. These are in the focus of the present paper. In this case, one can no longer exploit the special structure of the interaction kernel in (1.2). It turns out that one can still cover large portions (if not most) of the ranges of admissible strengths κ of interactions and establish, in particular, global weak well-posedness of particle system (1.1), which however requires us to use some deep methods in the theory of elliptic and parabolic PDEs. This is done in Theorems 1 and 2(i)-(iii). For the model singular attracting kernel in (1.10) we prove a necessarily non-Gaussian upper bound on the heat kernel (\equiv the density of the law) of particle system (1.1), see Theorem 2(iv). We believe that this bound is optimal in the regions where the particles are close to each other.

In Section 1.2 we comment on the existing literature on particle systems with general singular interactions.

We focus on weak solutions and exploit the connection of (1.1) to the Kolmogorov backward equation

$$\left(\partial_t - \Delta_x + \frac{1}{N} \sum_{i=1}^N \sum_{j=1, j \neq i}^N K_{ij}(x_i - x_j) \cdot \nabla_{x_i}\right) v = 0, \quad v(0, \cdot) = f(\cdot), \tag{1.6}$$

i.e. $v(t, x_1, \ldots, x_N) = \mathbf{E}_{X_1(0)=x_1, \ldots, X_N(0)=x_N}[f(X_1(t), \ldots, X_N(t))]$. Our main instruments in this paper are:

- De Giorgi's method, but ran in L^p with p chosen sufficiently large, in order to relax the assumptions on the strength of interactions.
- A "desingularization theorem" obtained, using ideas of Nash, in the paper with Semënov and Szczypkowski [38].

We impose conditions on the interaction kernels K_{ij} stated in the form of quadratic form inequalities, see (1.7) and (2.10), (2.12). The reason for this is two-fold. First, as we explain below, such conditions are ultimate in the sense that they provide a minimal PDE theory for the Kolmogorov equation (1.6); at the same time, there is a well developed machinery that allows to verify these conditions, see Example 1. Second, these conditions provide a natural setting for controlling the strength of interactions when N is large, see discussion after Theorem A.

Our class of general interaction kernels is given by Definition 1. We postpone the definition of our class of repulsing-attracting interaction kernels until the next section.

Let $L^p = L^p(\mathbb{R}^d)$ denote the Lebesgue spaces endowed with the norm $\|\cdot\|_p$. Let $W^{1,p}$ be the corresponding Sobolev spaces. Denote by $[L^p]^d$ the space of vector fields $\mathbb{R}^d \to \mathbb{R}^d$ with entries in L^p .

Definition 1. A Borel measurable vector field $K : \mathbb{R}^d \to \mathbb{R}^d$ is said to be form-bounded if $K \in [L^2_{loc}]^d$ and there exists constant κ ("form-bound of K") such that

$$||K\varphi||_2^2 \le \kappa ||\nabla \varphi||_2^2 + c_\kappa ||\varphi||_2^2 \quad \forall \varphi \in W^{1,2}$$

$$\tag{1.7}$$

for some $c_{\kappa} < \infty$.

In other words,

$$|K|^2 < \kappa(-\Delta) + c_{\kappa}$$

in the sense of quadratic forms in L^2 , which yields upon applying Cauchy-Schwarz inequality

$$K \cdot \nabla \leq \sqrt{\kappa}(-\Delta) + \frac{c_{\kappa}}{2\sqrt{\kappa}}.$$

We abbreviate (1.7) as $K \in \mathbf{F}_{\kappa}$.

The class \mathbf{F}_{κ} is a well known in the PDE literature condition on first-order perturbation in elliptic and parabolic operators. Moreover, unlike, say, the optimal Lebesgue class $|K_{ij}| \in L^d$, a larger class $K_{ij} \in \mathbf{F}_{\kappa}$ is in a sense ultimate from the PDE perspective: assuming $\kappa < (\frac{N}{N-1})^2$, it provides coercivity of the quadratic form of the corresponding Kolmogorov backward operator in $L^2(\mathbb{R}^{Nd})$:

$$\Lambda = -\Delta_x + \frac{1}{N} \sum_{i=1}^N \sum_{j=1, j \neq i}^N K_{ij}(x_i - x_j) \cdot \nabla_{x_i}, \quad x = (x_1, \dots, x_N) \in \mathbb{R}^{Nd}.$$
 (1.8)

Keeping in mind the connection between (1.1) and the Kolmogorov backward equation, it is natural for us to focus on the assumptions on K_{ij} that provide some "minimal theory" of Kolmogorov operator (1.8).

As the next Example 1(2) shows, the strength of attraction κ in model system (1.2) is a particular case of the form-bound κ in Definition 1.

Example 1. The following are some sufficient conditions for $K \in \mathbf{F}_{\kappa}$ stated in elementary terms:

$$|K| \in L^d \quad \Rightarrow \quad K \in \mathbf{F}_{\kappa},$$
 (1.9)

with κ that can be chosen arbitrarily small. (As a consequence, we obtain that the blow up effects described above in (a)-(c) can not be observed if we restrict our attention to $|K| \in L^d$. That said, the situation with the Lebesgue class drifts in the Keller-Segel model (d) is different because the regularity of the nonlinear drift $K_1 * \rho$ also depends on the regularity of the initial condition ρ_0 and is improving as q in $\rho_0 \in L^q$ increases.)

Indeed, for every $\varepsilon > 0$ we can represent $K = K_1 + K_2$ with $||K_1||_d < \varepsilon$ and $||K_2||_{\infty} < \infty$. So, we obtain, using the Sobolev embedding theorem,

$$||K\varphi||_2^2 \le 2||K_1||_d^2 ||\varphi||_{\frac{2d}{d-2}}^2 + 2||K_2||_{\infty}^2 ||\varphi||_2^2$$

$$\le C_S 2||K_1||_d^2 ||\nabla \varphi||_2^2 + 2||K_2||_{\infty}^2 ||\varphi||_2^2,$$

hence $K \in \mathbf{F}_{\kappa}$ with $\kappa = C_S 2\varepsilon$ and $c_{\kappa} = 2||K_2||_{\infty}^2$.

Of course, above κ can be chosen arbitrarily small at expense of increasing c_{κ} . Although in some questions the value of constant c_{κ} is important (e.g. in the study of long term behaviour of solution of (1.1)), they are not related to the problem of the blow up versus well-posedness in (1.1).

2. (Critical point singularities) The model singular interaction kernel

$$K(y) = \pm \sqrt{\kappa} \frac{d-2}{2} \frac{y}{|y|^2}, \quad y \in \mathbb{R}^d, \tag{1.10}$$

(+ is the attraction, - is the repulsion) is in \mathbf{F}_{κ} with $c_{\kappa} = 0$. This is a re-statement of the well known Hardy inequality:

$$\frac{(d-2)^2}{4} \||y|^{-1}\varphi\|_2^2 \le \|\nabla\varphi\|_2^2, \quad \forall \, \varphi \in W^{1,2}(\mathbb{R}^d).$$

This inequality is sharp: $K \notin \mathbf{F}_{\kappa'}$ for any $\kappa' < \kappa$ regardless of the value of $c_{\kappa'}$.

A finer example is given by the weighted Hardy inequality of [21]. Fix $0 \le \Phi \in L^q(S^{d-1})$ for some $q \ge \frac{2(d-2)^2}{2(d-1)} + 1$, where S^{d-1} is the unit sphere in \mathbb{R}^d . If

$$|K(y)|^2 \le \kappa \frac{(d-2)^2}{4} c \frac{\Phi(y/|y|)}{|y|^2}, \quad \text{where } c := \frac{|S^{d-1}|^{\frac{1}{q}}}{\|\Phi\|_{L^q(S^{d-1})}},$$

then $K \in \mathbf{F}_{\kappa}$ with $c_{\kappa} = 0$. Using this example, one can e.g. cut off a wedge in the model interaction kernel (1.10) while still controlling the value of the strength of interaction κ .

3. (Weak L^d class interaction kernels) More generally, vector fields K in $L^{d,\infty}$, i.e. such that

$$||K||_{d,\infty} := \sup_{s>0} s |\{y \in \mathbb{R}^d : |K(y)| > s\}|^{1/d} < \infty$$
(1.11)

are in \mathbf{F}_{κ} with $\sqrt{\kappa} = ||K||_{d,\infty} |B_1(0)|^{-\frac{1}{d}} \frac{2}{d-2}$, see [40]. When applied to (1.10), this inclusion gives the constant in Hardy's inequality.

4. (Morrey class interaction kernels) The scaling-invariant Morrey class $M_{2+\varepsilon}$, with $\varepsilon > 0$ fixed arbitrarily small, consists of vector fields $K \in [L_{\text{loc}}^{2+\varepsilon}]^d$ such that

$$||K||_{M_{2+\varepsilon}} := \sup_{r>0, y \in \mathbb{R}^d} r \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} |K|^{2+\varepsilon} dy \right)^{\frac{1}{2+\varepsilon}} < \infty.$$
 (1.12)

By one of the results in [14], if $K \in M_{2+\varepsilon}$, then $K \in \mathbf{F}_{\kappa}$ with $\kappa = c \|K\|_{M_{2+\varepsilon}}$ for a constant $c = c(d, \varepsilon)$ that depends on the constants in some classical inequalities of Harmonic Analysis.

This sufficient condition for form-boundedness can be further refined by considering the Chang-Wilson-Wolff class [11]: $K \in [L^2_{loc}]^d$ satisfies

$$||K||_{\xi} := \sup_{r>0, y \in \mathbb{R}^d} \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} |K|^2 r^2 \xi (|K|^2 r^2) dy \right)^{\frac{1}{2}} < \infty,$$

where $\xi:[0,\infty[\to[1,\infty[$ is an increasing function such that $\int_1^\infty \frac{ds}{s\xi(s)} < \infty$.

On the other hand, a simple argument with cutoff functions shows that the class of form-bounded vector fields \mathbf{F}_{κ} (say, $c_{\kappa} = 0$) is contained in the Morrey class M_2 .

It should be added that the cited results in [14, 11] appeared as a part of broader efforts to find necessary and sufficient conditions for form-boundedness stated in elementary terms (in the context related to study of Schrödinger operators with singular potentials, including self-adjointness, estimates on the number of bound states, resolvent convergence).

5. (Hypersurface singularities) Any interaction kernel K satisfying

$$|K(y)|^2 = C \frac{c(y)\mathbf{1}_{\{\frac{1}{2} \le |y| \le \frac{3}{2}\}}}{||y| - 1|(-\ln||y| - 1|)^{\beta}}, \quad \beta > 1.$$
(1.13)

is form-bounded, which can be seen from the previous example by arguing locally. (Note that the components of K are not in $L^{2+\epsilon}_{loc}$ for any $\epsilon > 0$; one can compare this with example (i).)

The class of form-bounded vector fields \mathbf{F}_{κ} is closed with respect to addition and multiplication by functions from L^{∞} (up to change of κ and c_{κ}). So, one can combine the previous examples.

Our main results, stated briefly, are as follows (omitting for now the repulsing-attracting interaction kernels):

Theorem A. (i) Let

$$K_{ij} \in \mathbf{F}_{\kappa}, \quad \kappa < 4 \left(\frac{N}{N-1}\right)^2.$$

Then there exists a strong Markov family of martingale solutions of particle system (1.1) that delivers a unique (in appropriate sense) weak solution to Cauchy problem for the Kolmogorov backward equation (1.6).

- (ii) If κ is smaller than a certain explicit constant $c_{d,N}$, then, moreover, conditional weak uniqueness and strong existence hold for (1.1).
 - (iii) In the special case

$$K_{ij}(y) = \sqrt{\kappa} \frac{d-2}{2} \frac{y}{|y|^2} + K_{0,ij}(y), \quad K_{0,ij} \in \mathbf{F}_{\nu},$$

if the strength of attraction satisfies only $\kappa < 16$ and ν is sufficiently small, then the first assertion in (i) still holds. Moreover, in the model attracting case

$$K_{ij}(y) = \sqrt{\kappa} \frac{d-2}{2} \frac{y}{|y|^2}, \quad \kappa < 16$$

the heat kernel of (1.1) satisfies an explicit non-Gaussian upper bound that we believe to be optimal in the regions where the particles are close to each other. (The constant 16 can be somewhat improved in low dimensions.)

For detailed statements, see Theorems 1 and 2.

The improvement in the assumptions on κ in (iii) is due to the use of many-particle Hardy inequality (1.20). In assertion (ii) constant $c_{d,N} \downarrow 0$ as the number of particles $N \uparrow \infty$, which, of course, is not what we are after in this paper. But we included this assertion anyway for the

sake of completeness and to demonstrate that as the strength of interactions becomes smaller the theory of (1.1) becomes more detailed.

We address the problem of well-posedness of stochastic particle system (1.1) directly, by rewriting (1.1) as SDE

$$dZ = -b(Z)dt + \sqrt{2}dB$$
, B is a Brownian motion in \mathbb{R}^{Nd} (1.14)

with $Z = (X_1, \ldots, X_N)$ and drift

$$b = (b_1, \dots, b_N) : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd},$$

where
$$b_i(x) := \frac{1}{N} \sum_{j=1, j \neq i}^{N} K_{ij}(x_i - x_j), \quad x = (x_1, \dots, x_N) \in \mathbb{R}^{Nd}, \quad 1 \le i \le N,$$
 (1.15)

and then applying results on well-posedness for SDEs with general drifts, in particular, our Theorem 3 below. Until recently, the results on general singular SDEs could not compete, in terms of the admissible point singularities of the drift, with the results on particle systems with singular interactions. However, in the past few years, there was a substantial progress in proving weak and strong well-posedness of SDE (1.14) with general drift b, which now can have critical-order singularities (i.e. reach blow up effects), see [31, 32, 29, 28, 42, 43, 44, 48]. That said, to apply these results to particle system (1.1) when the number of particles is large in a way that would allow to control the strength of interactions (measured, in our case, by constant κ), one needs to keep track of the strength of the singularities of the drift b (in our case, measured by its own form-bound with respect to the Laplacian in \mathbb{R}^{Nd}). In Lemma 1 we show that if $K_{ij} \in \mathbf{F}_{\kappa}(\mathbb{R}^d)$, then b satisfies

$$\begin{cases} b \in \mathbf{F}_{\delta}(\mathbb{R}^{Nd}) \\ \text{with } \delta = \frac{(N-1)^2}{N^2} \kappa, \quad c_{\delta} = \frac{(N-1)^2}{N} c_{\kappa}. \end{cases}$$
 (1.16)

(Note that if $c_{\kappa} = 0$, as is the case for (1.10), then $c_{\delta} = 0$.) Thus, we obtain our Theorem 1 and Theorem 2(i)-(iii) for particle system (1.1) from our results on the general singular SDE (1.14) in \mathbb{R}^{Nd} , which are Theorems 3, 4.

In this approach, it is crucial that the assumptions on the form-bound δ of drift b in Theorem 3 stay dimension-independent, so that when Theorem 3 is applied to particle system (1.1) the resulting assumption on κ would not depend on the number of particles N (or, rather, would tend to a strictly positive value as the number of particles goes to infinity). This is achieved by means of De Giorgi's method ran in L^p which allows us to "decouple" the proof of the tightness estimate needed to establish the existence of a martingale solution (cf. (2.23)) from any strong gradient bounds on solutions of the corresponding elliptic or parabolic equations that, generally speaking, introduce a dependence on the dimension in the assumptions on the form-bound of b.

Running De Giorgi's method in L^p with $p \gg 2$ allows to maximize admissible values of the form bounds/strengths of interactions in particle system (1.1). At the level of strongly continuous semigroups, the observation that working in L^p with p large allows to relax the assumptions on the form-bound of the drift was made even earlier in [41].

By the way, in Remark 5 we discuss the theory of the backward Kolmogorov equation (1.6) in the case when the strength of the interactions reaches κ the borderline value, which requires us to work in the Orlicz space with "critical" gauge function $\cosh -1$ (that is, in some sense, a limit of L^p as $p \uparrow \infty$).

The strong existence in Theorem 1(iv) (or in Theorem A(ii)) follows from the result in [29] whose proof, in turn, is a modification of the method of Röckner-Zhao [49].

Theorems 3, 4 on the general singular SDE (1.14) are of interest on their own.

Theorem 3 deals with the existence and uniqueness of a strong Markov family of martingale solutions of SDE (1.14). In a number of ways, Theorem 3 continues the paper with Semënov [32], see further discussion in Section 2.3.

Theorem 4 deals with conditional weak uniqueness for (1.14), i.e. the uniqueness among weak solutions satisfying a rather natural condition (Krylov-type bound). The main novelty of Theorem 4 is related to condition (\mathbb{B}_2) that takes into account the repulsion-attraction structure of the drift. However, in its present form this condition, when applied to particle system (1.1), imposes not so natural conditions on the repulsing part of the interactions (i.e. admissible strength of repulsion depends on the number of particles, see the last comment before Section 2.3), so for now we leave this result at the level of general singular SDEs.

1.1. **About the proofs.** The analytic core of the paper are Theorems 5, 6 and 7 from which Theorems 3 and 4 for general singular SDEs follow.

In Theorem 5 we prove Hölder continuity of solutions to the elliptic counterpart of the Kolmogorov backward equation (1.6), i.e. $(\lambda - \Delta + b \cdot \nabla)u = f$, $f \in C_c^{\infty}$, where b can, in particular, be defined by (1.15). This is needed to prove the strong Markov property for the martingale solutions in Theorem 3. Theorem 5 is proved by showing that solution of the elliptic Kolmogorov equation u belongs to appropriate L^p De Giorgi's classes and then following De Giorgi's method. These De Giorgi classes, however, are somewhat different from the L^p De Giorgi classes found in the literature (cf. [17]), i.e. they contain the integrals of

$$|\nabla (u-k)_{+}^{p/2}|^2, \quad k \in \mathbb{R},$$
 (1.17)

rather than the integrals of $|\nabla (u-k)_+|^p$.

Theorem 6, i.e. an embedding theorem for a family of non-homogeneous elliptic Kolmogorov equations that includes

$$(\lambda - \Delta + b \cdot \nabla)u = |b|f, \quad f \in C_c^{\infty}, \tag{1.18}$$

is needed to construct martingale solutions in Theorem 3. It also has other uses e.g. we apply it in subsequent paper [37] to construct strongly continuous Feller semigroup with general form-bounded drift with form-bound in the critical range $\delta < 4$. The proof of Theorem 6 also uses De Giorgi's method. Although the assertion of Theorem 6 is a global L^{∞} estimate on u in terms of a certain L^p norm of the right-hand side, its proof is local. Otherwise we would have to impose an additional global condition on b that would be difficult to verify for b given by (1.15). Also, we will need an intermediate result in the proof of Theorem 6 in order to establish a "separation property", i.e. that u is small far away from the support of f.

The point of departure of De Giorgi's method is the Caccioppoli inequality. To prove it under the repulsing-attracting form-boundedness type condition of Definitions 3, 4, we extend the iteration procedure ("Caccioppoli's iterations") introduced in an earlier paper with Vafadar [39] to the non-homogeneous L^p setting, as is needed to handle weak well-posedness of SDEs.

Finally, Theorem 7, needed to prove Theorem 4 on conditional weak uniqueness, contains rather strong gradient bounds on solution of (1.18). Its proof uses a quite ingeniously constructed test function of [41], see comments after Theorem 7.

1.2. More on the existing results. (i) Gradient form interaction kernels

$$K = \nabla V : \mathbb{R}^d \to \mathbb{R}^d \tag{1.19}$$

for some potential V on \mathbb{R}^d play a crucial role in Statistical Physics. We refer to [3, 45], see also references therein. In particular, in [45] the authors proved strong well-posedness of the particle system in $\mathbb{R}^{Nd} \setminus \bigcup_{1 \leq i < j \leq N} \{((x_1, \ldots, x_N) \in \mathbb{R}^{Nd} \mid x_i = x_j)\}$ for very singular interaction potentials satisfying some fairly general assumptions (however, excluding purely attracting singular interactions such as the ones in (1.2), covered as a special case by Theorems 1, 2). For instance, the result in [45, Sect. 9.2] yields strong well-posedness of the particle system for potential

$$V(x) = |x|^{-10} \left(2 + \sin \frac{1}{|x|} \right).$$

The corresponding interaction kernel $K = -10|x|^{-12}x\left(2+\sin(\frac{1}{|x|})\right) - |x|^{-13}x\cos(\frac{1}{|x|})$ oscillates between the repulsion and the attraction as x approaches the origin. The repulsion on average dominates the attraction. Still, our results do not cover such interactions. In fact, although in Theorem 2 our condition on the repulsing part of K is much weaker than the condition on the attracting part of K, it is still a global condition: $(\operatorname{div} K)_- \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$.

See also [9, 10] regarding the Dirichlet form approach to the problem of well-posedness of particle systems with gradient form interactions.

We also mention [8] where the authors work at the PDE level on the torus, consider interaction kernels of gradient form with the interaction potential V pointwise comparable to $\sqrt{\kappa} \frac{d-2}{2} \log |x|$ (which thus includes the attracting kernel in (1.10)) and, importantly, obtain quantitative estimates on the propagation of chaos for the McKean-Vlasov PDE for all $\kappa < 16(\frac{d}{d-2})^2$.

(ii) The present paper deals with general singular interactions, i.e. not having a particular structure such as gradient form. In particular, we refer to [19, 52] where the authors prove, as a part of their results on the propagation of chaos, well-posedness of particle system (1.1) for interaction kernels K in the sub-critical Ladyzhenskaya-Prodi-Serrin class. Applied to (1.1) (with $K_{ij} = K$), their condition reads as

$$|K| \in L^p + L^\infty$$
 (i.e. sum of two functions), $p > d$.

See [9] regarding time-homogeneous critical LPS class

$$|K| \in L^d + L^\infty.$$
 (LPS)

The class of form-bounded interactions kernels \mathbf{F}_{κ} is larger than (LPS) and, moreover, contains some interaction kernels that are strictly more singular than the ones in (LPS), such as (1.10).

However, here we are not comparing our results with papers [9, 19, 52] since we do not prove the existence of a mean field limit and its uniqueness.

Let us also make the following two comments regarding the relationship between class (LPS) and class \mathbf{F}_{κ} :

One advantage of the Lebesgue scale condition (LPS) is that it is easy to verify. However, it is not necessarily easy to deal with when one considers particle systems of type (1.1) for N large. Indeed, if, in order to prove well-posedness of (1.1) we were to consider this particle system as a special case of general SDE (1.14) in \mathbb{R}^{Nd} , then the well-posedness results on the Lebesgue scale for (1.14) would require $|b| \in L^q(\mathbb{R}^{Nd}) + L^{\infty}(\mathbb{R}^{Nd})$, q > Nd (see [45]) or q = Nd (see [7]). Clearly, this severely restricts the class of admissible interaction kernels $K_{ij} = K$ in (1.15). There is a finer argument due to [19] that still allows to prove strong well-posedness of (1.1) for $K \in L^p(\mathbb{R}^d) + L^{\infty}(\mathbb{R}^d)$, p > d, regardless of the number of particles N, but it requires extra work.

On the other hand, form-boundedness handles transition from from-bounded K_{ij} on \mathbb{R}^d to form-bounded b given by (1.15) on \mathbb{R}^{Nd} rather effortlessly, see (1.16). Moreover, crucially for particle systems, it allows to keep track of the values of the form-bounds (= strengths of interactions) regardless of the number of particles N. (To borrow an expression from [9], the present work can be viewed as a "propaganda piece" for form-boundedness and similar conditions in the context of particle systems and singular SDEs.)

- Consider drift $b: \mathbb{R}^d \to \mathbb{R}^d$. If u is a weak solution of the elliptic equation $(\lambda - \Delta + b \cdot \nabla)u = f$, $\lambda > 0$, $f \in C_c^{\infty}$ with $b \in L^d + L^{\infty}$ and $u \in W^{1,r}$ (e.g. using Theorem 7) for r large, then, by Hölder's inequality,

$$\Delta u \in L_{\text{loc}}^{\frac{rd}{d+r}}$$
.

However, for $b \in \mathbf{F}_{\delta}$, one can only say that

$$\Delta u \in L^{\frac{2d}{d+2}}_{\mathrm{loc}}$$

(in fact, one can show that $u \in W^{2,2}$). That is, if b is only form-bounded then there are no $W^{2,p}$ estimates on u for p large.

Regarding general singular interactions, let us also mention a model of the dynamics of neuroreceptors considered in [46] where the fact that a neurotransmitter, after it gets attached to a fixed neuroreceptor, prevents other neurotransmitters from entering, is modelled by introducing singular repulsing interactions between neurotransmitters in some regions on space. It is thus desirable to be able to handle interaction kernels with critical singularities that stay admissible after one multiplies them by indicator functions (so that the interactions can be turned on or turned off depending on the positions of the particles relative to each other and in space), as e.g. the class of form-bounded interaction kernels considered in the present work.

(iii) De Giorgi's method was used earlier in the context of singular SDEs in [48, 56, 58]. There the authors considered singular drifts arising in the study of 3D Navier-Stokes equations.

(iv) In dimensions $d \geq 3$ one does not obtain the Keller-Segel equation (1.5) as the mean field limit of particle system (1.2) since, evidently, there is a gap between the singularity of kernel $K_1(y) = c_d |y|^{-d} y$ in (1.5) and the singularity of kernel $K(y) = c_d |y|^{-2} y$ in (1.2) (that, we know, is already critical). Nevertheless, it is known in the literature on the Keller-Segel equation [23, 13] that requiring extra regularity of the initial distribution $\rho_0 \in L^{d/2}$, one can extend it to $\rho \in L^{\infty}(\mathbb{R}_+, L^{d/2})$, in which case, by Young's inequality,

$$(K_1 * \rho)(t, \cdot) \in L^d, \quad t \ge 0,$$

i.e. the drift belongs to still admissible critical time-inhomogeneous Ladyzhenskaya-Prodi-Serrin class. (By the way, repeating the argument in Example 1(1), one sees that drift $(K_1 * \rho)(t, \cdot)$ belongs to the class of time-inhomogeneous form-bounded vector fields, i.e. for a.e. $t \in \mathbb{R}_+$

$$||b(t,\cdot)\varphi||_2^2 \le \delta ||\nabla \varphi||_2^2 + c_\delta ||\varphi||_2^2 \quad \forall \varphi \in W^{1,2},$$

which, in principle, puts the corresponding Keller-Segel equation within the reach of our methods, at least at the level of a priori Sobolev regularity estimates, see Remark 4.)

The observation that to handle the d-dimensional Keller-Segel model one can use energy methods in L^p with p large (larger than $\frac{d}{2}$) goes back already to [23, 13].

We also use energy methods in L^p with p large, but we do it for a different purpose, i.e. to relax the assumption of the strength of interactions κ . Furthermore, in the present paper we face another situation where one needs to work in L^p with large p. That is, in presence of repulsing-attracting structure in the drift b we can replace the form-boundedness requirement by a more general condition ("multiplicative form-boundedness", cf. Theorem 3). Now, to treat the right-hand side of nonhomogeneous equation (1.18), which is the analytic object behind the SDE with drift b, we need an additional condition

$$|b|^{\frac{1+\alpha}{2}} \in \mathbf{F}_{\chi}$$
 for some $\chi < \infty, \alpha \in]0,1[$,

where $p' := \frac{p}{p-1} \le 1 + \alpha$. This extra condition is least restrictive if α is small, which forces us to consider large p. See Remark 3 for more details.

- (v) We also mention recent results in [12] on interacting particle systems and McKean-Vlasov SDEs with distributional interaction kernels in Besov spaces (see also references therein). The assumptions of [12] are somewhat orthogonal to the present work and, at least at the moment, do not include the model interaction kernels (1.10) (while including other quite irregular distributional kernels) or keep track of the strength of interactions κ .
- (vi) In what follows, we refer to a well known in the literature on parabolic PDEs and singular SDEs classification of drifts:
 - Sub-critical case if, upon zooming into small scales, i.e. applying parabolic scaling in

$$(\partial_t - \Delta + b \cdot \nabla)v = 0$$
 in \mathbb{R}^d

or in

$$Y_t = y - \int_0^t b(Y_s)ds + \sqrt{2}B_t, \quad y \in \mathbb{R}^d,$$

the drift term vanishes. For instance, $b \in [L^q]^d$, q > d is sub-critical.

- Critical case if zooming into small scales does not change the "norm" of the drift.

For instance, parabolic scaling does not change the form-bound of the drift or its L^d norm. So, both \mathbf{F}_{δ} and $[L^d]^d$ are critical classes. Note that this classification does not distinguish between critical drifts that reach blow up effects, such as $b \in \mathbf{F}_{\delta}$, and drifts that do not reach blow up effects, such as $b \in [L^d]^d$. In other words, one can multiply the latter by arbitrarily large constant without affecting well-posedness of the SDE, while form-bounded drifts can in general "sense" this multiplication (since it, obviously, changes the form-bound, which cannot be too large, see the beginning of the introduction). In order to distinguish between these two very different cases, we say that the former have critical-order singularities.

We also consider in the present paper other *critical* classes of drifts, such as multiplicatively form-bounded drifts (Definition 2.12) and weakly form-bounded drifts (Remark 9) that expand the class of form-bounded vector fields \mathbf{F}_{δ} rather substantially.

- Super-critical case if zooming into small scales actually increases the "norm" of the drift. For instance, $b \in L^q$, q < d, is super-critical. Let us add that all known results on super-critical drifts b require critical positive part of div b. This, of course, includes important case div b = 0.

In Remark 10 we comment on the existing literature on PDEs and SDEs with super-critical drifts. Briefly, super-criticality of the drift destroys many basic regularity results, but some parts of the theory can be salvaged.

(vii) As was indicated above, the proof of Theorem 2(iii) uses the many-particle Hardy inequality of [22]: for $d \geq 3$, all $N \geq 2$,

$$C_{d,N} \sum_{1 \le i \le j \le N} \int_{\mathbb{R}^{Nd}} \frac{|\varphi(x)|^2}{|x_i - x_j|^2} dx \le \int_{\mathbb{R}^{Nd}} |\nabla \varphi(x)|^2 dx, \quad x = (x_1, \dots, x_N), \tag{1.20}$$

for all $\varphi \in W^{1,2}(\mathbb{R}^{Nd})$, where

$$C_{d,N} := (d-2)^2 \max \left\{ \frac{1}{N}, \frac{1}{1 + \sqrt{1 + \frac{3(d-2)^2}{2(d-1)^2}(N-1)(N-2)}} \right\}.$$

In the proof we replace constant $C_{d,N}$ with smaller constant $\frac{(d-2)^2}{N}$. However, the maximum for large N and $d \leq 6$ in the definition of $C_{d,N}$ is attained in the second argument. So, the constraint $\kappa < 16$ in Theorem 2(iii) (or in Theorem A(iii)) can be somewhat relaxed for $d \leq 6$.

The authors of [22] also provide, among other results, an upper bound on the constant in (1.20). At the moment of writing of this article, to the best of author's knowledge, the optimal constant in (1.20) is not known.

It is natural to expect that the relationship between Theorem 2(iii) and the many-particle Hardy inequality (1.20) goes both ways, i.e. there is a direct relationship between the optimal constant in many-particle Hardy inequality (1.20) and the critical threshold value of κ that separates well-posedness of particle system (1.2) from a blow up, in which case Monte-Carlo simulations for (1.2) should produce the optimal $C_{d,N}$ in (1.20); we pursue this in [18].

1.3. **Notations.** Put

$$\langle f \rangle := \int_{\mathbb{R}^d} f(y) dy, \quad \langle f, g \rangle := \langle fg \rangle$$

(all functions in this paper are real-valued). For vector fields b, $f: \mathbb{R}^d \to \mathbb{R}^d$, we put

$$\langle b, f \rangle := \langle b \cdot f \rangle$$
 (· is the inner product in \mathbb{R}^d).

Let $\|\cdot\|_{p\to q}$ denote the $L^p\to L^q$ operator norm. Let C_∞ denote the space of continuous functions on \mathbb{R}^d vanishing at infinity, endowed with the sup-norm. Let $B_R(y)\subset\mathbb{R}^d$ be the open ball of radius R centered at $y\in\mathbb{R}^d$, $|B_R(x)|$ denotes its volume. Set $B_R:=B_R(0)$. Given a function f, we denote its positive and negative parts by

$$(f)_+ := f \vee 0, \quad (f)_- := -(f \wedge 0).$$

Set

$$\gamma(x) := \left\{ \begin{array}{ll} c \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{ if } |x| < 1, \\ 0, & \text{ if } |x| \geqslant 1, \end{array} \right.$$

where c is adjusted to $\int_{\mathbb{R}^d} \gamma(x) dx = 1$, and put $\gamma_{\varepsilon}(x) := \frac{1}{\varepsilon^d} \gamma\left(\frac{x}{\varepsilon}\right), \ \varepsilon > 0, \ x \in \mathbb{R}^d$. Define the Friedrichs mollifier of a function $h \in L^1_{\text{loc}}$ (or a vector field with entries in L^1_{loc}) by

$$E_{\varepsilon}h := \gamma_{\varepsilon} * h.$$

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2. Particle systems

For brevity, we will consider first the particle system without the drift terms $M(X_i)$:

$$X_i(t) = x_i - \frac{1}{N} \sum_{j=1, j \neq i}^{N} \int_0^t K_{ij} (X_i(s) - X_j(s)) ds + \sqrt{2}B_i(t), \quad 1 \le i \le N, \quad t \in [0, T], \quad (2.1)$$

where $x = (x_1, ..., x_N) \in \mathbb{R}^{Nd}$, $N \geq 2$. However, we will explain in Remark 6 below how to put the drifts back there.

Let $e_t: C([0,T],\mathbb{R}^{Nd}) \to \mathbb{R}^{Nd}$ be defined by

$$e_t(\omega) := \omega_t$$
.

Recall that a probability measure \mathbb{P}_x $(x \in \mathbb{R}^{Nd})$ on the canonical space of continuous trajectories $\omega = (\omega^1, \dots, \omega^N)$ in \mathbb{R}^{Nd} is called a martingale solution to particle system (2.1) on [0, T] if 1)

$$\mathbb{P}_{x,0} = \delta_x,$$

where $\mathbb{P}_{x,t} := \mathbb{P} \circ e_t^{-1}$ (on \mathbb{R}^{Nd}),

2)

$$\mathbb{E}_x \sum_{i=1}^N \sum_{j=1, j \neq i}^N \int_0^T |K_{ij}(\omega_t^i - \omega_t^j)| dt < \infty,$$

3) for every $\phi \in C_c^2(\mathbb{R}^{Nd})$ the process

$$[0,T] \ni r \mapsto \phi(\omega_r) - \phi(x) + \int_0^r \left(-\Delta_y \phi(\omega_t) + \frac{1}{N} \sum_{i=1}^N \sum_{j=1, j \neq i}^N K_{ij} (\omega_t^i - \omega_t^j) \cdot \nabla_{y_i} \phi(\omega_t) \right) dt$$

is a martingale under \mathbb{P}_x .

We will also need the following definition. Let K satisfy (1.7), let $\{K^n\}$ be some sequence of vector fields (in what follows, K^n will be more regular than K).

Definition 2. Let us say that $\{K^n\}$ does not increase the form-bounds of K if for every $n \geq 1$

$$||K^n \varphi||_2^2 \le \kappa ||\nabla \varphi||_2^2 + c_{\kappa} ||\varphi||_2^2 \quad \forall \varphi \in W^{1,2}(\mathbb{R}^d),$$

i.e. $\{K^n\}$ satisfy (1.7) with the same constants as K.

2.1. General interaction kernels.

Theorem 1 (General interactions). Assume that the interaction kernels K_{ij} in particle system (2.1) satisfy

$$K_{ij} \in \mathbf{F}_{\kappa} \quad with \ \kappa < 4\left(\frac{N}{N-1}\right)^2$$
 (2.2)

(see Definition 1). Then the following are true:

- (i) There exists a strong Markov family of martingale solutions $\{\mathbb{P}_x\}_{x\in\mathbb{R}^{Nd}}$ of particle system (2.1).
- (ii) The function

$$u(x) := \mathbb{E}_{\mathbb{P}_x} \int_0^\infty e^{-\lambda s} f(\omega_s^1, \dots, \omega_s^N) ds, \quad x \in \mathbb{R}^{Nd}, \quad f \in C_c^\infty(\mathbb{R}^{Nd}), \tag{2.3}$$

where λ is assumed to be sufficiently large, is a locally Hölder continuous weak solution to elliptic Kolmogorov equation

$$\left(\lambda - \Delta + \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} K_{ij}(x_i - x_j) \cdot \nabla_{x_i}\right) u = f, \quad x = (x_1, \dots, x_N), \tag{2.4}$$

see definitions in Remark 7 where we also discuss the uniqueness of u.

(iii) Fix $p > \frac{2}{2-\frac{N-1}{N}\sqrt{\kappa}}$. The family of operators $\{P_t\}_{t\geq 0}$ defined by

$$P_t f(x) := \mathbb{E}_{\mathbb{P}_x}[f(\omega_t^1, \dots, \omega_t^N)], \quad f \in C_c^{\infty}(\mathbb{R}^{Nd}),$$

admits extension by continuity to a strongly continuous quasi contraction Markov semigroup on L^p of integral operators, say $P_t =: e^{-t\Lambda_p}$, such that

$$||e^{-t\Lambda_p}||_{p\to q} \le cw^{\omega t} t^{-\frac{Nd}{2}(\frac{1}{p} - \frac{1}{q})}, \quad p \le q \le \infty$$
 (2.5)

for appropriate constants c and ω . In view of (2.5), Dunford-Pettis' theorem yields that $e^{-t\Lambda_p}$ is a semigroup of integral operators. Their integral kernel $e^{-t\Lambda}(x,z)$ does not depend on p and is defined to be the heat kernel of particle system (2.1).

If p = 2, then we have

$$\Lambda_2 \supset -\Delta + \frac{1}{N} \sum_{i=1}^N \sum_{j=1, j \neq i}^N K_{ij}(x_i - x_j) \cdot \nabla_{x_i} \upharpoonright C_c^{\infty}(\mathbb{R}^{Nd}).$$

The semigroup $e^{-t\Lambda_p}$ is unique among semigroups that can be constructed via approximation, i.e. for any sequence of bounded smooth interaction kernels $\{K_{ij}^n\}$,

$$K_{ij}^n \to K_{ij}$$
 in $[L_{loc}^2(\mathbb{R}^d)]^d$,

that do not increase the form-bounds of K, for every $f \in C_c^{\infty}(\mathbb{R}^{Nd})$ solutions $\{v_n\}$ to

$$(\partial_t - \Delta + \frac{1}{N} \sum_{i=1}^N \sum_{j=1, j \neq i}^N K_{ij}^n (x_i - x_j) \cdot \nabla_{x_i}) v_n = 0, \quad v_n(0) = f$$

converge to the same limit $e^{-t\Lambda_p}f$ in $L^p(\mathbb{R}^{Nd})$ loc. uniformly in $t\geq 0$.

(iv) If, furthermore,

$$\kappa < \frac{1}{(N-1)^2 d^2},$$

then for every initial configuration $x = (x_1, ..., x_N) \in \mathbb{R}^{Nd}$ martingale solution \mathbb{P}_x satisfies for a given $q \in]Nd, \frac{N}{N-1}\kappa^{-\frac{1}{2}}[$ Krylov-type bounds

$$\mathbb{E}_{\mathbb{P}_{x}} \int_{0}^{T} |h(s, \omega_{s}^{1}, \dots, \omega_{s}^{N})| ds \leq c \|h\|_{L^{q}([0, T] \times \mathbb{R}^{Nd})}$$
 (2.6)

and

$$\mathbb{E}_{\mathbb{P}_x} \int_0^T |b(\omega_s^1, \dots, \omega_s^N)| |h(\tau, \omega_s^1, \dots, \omega_s^N)| ds \le c \|b|h|^{\frac{q}{2}} \|_{L^2([0, T] \times \mathbb{R}^{Nd})}^{\frac{2}{q}}, \tag{2.7}$$

for all $h \in C_c([0,T] \times \mathbb{R}^{Nd})$, for some constant c > 0, where vector field $b = (b_1, \ldots, b_N)$: $\mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ is defined by

$$b_i(x) := \frac{1}{N} \sum_{j=1, j \neq i}^{N} K_{ij}(x_i - x_j), \quad x = (x_1, \dots, x_N) \in \mathbb{R}^{Nd}, \quad 1 \le i \le N.$$

Moreover, \mathbb{P}_x is the only martingale solution to (2.1) that satisfies (2.6), (2.7) ("conditional weak uniqueness").

(v) There exists constant C < 1 such that if K_{ij} is of the form

$$K_{ij}(x_i, x_i - x_j) = \zeta(x_i) K_{ij}^0(x_i - x_j),$$
(2.8)

with ζ having compact support, $\|\zeta\|_{\infty} \leq 1$, and $K_{ij}^0 \in \mathbf{F}_{\kappa}$ with

$$\kappa < \frac{C}{(N-1)^2 d^2} \tag{2.9}$$

(the previous assertions are valid for such interaction kernels as well), then for every initial configuration $(x_1, \ldots, x_N) \in \mathbb{R}^{Nd}$ particle system (2.1) has a strong solution on [0, T] that is unique among all strong solutions defined on the same probability space satisfying (2.6), (2.7).

We recall from the discussion in the introduction that if the strength of interactions κ is taken to be too large then a weak solution to the particle system (2.1) can cease to exist. So, in Theorem 1(i) we are dealing with the critical scale of the strength of interactions.

Let us emphasize that as the strength of interactions κ becomes smaller, the theory of particle system (2.1) in Theorem 1 becomes more detailed.

We are rather satisfied with the conditions on the interaction kernels K_{ij} in Theorem 1(i)-(iii) where the assumption on the strength of interactions κ "stabilizes" to a positive value as the number of particles $N \to \infty$, so, in principle, this opens up a possibility of studying the existence of the mean field limit (see Remark 2). However, in assertions (iv), (v) of Theorem 1 the assumption on κ degenerates to zero as N goes to infinity, which seems to be a by-product of our method of embedding particle system (2.1) in the general SDE (4.1). We comment more on this below.

2.2. Attraction and repulsion. We now turn to the interaction kernels having a repulsion-attraction structure. While the repulsion between the particles, in a sense, contributes towards well-posedness of particle system (2.1) by preventing collisions, the attraction can lead to the blow up effects (see the discussion in the introduction). We take into account the attraction between the particles by looking at the positive part of the divergence of the interaction kernels K_{ij} in (2.1).

Definition 3. $(\operatorname{div} K)_+ \in L^1_{\operatorname{loc}}$ is said to be form-bounded if there exists constant κ_+ such that

$$\langle (\operatorname{div} K)_{+} \varphi, \varphi \rangle \leq \kappa_{+} \|\nabla \varphi\|_{2}^{2} + c_{\kappa_{+}} \|\varphi\|_{2}^{2}, \quad \forall \varphi \in W^{1,2}, \tag{2.10}$$

for some $c_{\kappa_{+}}$.

We abbreviate (2.10), with a slight abuse of notation, as

$$(\operatorname{div} K)_{+}^{\frac{1}{2}} \in \mathbf{F}_{\kappa_{+}}.$$

For example, the previous condition is satisfied if $(\operatorname{div} K)_+ \in L^{\frac{d}{2},\infty}$ (weak $L^{\frac{d}{2}}$ class). This includes, of course, $(\operatorname{div} K)_+ \in L^{\frac{d}{2}}$, in which case κ_+ can be chosen arbitrarily small (cf. Example 1(1)).

Example 2. Let K be the model singular attracting kernel (1.10), i.e. $K(y) = \sqrt{\kappa} \frac{d-2}{2} \frac{y}{|y|^2}$. Then

$$\operatorname{div} K = \sqrt{\kappa} \frac{(d-2)^2}{2} |y|^{-2},$$

so, by Hardy's inequality $(\operatorname{div} K)_+^{\frac{1}{2}} \in \mathbf{F}_{\kappa_+}, \ \kappa_+ = 2\sqrt{\kappa}, \ c_{\kappa_+} = 0.$

Regarding the negative part $(\operatorname{div} K)_-$, which is responsible for the repulsion between the particles, we will only impose a rather quite mild condition that $(\operatorname{div} K)_-$ can be represented as the sum of a function in $L^1(\mathbb{R}^d)$ and a bounded function.

Already the hypothesis $(\operatorname{div} K_{ij})_+^{\frac{1}{2}} \in \mathbf{F}_{\kappa_+}$ allows one to easily prove, integrating by parts and using Lemma 2, that solution v of the backward Kolmgorov equation for particle system (2.1)

$$\left(\partial_t - \Delta + \frac{1}{N} \sum_{i=1}^N \sum_{j=1, j \neq i}^N K_{ij}(x_i - x_j) \cdot \nabla_{x_i}\right) v = 0, \quad v(0, \cdot) = f(\cdot) \text{ in } \mathbb{R}^{Nd}$$

satisfies a quasi contraction estimate

$$||v(t)||_{p} \le e^{\omega_{p}t} ||f||_{p}, \quad t > 0$$
 (2.11)

provided κ_+ is not too large, for appropriate p and ω_p . However, without any additional assumptions on the interaction kernels K_{ij} themselves, there is no hope of advancing substantially farther than (2.11). In fact, without conditions on K_{ij} , even requiring div $K_{ij} = 0$, puts us firmly in the super-critical regime (cf. (vi) the introduction), so even the proof of a priori Hölder continuity of solution v or of solution to the corresponding elliptic equation becomes out of reach. We need a condition on K_{ij} that will put us back in the critical regime.

Definition 4. A vector field $K \in [L^1_{loc}]^d$ is said to be multiplicatively form-bounded if there exists constant κ_0 ("multiplicative form-bound") such that

$$\langle |K|\varphi,\varphi\rangle \le \kappa_0 \|\nabla\varphi\|_2 \|\varphi\|_2 + c_{\kappa_0} \|\varphi\|_2^2, \quad \forall \varphi \in W^{1,2}. \tag{2.12}$$

We abbreviate (2.12) as

$$K \in \mathbf{MF}_{\kappa_0}$$
.

It will be clear from the results below that the actual value of κ_0 is not important for well-posedness of the particle system (2.1); it is the value of κ_+ that matters.

Note that the class of form-bounded vector field \mathbf{F}_{κ_0} is also a critical class, so we could use it here as well. However, our ultimate goal is to identify the optimal (least restrictive) assumptions on K_{ij} , so we will work with the broader class \mathbf{MF}_{κ_0} :

- **Example 3.** (i) Every form-bounded vector field is multiplicatively form-bounded, but not vice versa, see Remark 8. In particular, all vector fields listed in Example 1 are multiplicatively form-bounded.
 - (ii) The class \mathbf{MF}_{κ_0} contains the largest possible, up to the strict inequality in $\varepsilon > 0$, scaling-invariant Morrey class $M_{1+\varepsilon}$: if

$$||K||_{M_{1+\varepsilon}} := \sup_{r>0} r \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} |K|^{1+\varepsilon} dy \right)^{\frac{1}{1+\varepsilon}} < \infty,$$

then

$$K \in \mathbf{MF}_{\kappa_0}, \quad \kappa_0 = c(d, \varepsilon) \|K\|_{M_{1+\varepsilon}},$$

see details in Remark 8. Here one can already see the gain in comparison with the class of form-bounded vector field \mathbf{F}_{κ_0} , which contains only $M_{2+\varepsilon}$ (and itself is contained in M_2).

(iii) If the following Morrey class condition is satisfied:

$$\sup_{r>0,y\in\mathbb{R}^d}r^2\left(\frac{1}{|B_r(y)|}\int_{B_r(y)}|(\operatorname{div} K)_+|^{1+\varepsilon}dy\right)^{\frac{1}{1+\varepsilon}}<\infty,$$

then $(\operatorname{div} K)_{+}^{\frac{1}{2}} \in \mathbf{F}_{\kappa_{+}}$ with appropriate κ_{+} .

It was demonstrated in [50] that condition $b \in \mathbf{MF}_{\delta}$ under additional divergence-free hypothesis div b=0 provides two-sided Gaussian bounds on the heat kernel of operator $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ with uniformly elliptic measurable matrix a. (Of course, having $(\operatorname{div} b)^{\frac{1}{2}} \in \mathbf{F}_{\delta_+}$, as in the present paper, destroys both the upper and the lower Gaussian bounds on the heat kernel even of $-\Delta + b \cdot \nabla$.)

Put

$$\mathbf{F} := \{ K \mid K \in \mathbf{F}_{\kappa_0} \text{ for some } \kappa_0 < \infty \}$$

and

$$\mathbf{MF} := \{ K \mid K \in \mathbf{MF}_{\kappa_0} \text{ for some } \kappa_0 < \infty \}.$$

Definition 2 extends naturally to K satisfying (2.12), (2.10) or (2.15) below. In all these cases, in Section 6 we show that the vector fields K^n defined by

$$K^n := E_{\varepsilon_n} K, \quad \varepsilon_n \downarrow 0, \quad E_{\varepsilon} \text{ is the Friedrichs mollifier},$$
 (2.13)

are bounded, smooth and do not increase the corresponding form-bounds of K.

Theorem 2 (Repulsing-attracting interactions). The following are true:

(i) Assume that the interaction kernels K_{ij} in particle system (2.1) satisfy

$$K_{ij} \in \mathbf{MF}, \qquad \begin{cases} (\operatorname{div} K_{ij})_{-} \in L^{1} + L^{\infty}, \\ (\operatorname{div} K_{ij})_{+}^{\frac{1}{2}} \in \mathbf{F}_{\kappa_{+}} \text{ with } \kappa_{+} < 4 \frac{N}{N-1} \end{cases}$$
 $|K_{ij}|^{\frac{1+\alpha}{2}} \in \mathbf{F}$ (2.14)

for some $\alpha > 0$ fixed arbitrarily close to zero. The assertions (i), (ii) of Theorem 1 are valid for these interaction kernels as well.

(ii) Assume that K_{ij} satisfy a more restrictive condition than (2.14) in Theorem 2:

$$K_{ij} \in \mathbf{F}, \qquad \begin{cases} (\operatorname{div} K_{ij})_{-} \in L^{1} + L^{\infty}, \\ (\operatorname{div} K_{ij})_{+}^{\frac{1}{2}} \in \mathbf{F}_{\kappa_{+}} \text{ with } \kappa_{+} < 4 \frac{N}{N-1}. \end{cases}$$
 (2.15)

Fix $p > \frac{4}{4 - \frac{N-1}{N-1}\kappa_+}$. Then assertion (iii) of Theorem 1 also remains valid.

(iii) Let

$$K_{ij}(y) = \sqrt{\kappa} \frac{d-2}{2} |y|^{-2} y + K_{0,ij}(y), \quad y \in \mathbb{R}^d.$$
 (2.16)

If the strength of attraction

and $K_{0,ij}$ satisfy conditions (2.2) or (2.14) with sufficiently small form-bounds, then assertions (i)-(iii) of Theorem 1 with $p > \frac{4}{4-\sqrt{\kappa}}$ remain valid.

(iv) Furthermore, for the model attracting interaction kernel

$$K(y) = \sqrt{\kappa} \frac{d-2}{2} |y|^{-2} y, \quad \kappa < 16,$$

the previous assertions remain valid, the heat kernel $e^{-t\Lambda}(x,z)$ of particle system (2.1) satisfies, up to modification on a measure zero set, the heat kernel bound

$$e^{-t\Lambda}(x,z) \le Ct^{-\frac{Nd}{2}} \prod_{1 \le i < j \le N} \eta(t^{-\frac{1}{2}}|z_i - z_j|), \quad t \in]0,T],$$

for some $C = C_T$, for all $x \in \mathbb{R}^{Nd}$, $z = (z_1, \dots, z_N) \in \mathbb{R}^{Nd}$ provided $z_i \neq z_j$ $(i \neq j)$, for a fixed function $1 \leq \eta \in C^2(]0, \infty[)$ such that

$$\eta(r) = \begin{cases} r^{-\sqrt{\kappa} \frac{d-2}{2} \frac{1}{N}} & 0 < r < 1, \\ 2, & r > 2. \end{cases}$$

Remark 1. The additional right-most condition on K_{ij} in (2.14) is, generally speaking, much weaker than the left-most condition (informally, the former treats |K| as a potential, while a proper "potential analogue" of the drift perturbation $K \cdot \nabla$ would be $|K|^2$). For instance, if we were to state condition (2.14) on the scale of L^p spaces, then it would become

$$|K| \in L^d + L^{\infty}, \qquad \begin{cases} (\operatorname{div} K)_- \in L^1 + L^{\infty}, \\ (\operatorname{div} K)_+ \in L^{\frac{d}{2}} + L^{\infty} \end{cases} \qquad |K| \in L^{\frac{d}{2}(1+\alpha)} + L^{\infty},$$

where, recall, $\alpha > 0$ is fixed arbitrarily small, i.e. the right-most condition follows from the left-most one. The same would happen if we were working on the scale of scaling-invariant Morrey spaces (cf. Example 1(4)).

The improvement of the assumptions on κ in Theorem 2(iii), compared to Theorem 1 and Theorem 2(i), (ii), is due to a refinement of Lemma 2 by means of the many-particle Hardy inequality (1.20) of [22].

The heat kernel bound in Theorem 2(iv) is not unexpected (although we could not find it in the literature). Indeed, an elementary calculation shows that

$$\psi(x) := \prod_{1 \le i < j \le N} |x_i - x_j|^{-\sqrt{\kappa} \frac{d-2}{2} \frac{1}{N}}.$$

is a Lyapunov function of the formal adjoint of $\Lambda = -\Delta_x - \sqrt{\kappa} \frac{d-2}{2} \frac{1}{N} \sum_{i=1}^N \sum_{j=1, j \neq i}^N \frac{x_i - x_j}{|x_i - x_j|^2} \cdot \nabla_{x_i}$, i.e. the following identity holds:

$$-\Delta_x \psi + \sqrt{\kappa} \frac{d-2}{2} \frac{1}{N} \sum_{i=1}^N \nabla_{x_i} \left(\sum_{j=1, j \neq i}^N \frac{x_i - x_j}{|x_i - x_j|^2} \psi \right) = 0.$$
 (2.17)

One can expect that such Lyapunov function will appear as a multiple in the heat kernel bounds. That said, the question of how to prove such an estimate is non-trivial due to singularities in the drift. An interesting aspect of Theorem 2(iv) is its proof, which uses an abstract desingularization result from [38], see Appendix A.

In Theorem 2(iv), we expect to have two-sided bound

$$C_1 t^{-\frac{Nd}{2}} e^{-\frac{|x-y|^2}{c_2 t}} \varphi_t(y) \le e^{-t\Lambda}(x, y) \le C_3 t^{-\frac{Nd}{2}} e^{-\frac{|x-y|^2}{c_4 t}} \varphi_t(y), \tag{2.18}$$

where

$$\varphi_t(y) := \prod_{1 \le i < j \le N} \eta(t^{-\frac{1}{2}} | y_i - y_j |),$$

as is suggested by the analogous results for Kolmogorov operator $-\Delta - \sqrt{\kappa}|x|^{-2}x \cdot \nabla$, $0 < \kappa < 4$ on \mathbb{R}^d , see [47]. Moreover, there should be an analogous to Theorem 2(iv) and (2.18) result in the case of attracting interactions, see [47] and [36] regarding $-\Delta + \sqrt{\kappa}|x|^{-2}x \cdot \nabla$, $0 < \kappa < \infty$. ([36, 38] deal with the fractional Laplacian $(-\Delta)^{\alpha/2}$ perturbed by the model singular drift term $c|x|^{-\alpha}x \cdot \nabla$, $1 < \alpha < 2$.)

One drawback of assertions (iv), (v) of Theorem 1 is the difficulty with taking into account the repulsion/attraction structure of the interaction kernel K simply by looking at the divergence of K, as we do in Theorem 2. That said, in what concerns conditional weak uniqueness for the particle system (as in Theorem 1(iv)), in Theorem 4 we consider the general SDE (4.1) and propose another condition on the drift b that provides conditional weak uniqueness for (4.1) while taking into account the repulsion/attraction. We show in Example 4 that there is some truth to this condition: it is always satisfied in dimensions $d \geq 4$ for the model repulsing drift $b(x) = -\sqrt{\delta} \frac{d-2}{2}|x|^{-2}x$, regardless of the value for the form-bound $\delta > 0$, as one would expect. This requires us to obtain gradient bounds in L^q starting with q > d - 2, hence the need to work in the elliptic setting. (In the parabolic setting we would need q > d.) Nevertheless, this result, when applied via Lemma 2 to drift (2.19) with repulsing interactions $K_{ij}(y) = -\sqrt{\kappa} \frac{d-2}{2}|y|^{-2}y$, leads to a condition on κ that still depends on the number of particles N. So, there is still work to be done to find a proper analogue of Theorem 4 for particle system (2.1).

2.3. Comments on the proofs of Theorems 1 and 2. It is not difficult to modify the proofs of Theorems 1 and 2 to extend them to the sums of the interaction kernels satisfying (2.2) and (2.14), under properly adjusted assumptions on the form-bounds.

We prove Theorems 1 and 2 by embedding particle system (2.1) in the general SDE (4.1) considered in \mathbb{R}^{Nd} , with drift $b = (b_1, \dots, b_N) : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ defined by

$$b_i(x) := \frac{1}{N} \sum_{j=1, j \neq i}^{N} K_{ij}(x_i - x_j), \quad x = (x_1, \dots, x_N) \in \mathbb{R}^{Nd}, \quad 1 \le i \le N.$$
 (2.19)

Lemma 1. If $K_{ij} \in \mathbf{F}_{\kappa}(\mathbb{R}^d)$, then b defined by (2.19) satisfies

$$\begin{cases} b \in \mathbf{F}_{\delta}(\mathbb{R}^{Nd}) \\ with \ \delta = \frac{(N-1)^2}{N^2} \kappa, \quad c_{\delta} = \frac{(N-1)^2}{N} c_{\kappa}. \end{cases}$$

Lemma 2. If $K_{ij} \in \mathbf{MF}_{\kappa}(\mathbb{R}^d)$, $(\operatorname{div} K_{ij})_+^{\frac{1}{2}} \in \mathbf{F}_{\kappa_+}(\mathbb{R}^d)$, $|K_{ij}|^{\frac{1+\alpha}{2}} \in \mathbf{F}_{\sigma}(\mathbb{R}^d)$, $\alpha \in [0,1]$, then b defined by (2.19) satisfies

$$\begin{cases}
b \in \mathbf{MF}_{\delta}(\mathbb{R}^{Nd}) \\
with \ \delta = \frac{N-1}{\sqrt{N}}\kappa, \quad c_{\delta} = (N-1)c_{\kappa},
\end{cases}$$
(2.20)

$$\begin{cases}
(\operatorname{div} b)_{+}^{\frac{1}{2}} \in \mathbf{F}_{\delta_{+}}(\mathbb{R}^{Nd}), \\
with \, \delta_{+} = \frac{N-1}{N}\kappa_{+}, \quad c_{\delta_{+}} = (N-1)c_{\kappa_{+}},
\end{cases}$$
(2.21)

$$\begin{cases} |b|^{\frac{1+\alpha}{2}} \in \mathbf{F}_{\chi}(\mathbb{R}^{Nd}), \\ with \ \chi = \frac{(N-1)^{1+\alpha}}{N^{1+\alpha}}\sigma, \quad c_{\chi} = \frac{(N-1)^{1+\alpha}}{N^{\alpha}}c_{\sigma}. \end{cases}$$
 (2.22)

Lemmas 1, 2 allow us to obtain the existence of a strong Markov family of martingale solutions to (2.1) in Theorem 1(i), Theorem 2(i) from Theorem 3(i) for general SDE (4.1). Theorem 3, and other results in Section 4 dealing with general singular drifts, are of interest on their own.

In Theorem 3 the family of martingale solutions for (4.1) is constructed by applying a tightness argument where the central role belongs to the estimate

$$\mathbf{E} \int_{t_0}^{t_1} |b_{\varepsilon}(Y_{\varepsilon}(s))| ds \le C(t_1 - t_0)^{\frac{\gamma}{1 + \gamma}}, \quad t_0, t_1 \in [0, T]$$

$$\tag{2.23}$$

(this is (10.5)), where b_{ε} is a regularization of b that does not increase form-bounds δ , δ_{+} (see Definition 2) in Lemmas 1, 2, and Y_{ε} is the strong solution of (4.1) with drift b_{ε} . Constants C, $\gamma > 0$ are independent of ε .

To prove (2.23) and, furthermore, to prove the strong Markov property, we establish regularity results for non-homogeneous elliptic PDEs (5.1) and (5.6). These are Theorems 5 and 6, obtained via De Giorgi's method ran in L^p , where p depends on the values of form-bounds δ and δ_+ . Theorems 5 and 6 are the main analytic results in the present paper. We prove Theorem 5 by showing that u belongs to L^p De Giorgi's classes and then following the arguments in [17, Ch. 7], that is, applying De Giorgi's method. As was mentioned in the introduction, we deal with L^p De Giorgi classes that are somewhat different from the L^p De Giorgi classes found in the literature (cf. [17]).

Remark 2 (On the number of particles $N \to \infty$). Let interaction kernel K satisfy (2.2). Then in (2.23) $\gamma = \sqrt{2} - 1$ (see the proof of Theorem 3) and, by Lemma 1,

$$\delta = \frac{(N-1)^2}{N^2} \kappa, \quad c_{\delta} = \frac{(N-1)^2}{N} c_{\kappa}.$$

Let $c_{\kappa} = 0$ (as is the case for the model singular interactions (1.10)), then $c_{\delta} = 0$. In turn, as $N \to \infty$, constant δ tends to κ . Thus, our assumptions on the form-bound withstand the passage to the limit $N \to \infty$. However, in the tightness estimate (2.23) applied to (2.19) the constant C depends on N (this is because in the proof of (2.23) via De Giorgi's method we apply Sobolev's embedding theorem on \mathbb{R}^{Nd} , which becomes weaker as the dimension of the spaces increases, and hence De Giorgi's iterations converge slower). So, (2.23) does not allow to conclude the existence of a mean field limit by arguing as e.g. in Fournier-Jourdain [15]. This is not surprising since (2.23), as it is proved now, does not take into account the exchangeability hypothesis on (2.1) even if we were to impose it.

In [32], we proved, using De Giorgi's iterations in L^p , that the general SDE (4.1) with $b \in \mathbf{F}_{\delta}$, $\delta < 4$ has a martingale solution for every initial point. This result yields the existence of a martingale solution part of Theorem 3 under condition (\mathbb{A}_1) on b, which we included in Theorem

3 for the sake of completeness. In what concerns (\mathbb{A}_1) , in the present paper we make the next step and prove the strong Markov property.

Remark 3. One of the main observations of the present paper is related to condition (\mathbb{A}_2) of Theorem 3. This condition dictates the multiplicative form-boundedness assumption (2.14) on the interaction kernel K when the latter has repulsion-attraction structure. In (\mathbb{A}_2), we relax the a priori condition $|b| \in L^2_{loc}$ as in (\mathbb{A}_1) to $|b| \in L^{1+\alpha}_{loc}$ for $\alpha > 0$ fixed arbitrarily small, aiming at stronger hypersurface singularities of b (and thus of K). To achieve this, we once again need to work in L^p for p large. In fact, when dealing with the right-hand side of non-homogeneous equation

$$(\mu - \Delta + b \cdot \nabla)u = |b|f \quad (f \in C_c^{\infty}),$$

as is needed to prove weak well-posedness of the general SDE (4.1), we need to impose an extra condition

$$|b|^{\frac{1+\alpha}{2}} \in \mathbf{F}_{\chi}$$
 for some $\chi < \infty, \alpha \in]0,1[$,

which is related to p via inequality

$$p' = \frac{p}{p-1} \le 1 + \alpha$$

(cf. Theorem 6). If we were to consider this non-homogeneous equation in L^2 , we would have to take $\alpha = 1$, and so (\mathbb{A}_2) and (2.14) would force the old form-boundedness assumption on drift b, i.e. as in (\mathbb{A}_1) . This is another situation where one needs to work in L^p with p large, not related to maximizing admissible values of the form-bounds.

Another technical novelty of the paper is Theorem 6, i.e. the embedding theorem, which has applications beyond this paper, see [37].

As we already mentioned in the introduction, the proof of Caccioppoli's inequality (Proposition 4) under assumption (2.14) uses an extension of the iteration procedure introduced in [39]. In [39], the authors worked in L^2 and used Moser's method to prove the Harnack inequality for positive solutions of $(-\nabla \cdot a \cdot \nabla + b \cdot \nabla)u = 0$ with measurable uniformly elliptic matrix a and $b \in \mathbf{MF}_{\delta}$, $\delta < \infty$, provided that the form-bounds of the positive and the negative parts of div b satisfy some sub-critical assumptions.

Assertion (iv) of Theorem 1 follows, after applying Lemma 1, from the result in [29] whose proof, in turn, follows the method of Röckner-Zhao [49]. In [29] we needed a technical hypothesis that b has compact support, hence the condition in (iv) on the support of ζ . That said, this hypothesis can be removed in [29] by working with weights vanishing at infinity, which we plan to pursue elsewhere.

3. Other remarks on Theorems 1, 2

Remark 4 (On McKean-Vlasov equation with form-bounded interaction kernel). If $K \in \mathbf{F}_{\kappa}$, then in the McKean-Vlasov PDE

$$\partial_t \rho - \Delta \rho - \operatorname{div}(\rho \tilde{K}) = 0, \quad \tilde{K}(t, \cdot) = K(\cdot) * \rho(t, \cdot),$$
 (3.1)

with the initial condition $\rho_0 \geq 0$, $\langle \rho_0 \rangle = 1$, the drift \tilde{K} is a time-inhomogeneous form-bounded vector field $[0, \infty[\times \mathbb{R}^d \to \mathbb{R}^d \text{ having the same form-bound as } K$, i.e. for a.e. $t \in [0, \infty[$, for all $\varphi \in W^{1,2}$,

$$\begin{split} \langle |\tilde{K}(t)|^2 \varphi^2 \rangle &= \left\langle |\langle K(\cdot - z) \rho(t,z) \rangle_z|^2 \varphi^2 \right\rangle \\ & \text{(apply Cauchy-Schwartz' inequality and use } \langle \rho(t,z) \rangle_z = 1) \\ &\leq \left\langle \langle |K(\cdot - z)|^2 \rho(t,z) \rangle_z \varphi^2 \right\rangle = \left\langle \langle |K(\cdot - z)|^2 \varphi^2 \rangle \rho(t,z) \right\rangle_z \\ & \text{(apply } K \in \mathbf{F}_\kappa \text{ and use again } \langle \rho(t,z) \rangle_z = 1) \\ &\leq \kappa \langle |\nabla \varphi|^2 \rangle + c_\kappa \langle \varphi^2 \rangle. \end{split}$$

Thus, in particular, all a priori estimates on solutions of the Kolmogorov forward equation with time-inhomogeneous form-bounded drifts (which can be obtained e.g. using the dual version of the method of [28]) transfer to solutions of McKean-Vlasov equation (3.1).

Remark 5 (Borderline strengths of interactions). Applying the result of [26] for general form-bounded drifts $b \in \mathbf{F}_{\delta}$, $\delta \leq 4$, one can reach the borderline values of the strengths of interactions

$$\kappa = 4\left(\frac{N}{N-1}\right)^2$$
 if (2.2) holds, or

$$\kappa_{+} = 4 \frac{N}{N-1}$$
 if (2.15) holds

by considering the corresponding to (2.1) Kolmogorov backward equation in the Orlicz space with gauge function $\Phi = \cosh -1$. This space is situated between all L^p and L^∞ (paper [26] deals with the dynamics of the torus but, as we show in subsequent paper [37], one can also work on \mathbb{R}^d , although at expense of requiring a fast vanishing of the drift at infinity).

This result can be viewed as some sort of dual variant of the theory of entropy solutions of the forward Kolmogorov equation (regarding entropy solutions, see [9]). That said, it seems like one can prove more by working with the backward Kolmogorov equation, e.g. construct strongly continuous semigroup for the borderline value of the form-bound in addition to the energy inequality and the uniqueness of weak solution, see [26].

Remark 6 (Drifts). One can easily extend the proofs of Theorems 1, 2 to include particle system

$$dX_i = M_i(X_i)dt - \frac{1}{N} \sum_{j=1, j \neq i}^{N} K_{ij}(X_i - X_j)dt + \sqrt{2}dB_i, \quad 1 \le i \le N,$$

having singular drift terms

$$M_i \in \mathbf{F}_{\mu},$$
 .

Let us discuss for simplicity the case when K_{ij} satisfy (2.2). We require that μ , κ satisfy

$$\left(\sqrt{\mu} + \frac{N-1}{N}\sqrt{\kappa}\right)^2 < 4.$$

We only need to embed this particle system into (4.1), i.e. prove an analogue of Lemma 1 for vector field $b = b^M + b^K$ with b^M , $b^K : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ having components

$$b_i^M(x) := M_i(x_i), \quad b_i^K(x) := \frac{1}{N} \sum_{j=1, j \neq i}^N K_{ij}(x_i - x_j), \quad 1 \le i \le N,$$
 (3.2)

and then e.g. use Theorem 3 for the general SDE (4.1). Repeating the proof of Lemma 1, we obtain right away that

$$b^K \in \mathbf{F}_{\delta^K}(\mathbb{R}^{Nd})$$
 with $\delta^K = \frac{(N-1)^2}{N^2} \kappa$, $c_{\delta^K} = \frac{(N-1)^2}{N} c_{\kappa}$

and

$$b^M \in \mathbf{F}_{\delta^M}(\mathbb{R}^{Nd})$$
 with $\delta^M = \mu$, $c_{\delta^M} = Nc_{\mu}$,

see Remark 11 in Section 7 for the proof. It remains to note that the sum of two form-bounded vector fields is form-bounded, i.e. $b = b^M + b^K$ is in \mathbf{F}_{δ} with $\sqrt{\delta} = \sqrt{\delta^M} + \sqrt{\delta^K}$, and δ must be strictly less than 4, cf. Theorem 3.

Arguing similarly, one can treat general drifts $M_i(X_1, ..., X_N)$ $(1 \le i \le N)$ in the particle system after adjusting the hypothesis on the form-bound, i.e. now $\delta^M = N\mu$.

Remark 7 (On the uniqueness of weak solution to elliptic Kolmogorov PDE). Our most complete uniqueness result for the Kolmogorov elliptic equation in (2.4) with the interaction kernels satisfying (2.2) or (2.15) is proved in [26] on the torus, see Remark 5. Speaking of \mathbb{R}^d , let us first say a few words about the case of very sub-critical strengths of interactions.

Definition 5. If K satisfies (2.2) with $\kappa < (\frac{N}{N-1})^2$, we say that u is a weak solution of (2.4) if $u \in W^{1,2} \cap L^{\infty}$ and

$$\mu\langle u,\varphi\rangle + \langle \nabla u, \nabla \varphi \rangle + \frac{1}{N} \left\langle \sum_{i=1}^{N} \sum_{j=1, i \neq i}^{N} K_{ij}(x_i - x_j) \cdot \nabla_{x_i} u, \varphi \right\rangle = \langle f, \varphi \rangle$$

for every $\varphi \in W^{1,2}$. (Recall that in (2.4) the initial function is bounded, so the solution is bounded as well.)

Definition 6. If K satisfies (2.15) with $\kappa_+ < 2\frac{N}{N-1}$, then u is a weak solution to (2.4) if $u \in W^{1,2} \cap L^{\infty}$ and

$$\mu \langle u, \varphi \rangle + \langle \nabla u, \nabla \varphi \rangle$$

$$- \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1, i \neq i}^{N} \left[\langle \operatorname{div} K_{ij}(x_i - x_j) u, \varphi \rangle + \langle K_{ij}(x_i - x_j) u, \nabla_{x_i} \varphi \rangle \right] = \langle f, \varphi \rangle$$

for all $\varphi \in W^{1,2}$.

In both cases the uniqueness of the weak solution follows upon applying Lemmas 1, 2 and the Lax-Milgram theorem in L^2 , i.e. we can take p=2 in Theorem 1(iii), Theorem 2(ii).

In the general case, we need to consider (2.4) in L^p , where p is as in Theorem 1(iii) or Theorem 2(ii). In this regard, we refer to [50] for the definition of weak solution and results on weak solutions of parabolic equations in L^p .

If K satisfies (2.14), then we can prove that u constructed in Theorem 2(i) is a weak solution of (2.4) e.g. in the following sense.

Definition 7. If K satisfies (2.14), then we say that u is a weak solution of (2.4) if $u \in W^{1,2}_{loc} \cap L^{\infty}$ and

$$\mu \langle u, \varphi \rangle + \langle \nabla u, \nabla \varphi \rangle$$

$$- \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \left[\langle \operatorname{div} K_{ij} (x_i - x_j) u, \varphi \rangle + \langle K_{ij} (x_i - x_j) u, \nabla_{x_i} \varphi \rangle \right] = \langle f, \varphi \rangle$$

for all $\varphi \in W^{1,2}_{loc} \cap L_c^{\infty}$ (L_c^{∞} are bounded functions with compact supports).

The latter is a way to establish a link between function u defined by (2.3) and the formal elliptic equation (2.4). However, the proof of uniqueness of such a weak solution under condition (2.14) remains elusive. Still, we can prove that u given by (2.3) is unique among weak solutions that can be obtained via a reasonable regularization of K, see Theorem 3(v). Alternatively, we can restrict our attention to the subclass of weakly form-bounded vector fields, see Remark 8, and prove uniqueness via the Lax-Milgram theorem in the triple of Bessel potential spaces

$$\mathcal{W}^{\frac{1}{2},2} \subset \mathcal{W}^{-\frac{1}{2},2} \subset \mathcal{W}^{-\frac{3}{2},2}$$

where $W^{p,\alpha} := (\lambda - \Delta)^{-\frac{\alpha}{2}} L^p$ (rather than the standard $W^{1,2} \subset L^2 \subset W^{-1,2}$), see [35], although this comes at the cost of requiring that the weak form-bound of K (and therefore its multiplicative form-bound κ_0 , cf. (3.4)) must be strictly less than 1.

Remark 8 (Sufficient condition for multiplicative form-boundedness). A Borel measurable vector field $K: \mathbb{R}^d \to \mathbb{R}^d$ is said to belong to the class of weakly form-bounded vector fields $\mathbf{F}_{\kappa}^{1/2}$ if $|K| \in L^1_{loc}$ and

$$||K|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}}||_{2 \to 2} \le \sqrt{\kappa} \quad (L^2 \to L^2 \text{ operator norm})$$
 (3.3)

for some $\lambda > 0$. We have

$$\mathbf{F}_{\kappa}^{1/2} \subset \mathbf{MF}_{\kappa}.$$
 (3.4)

Indeed, if $K \in \mathbf{F}_{\kappa}^{1/2}$, then, arguing as in [50], we have

$$\begin{split} \langle |K|\varphi,\varphi\rangle &\leq \kappa \langle (\lambda-\Delta)^{\frac{1}{2}}\varphi,\varphi\rangle \leq \kappa \|(\lambda-\Delta)^{\frac{1}{2}}\varphi\|_2 \|\varphi\|_2 \\ &= \kappa \sqrt{\|\nabla\varphi\|_2^2 + \lambda \|\varphi\|_2^2} \|\varphi\|_2 \leq \kappa \|\nabla\varphi\|_2 \|\varphi\|_2 + \kappa \sqrt{\lambda} \|\varphi\|_2^2, \end{split}$$

i.e. $K \in \mathbf{MF}_{\kappa}$.

The class $\mathbf{F}_{\kappa}^{1/2}$ (and therefore \mathbf{MF}_{κ}) contains the largest possible up to the strict inequality in $\varepsilon > 0$ scaling-invariant Morrey class $M_{1+\varepsilon}$, i.e. if

$$||K||_{M_{1+\varepsilon}} := \sup_{r>0, x \in \mathbb{R}^d} r \left(\frac{1}{|B_r|} \int_{B_r(x)} |K|^{1+\varepsilon} dx \right)^{\frac{1}{1+\varepsilon}} < \infty,$$

then $M_{1+\varepsilon} \subset \mathbf{F}_{\kappa}^{1/2}$ with $\kappa = c(d,\varepsilon) \|K\|_{M_{1+\varepsilon}}$ [1].

It is easily seen that $M_{1+\varepsilon}$ is larger than M_2 , which, in turn, contains \mathbf{F}_{κ} . That said, we also need to control the form-bounds. In fact, we have

$$\mathbf{F}_{\kappa} \subset \mathbf{F}_{\sqrt{\kappa}}^{1/2}.\tag{3.5}$$

Indeed, rewriting $K \in \mathbf{F}_{\kappa}$ as

$$||K|(\lambda - \Delta)^{-\frac{1}{2}}||_{2 \to 2} \le \sqrt{\kappa}$$

(with $\lambda = c_{\kappa}/\kappa$), we obtain the required result by applying the Heinz inequality. In (3.5) we have a proper inclusion because the class of weakly form-bounded vector fields also contains the Kato class of vector fields $||K|(\lambda - \Delta)^{-\frac{1}{2}}||_{\infty} \leq \sqrt{\kappa}$ while \mathbf{F}_{κ} does not (see [24, 33]).

Remark 9 (Stronger hypersurface singularities). We refer to [28] and [31] for the results on weak well-posedness of general SDE (4.1) with drift $b \in \mathbf{F}_{\delta}^{1/2}$ or with b in the time-inhomogeneous analogue of the Morrey class $M_{1+\varepsilon}$ (it is quite close to $\mathbf{F}_{\delta}^{1/2}$ but also covers critical-order singularities of the drift in time). This allows to treat

$$b(x) = \pm \frac{cx}{||x| - 1|^{1 - \gamma}} \eta(x),$$

for a fixed $0 < \gamma < 1$, $c \in \mathbb{R}$ and $0 \le \eta \in C_c^{\infty}$, i.e. hypersurface singularities that are essentially twice more singular than (1.13). That said, in these results the assumptions on the form-bound δ are dimension-dependent. So, if we were to apply them to particle system (2.1) we would arrive at the assumptions on the strength of interaction κ that degenerate to zero as $N \uparrow \infty$ (i.e. are of the form $\kappa < \frac{C}{(Nd)^2}$), which is not what we are after in the present work.

4. SDEs with general singular drifts

Theorem 1 and Theorem 2 (excluding the heat kernel bound in assertion (iv)) are proved by embedding particle system (2.1) in general SDE (4.1) via (1.15) and then applying Theorem 3 below. The general SDE, which we consider here, to lighten the notations, in \mathbb{R}^d instead of \mathbb{R}^{Nd} , is

$$Y(t) = y - \int_0^t b(Y(s))ds + \sqrt{2}B(t), \quad t \in [0, T], \quad y \in \mathbb{R}^d,$$
 (4.1)

where a priori $b \in [L^1_{loc}]^d$, $\{B(t)\}_{t>0}$ is a Brownian motion in \mathbb{R}^d ,

Let us add that once we put the drifts in particle system (2.1) using Remark 6, we can obtain most of Theorem 3 from Theorems 1 and 2 by taking all $K_{ij} = 0$. So, thanks to Lemmas 1 and 2, our results on particle systems and general SDEs are, to a large extent, equivalent. Of course, the heat kernel bound in Theorem 2(iv) is specific to particle systems. Also, Theorem 4 does not have at the moment a particle system counterpart, although we believe that it is of interest on its own.

Set

$$b_n := E_{\varepsilon_n} b, \quad \varepsilon_n \downarrow 0, \quad E_{\varepsilon} \text{ is the Friedrichs mollifier}, \quad \varepsilon_n \downarrow 0.$$
 (4.2)

Theorem 3. Assume that a Borel measurable vector field b in SDE (4.1) satisfies one of the following two conditions:

$$b \in \mathbf{F}_{\delta} \quad with \ \delta < 4$$
 (A₁)

or

$$b \in \mathbf{MF}, \qquad \begin{cases} (\operatorname{div} b)_{-} \in L^{1} + L^{\infty}, \\ (\operatorname{div} b)_{+}^{\frac{1}{2}} \in \mathbf{F}_{\delta_{+}} \text{ with } \delta_{+} < 4, \end{cases} \qquad |b|^{\frac{1+\alpha}{2}} \in \mathbf{F}$$
 (A₂)

for some $\alpha > 0$ fixed arbitrarily small. Then the following are true:

(i) There exists a strong Markov family $\{\mathbb{P}_y\}_{y\in\mathbb{R}^d}$ of martingale solutions of SDE (4.1), i.e. $\mathbb{P}_y[\omega_0=y]=1$,

$$\mathbb{E}_y \int_0^T |b(\omega_t)| dt < \infty$$

and for every $\phi \in C_c^2(\mathbb{R}^d)$ the process

$$[0,T] \ni r \mapsto \phi(\omega_r) - \phi(y) + \int_0^r (-\Delta + b \cdot \nabla)\phi(\omega_t)dt$$

is a martingale under \mathbb{P}_{y} .

(ii) The function

$$u(x) := \mathbb{E}_{\mathbb{P}_x} \int_0^\infty e^{-\lambda s} f(\omega_s) ds, \quad x \in \mathbb{R}^d, \quad f \in C_c^\infty(\mathbb{R}^d), \tag{4.3}$$

where λ is assumed to be sufficiently large, is a locally Hölder continuous weak solution to elliptic Kolmogorov equation $(\lambda - \Delta + b \cdot \nabla)u = f$ (see Remark 7 for the definitions).

In assertions (iii) and (iv) we replace condition (\mathbb{A}_2) with a somewhat more restrictive hypothesis

$$\begin{cases}
b \in \mathbf{F}, \\
(\operatorname{div} b)_{+}^{\frac{1}{2}} \in \mathbf{F}_{\delta_{+}} \text{ with } \delta_{+} < 4, \quad (\operatorname{div} b)_{-}^{\frac{1}{2}} \in L^{1} + L^{\infty}.
\end{cases}$$
(A₃)

If b satisfies (\mathbb{A}_1) , fix $p > \frac{2}{2-\sqrt{\delta}}$. If b satisfies (\mathbb{A}_3) , fix $p > \frac{4}{4-\delta_+}$.

(iii) ([41, 50], see also [33]) The family of operators

$$P_t f(x) := \mathbb{E}_{\mathbb{P}_x}[f(\omega_t)], \quad t > 0, \quad f \in C_c^{\infty}$$

admits extension by continuity to a strongly continuous quasi contraction Markov semigroup on L^p , say, $P_t =: e^{-t\Lambda_p}$, such that

$$||e^{-t\Lambda_p}||_{p\to q} \le ce^{\omega t}t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}, \quad p \le q \le \infty$$

for appropriate constants c and ω .

The following approximation uniqueness result holds: for any sequence of bounded smooth vector fields

$$b_n \to b$$
 in $[L_{loc}^2]^d$

that do not increase the form-bounds on b in (A_2) or (A_3) , the classical solutions v_n to

$$(\partial_t - \Delta + b_n \cdot \nabla)v_n = 0, \quad v_n(0) = f \in C_c^{\infty}$$

converge to the same limit $e^{-t\Lambda_p}f$ in L^p loc. uniformly in $t \geq 0$.

(iv) The resolvent $(\mu + \Lambda_p(b))^{-1}$ has Feller property, i.e. for each μ greater than some $\mu_0 > 0$ it extends by continuity to a bounded linear operator on C_{∞} :

$$R_{\mu}(b) := \left[(\mu + \Lambda_{p}(b))^{-1} \upharpoonright L^{p} \cap C_{\infty} \right]_{C_{\infty} \to C_{\infty}}^{\text{clos}} \in \mathcal{B}(C_{\infty}).$$

Moreover,

$$R_{\mu}(b_n) \to R_{\mu}(b)$$
 strongly in C_{∞} , $\mu \ge \mu_0$,

where $R_{\mu}(b_n)$ coincides with the resolvent of $-\Delta + b_n \cdot \nabla$ on C_{∞} , n = 1, 2, ...

(v) (Approximation uniqueness) If b satisfies

$$\begin{cases}
b \in \mathbf{MF}, \\
(\operatorname{div} b)_{-} \in L^{1} + L^{\infty}, & (\operatorname{div} b)_{+}^{\frac{1}{2}} \in \mathbf{F}_{\delta_{+}} \text{ with } \delta_{+} < 2,
\end{cases}$$
(4.4)

then there exist generic constants $\lambda_0 > 0$ and $\varkappa \in]0,1[$ such that if, additionally, $|b| \in L^{2-\varkappa}$, then, for any sequence b_n of bounded smooth vector fields satisfying (4.4) with the same constants as b and such that $b_n \to b$ in $L^{2-\varkappa}$, the sequence of the classical solutions u_n to

$$(\lambda - \Delta + b_n \cdot \nabla)u_n = f, \quad f \in C_c^{\infty}, \quad \lambda \ge \lambda_0$$

converge in L^2 to the same limit which, thus, does not depend on a particular choice of $\{b_n\}$.

In (iii) we can consider b_n defined by (4.2).

In subsequent paper [37] we strengthen assertion (iv) by constructing strongly continuous Feller semigroup in C_{∞} for $-\Delta + b \cdot \nabla$, $b \in \mathbf{F}_{\delta}$, for all $\delta < 4$.

Combining assertion (v) with Theorem 5, one can further show that the limit u is locally Hölder continuous and is a weak solution of $(\mu - \Delta + b \cdot \nabla)u = f$. Furthermore, one can construct the corresponding strongly continuous semigroup in L^p , but we will not pursue this here. We included assertion (v) to emphasize that there is, in principle, nothing pathological from the point of view of a posteriori estimates about condition $b \in \mathbf{MF}$ compared to $b \in \mathbf{F}$ used in (iii), (iv). That said, the proof of assertion (v) is somewhat more involved than the proof of the uniqueness of the limit in (iii) and uses Gehring's lemma. It is ultimately an L^2 argument, hence the need for a more restrictive condition $\delta_+ < 2$ in (\mathbb{A}_3) , see Remark 17 for details.

We need assertion (iv) of Theorem 3 in the proof of the following uniqueness result.

Theorem 4 (Krylov-type estimates and conditional uniqueness). Assume that a Borel measurable vector field b satisfies one of the following conditions:

$$b \in \mathbf{F}_{\delta} \text{ with } \delta < \left(\frac{2}{q}\right)^2 \wedge 1 \text{ for some } q > (d-2) \vee 2$$
 (\mathbb{B}_1)

or

$$\begin{cases} b \in \mathbf{F}_{\delta} \cap [W_{\mathrm{loc}}^{1,1}(\mathbb{R}^{d})]^{d} \text{ for some finite } \delta, \text{ has symmetric Jacobian } Db := (\nabla_{k}b_{i})_{k,i=1}^{d}, \\ \text{the normalized eigenvectors } e_{j} \text{ and eigenvalues } \lambda_{j} \geq 0 \text{ of the negative part of } Db - \frac{\mathrm{div}\,b}{q}I \\ \text{for some } q > (d-2) \vee 2 \text{ satisfy } \sqrt{\lambda_{j}}e_{j} \in \mathbf{F}_{\nu_{j}} \text{ with } \nu := \sum_{j=1}^{d} \nu_{j} < \frac{4(q-1)}{q^{2}}. \end{cases}$$

$$(\mathbb{B}_{2})$$

Then the following are true for the strong Markov family of martingale solutions of SDE (4.1) constructed in Theorem 3:

(i) For every $y \in \mathbb{R}^d$, martingale solution \mathbb{P}_y satisfies Krylov-type bound

$$\mathbb{E}_{\mathbb{P}_y} \int_0^\infty e^{-\lambda s} |\mathbf{g}f|(\omega_s) ds \le C \|\mathbf{g}|f|^{\frac{q}{2}} \|_2^{\frac{2}{q}}, \quad \forall \, \mathbf{g} \in \mathbf{F}, \quad \forall \, f \in C_c, \tag{4.5}$$

for $q > (d-2) \vee 2$ close to $(d-2) \vee 2$, for all λ sufficiently large.

- (i') $\{\mathbb{P}_y\}_{y\in\mathbb{R}^d}$ is the only Markov family of martingale solutions to (4.1) that satisfies Krylov-type bound in (i).
 - (ii) For every $y \in \mathbb{R}^d$, \mathbb{P}_x satisfies Krylov bound:

$$\mathbb{E}_{\mathbb{P}_y} \int_0^\infty e^{-\lambda s} |f(\omega_s)| ds \le C \|f\|_{\frac{qd}{d+q-2}}, \quad \forall f \in C_c$$
 (4.6)

for all λ sufficiently large.

(ii') We make (4.6) more restrictive by selecting q close to $(d-2)\vee 2$, so that in (4.6) $\frac{qd}{d+q-2}=\frac{d}{2-\varepsilon}\wedge\frac{2}{1-\varepsilon}$ for some $\varepsilon>0$ small. Let $\{\mathbb{P}^2_y\}_{t\in\mathbb{R}^d}$ be another Markov family of martingale solutions for (4.1) that satisfies Krylov bound

$$\mathbb{E}_{\mathbb{P}_{y}^{2}} \int_{0}^{\infty} e^{-\lambda s} |f|(\omega_{s}) ds \leq C \|f\|_{\frac{d}{2-\varepsilon} \wedge \frac{2}{1-\varepsilon}}, \quad \forall f \in C_{c}$$

$$\tag{4.7}$$

(one such family exists, it is $\{\mathbb{P}_y\}_{y\in\mathbb{R}^d}$ from above). Assume additionally that, for some $\varepsilon_1 \in]\varepsilon, 1[$ we have

$$(1+|x|^{-2})^{-\beta}|b|^{\frac{d}{2-\varepsilon_1}\vee\frac{2}{1-\varepsilon_1}}\in L^1$$

for some $\beta > \frac{d}{2}$ fixed arbitrarily large, and either (\mathbb{B}_1) holds with $\delta < \frac{4}{q_*^2} \wedge 1$, where

$$q_* := \begin{cases} \frac{d-2}{\varepsilon_1 - \varepsilon} & \text{if } d \ge 4, \\ 2\left(\frac{1}{3(\varepsilon_1 - \varepsilon)} \lor 1\right) & \text{if } d = 3, \end{cases}$$

or (\mathbb{B}_2) holds with $q = q_*$ and $\nu < \frac{4(q_*-1)}{q_*^2}$. Then $\{\mathbb{P}_y^2\}_{y \in \mathbb{R}^d}$ coincides with $\{\mathbb{P}_y\}_{y \in \mathbb{R}^d}$ from above. Some remarks are in order.

- 1. In the last assertion, the uniqueness class of martingale solutions satisfying Krylov bound (4.7), which depends on our choice of ε , determines the extra conditions that one needs to impose on b. Note that if in (\mathbb{B}_1) one has $|b| \in L^d$, or in (\mathbb{B}_2) the eigenvectors have entries in L^d , then the form-bounds δ and ν_j (j = 1, ..., d), respectively, can be chosen arbitrarily small, in which case these extra conditions on b are trivially satisfied.
- 2. In (\mathbb{B}_2) we require Jacobian Db to be symmetric, so $b = \nabla V$ for some potential V. Let us illustrate condition (\mathbb{B}_2) with the following example.

Example 4. Let $d \geq 4$. Let

$$b(x) = -\sqrt{\delta} \frac{d-2}{2} \frac{x}{|x|^2},$$

a drift that pushes solution Y_t of (4.1) away from the origin. Put for brevity $c := \sqrt{\delta} \frac{d-2}{2} > 0$. We have div $b = -c(d-2)|x|^{-2}$ and $\nabla_j b_i = c \left[-|x|^{-2} \delta_{ij} + 2x_i x_j |x|^{-4} \right]$. Therefore, for every $\xi = (\xi_i) \in \mathbb{R}^d$,

$$\xi^{\top}(Db - \frac{\operatorname{div} b}{q}I)\xi = \sum_{i,j=1}^{d} \xi_{j}[(\nabla_{j}b_{i}) - \frac{1}{q}(\operatorname{div} b)\delta_{ji}]\xi_{i} = c\left(\frac{d-2}{q} - 1\right)|x|^{-2}|\xi|^{2} + 2c|x|^{-4}(x \cdot \xi)^{2}$$
$$= \xi^{\top}(B_{+} - B_{-})\xi,$$

where $B_+ \geq 0$ is the matrix with entries $2cx_ix_j|x|^{-4}$, and $B_- := -c(\frac{d-2}{q}-1)|x|^{-2}I \geq 0$. Thus, constant ν in condition (\mathbb{B}_2) can be made as small as needed by selecting q > d-2 sufficiently close to d-2, and so for this b condition (\mathbb{B}_2) can be satisfied for any strength of repulsion from the origin.

In the previous example it is crucial that we can select q as close to d-2 as needed. By working in the parabolic setting we could obtain a stronger uniqueness result, i.e. for every fixed initial point x. However, the parabolic setting requires us to take q > d [30], and so the previous example becomes invalid: we have to require smallness of δ even in the case of repulsion.

Remark 10 (On some other classes of singular vector fields arising in the study of singular SDEs and PDEs). 1. A number of important results on the regularity theory of $-\Delta + b \cdot \nabla$ was obtained in [54, 55] which considered supercritical form-boundedness type conditions on b (in the context of the study of 3D Navier-Stokes equations). These are conditions of the type: there exists $\varepsilon \in]0,1]$ such that $|b| \in L^{1+\varepsilon}_{loc}([0,\infty[\times\mathbb{R}^d)])$ and

$$\int_{0}^{\infty} \int_{\mathbb{R}^{d}} |b(t,\cdot)|^{1+\varepsilon} \xi^{2}(t,\cdot) dt \leq \delta \int_{0}^{\infty} \|\nabla \xi(t,\cdot)\|_{2}^{2} dt + \int_{0}^{\infty} g(t) \|\xi(t,\cdot)\|_{2}^{2} dt$$
for all $\xi \in C_{c}^{\infty}([0,\infty[\times \mathbb{R}^{d})])$ (4.8)

for some $\delta > 0$ and $0 \le g \in L^1_{\text{loc}}([0, \infty[)$ under, necessarily, some assumptions on div b which cannot be too singular. Here super-criticality/criticality/sub-criticality refer to how the assumptions on b behave under rescaling the equation. In the super-critical case one has to sacrifice a large portion of the regularity theory of $-\Delta + b \cdot \nabla$ including the usual Harnack inequality and the Hölder continuity of solutions to the elliptic and parabolic equations. See also counterexample to the uniqueness in law for SDEs with super-critical drifts in [58]. However, some parts of the theory,

such as the local boundedness of weak solutions, can be salvaged, see cited papers, see also recent developments in [4, 20]. Let us also note that if we were to specify (4.8) to the critical case when the usual regularity theory is still valid, then we would need to take $\varepsilon = 1$, i.e. we would obtain condition (2.15), but not more general condition (2.14).

2. As was noted in [32], after supplementing (4.8) with condition (div b) $_{+}^{\frac{1}{2}} \in \mathbf{F}_{\nu}$ for some $\nu < 4$, one can still prove the existence of a martingale solution to SDE (4.1). In the present paper we work in the critical setting which allows us to preserve most of the important results in the regularity theory of elliptic equations that do not involve estimates on the second order derivatives of the solutions (which are destroyed by the form-boundedness assumptions), and thus allows to prove, e.g. the strong Markov property, approximation uniqueness or conditional weak uniqueness results for particle system (1.1) (see, however, [20] who constructed a Markov family of weak solutions in a super-critical setting using a selection procedure).

Let us also add that above super-criticality refers to the assumptions on b, but not on $(\operatorname{div} b)_+$. In fact, as the counterexample to weak solvability of (1.14) with the model attracting drift shows, one cannot go beyond the form-boundedness assumption (critical) on $(\operatorname{div} b)_+$.

5. Regularity results for PDEs

1. To prove Theorem 3, we need the regularity results of Theorems 5, 6 for non-homogeneous elliptic equations (5.1), (5.6), respectively In these results we assume additionally that the coefficients of (5.1), (5.6) are bounded and smooth. However, importantly, the constants in the regularity estimates are *generic*, i.e. they depend only on the structure parameters of the equation such as the dimension d, constant term λ and the form-bounds of the vector fields (but not on the smoothness or boundedness of the coefficients).

Theorem $5 \Rightarrow$ strong Markov property in Theorem 3.

Theorem $6 \Rightarrow$ existence of martingale solutions in Theorem 3.

Theorem 5 (Hölder continuity of solutions). Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a bounded smooth vector field such that either

$$b \in \mathbf{F}_{\delta} \text{ with } \delta < 4$$
 (this is (\mathbb{A}_1))

or

$$\begin{cases}
b \in \mathbf{MF}, \\
(\operatorname{div} b_{+})^{\frac{1}{2}} \in \mathbf{F}_{\delta_{+}} \text{ with } \delta_{+} < 4,
\end{cases} (\bar{\mathbb{A}}_{2})$$

where div $b = \text{div } b_+ - \text{div } b_-$ for some bounded smooth functions div $b_{\pm} \geq 0$. Let $f \in C_c^{\infty}$, $\lambda \geq 0$. Then the classical solution u to non-homogeneous equation

$$(\lambda - \Delta + b \cdot \nabla)u = f \tag{5.1}$$

is locally Hölder continuous with generic constants that also depend on $||f||_{\infty}$.

(The difference between $(\bar{\mathbb{A}}_2)$ and (\mathbb{A}_2) is that in the former case we do not require div b_{\pm} to be positive and negative parts of div b, which are continuous but not necessarily smooth.)

The fact that the constants are generic is of course the main point of Theorem 5.

Define weight

$$\rho(y) = (1 + k|y|^2)^{-\frac{d}{2} - 1}, \quad y \in \mathbb{R}^d, \tag{5.2}$$

where constant constant k > 0 will be chosen sufficiently small. This weight has property

$$|\nabla \rho| \le \left(\frac{d}{2} + 1\right) \sqrt{k\rho}.\tag{5.3}$$

For a fixed $x \in \mathbb{R}^d$, put $\rho_x(y) := \rho(y - x)$.

Theorem 6 (Embedding theorem). Let $b, h : \mathbb{R}^d \to \mathbb{R}^d$ be bounded smooth vector fields such that

$$b \in \mathbf{F}_{\delta} \text{ with } \delta < 4, \quad \mathsf{h} \in \mathbf{F}_{\chi} \text{ with } \chi < \infty$$
 (5.4)

or

$$\begin{cases}
b \in \mathbf{MF}_{\delta} \text{ for some } \delta < \infty, \\
(\operatorname{div} b_{+})^{\frac{1}{2}} \in \mathbf{F}_{\delta_{+}} \text{ with } \delta_{+} < 4,
\end{cases} |\mathsf{h}|^{\frac{1+\gamma}{2}} \in \mathbf{F}_{\chi} \quad \text{with } \chi < \infty \tag{5.5}$$

for some $\gamma > 0$ fixed arbitrarily small, where div $b = \text{div } b_+ - \text{div } b_-$ for some bounded smooth functions div $b_{\pm} \geq 0$. In the former case, fix $p > \frac{2}{2-\sqrt{\delta}}$, $p \geq 2$, and in the latter case fix $p > \frac{4}{4-\delta_+}$, $p' \leq 1 + \gamma$, $p \geq 2$.

Then, for a fixed $1 < \theta < \frac{d}{d-2}$, there exist generic constants λ_0 , k (in ρ), C and $\beta \in]0,1[$ such that the classical solution u to non-homogeneous equation

$$(\lambda - \Delta + b \cdot \nabla)u = |\mathbf{h}f|, \quad f \in C_c^{\infty} \tag{5.6}$$

on \mathbb{R}^d satisfies for all $\lambda \geq \lambda_0 \vee 1$:

$$||u||_{\infty} \leq C \sup_{x \in \frac{1}{2}\mathbb{Z}^{d}} \left((\lambda - \lambda_{0})^{-\frac{1}{p\theta}} \langle \left(\mathbf{1}_{|\mathsf{h}|>1} + |\mathsf{h}|^{p\theta} \mathbf{1}_{|\mathsf{h}|\leq 1} \right) |f|^{p\theta} \rho_{x} \rangle^{\frac{1}{p\theta}} + \lambda^{-\frac{\beta}{p}} \langle \left(\mathbf{1}_{|\mathsf{h}|>1} + |\mathsf{h}|^{p\theta'} \mathbf{1}_{|\mathsf{h}|\leq 1} \right) |f|^{p\theta'} \rho_{x} \rangle^{\frac{1}{p\theta'}} \right).$$

$$(5.7)$$

2. The following result is needed to prove conditional weak uniqueness in Theorem 4.

Theorem 7 (Gradient bounds). Assume that a bounded smooth vector field b satisfies either condition (\mathbb{B}_1) of Theorem 4 or

$$\begin{cases} b \in \mathbf{F}_{\delta} \cap [W_{\mathrm{loc}}^{1,1}(\mathbb{R}^{d})]^{d} & \text{with finite } \delta \text{ and symmetric Jacobian } Db := (\nabla_{k}b_{i})_{k,i=1}^{d}, \\ \text{and the negative part } B_{-} & \text{of matrix } Db - \frac{\mathrm{div}\,b}{q}I \text{ for some } q > (d-2) \vee 2 \\ \text{satisfies } \langle B_{-}h, h \rangle \leq \nu \langle |\nabla|h||^{2} \rangle + c_{\nu} \langle |h|^{2} \rangle & \text{for some } \nu < \frac{4(q-1)}{q^{2}}. \end{cases}$$

Then the following are true:

(i) For every $g \in \mathbf{F}$ there exist generic constants μ_0 and K such that, for every $\mu > \mu_0$, the classical solution u to elliptic equation $(\mu - \Delta - b \cdot \nabla)u = |g|f$, $f \in C_c^{\infty}$, satisfies

$$\|\nabla |\nabla u|^{\frac{q}{2}}\|_2 \le K \|\mathbf{g}|f|^{\frac{q}{2}}\|_2.$$

(ii) There exist generic constants μ_0 and K such that the classical solution u to elliptic equation $(\mu - \Delta - b \cdot \nabla)u = f$, $f \in C_c^{\infty}$, satisfies, for all $\mu > \mu_0$,

$$\|\nabla |\nabla u|^{\frac{q}{2}}\|_{2} \le K \|f\|^{\frac{q}{2}}_{\frac{qd}{d+q-2}}.$$

In both assertions, by the Sobolev embedding theorem, u is Hölder continuous, although using these gradient bounds to prove Hölder continuity would be excessive: in Theorem 5 we arrive at the same conclusion directly, using De Giorgi's method, under less restrictive conditions on b.

For example, condition $(\bar{\mathbb{B}}_2)$ holds if condition (\mathbb{B}_2) of Theorem 4 is satisfied, see Lemma 12.

Assuming that $b \in \mathbf{F}_{\delta}$, $\delta < (\frac{2}{d-2})^2 \wedge 1$, [41] proved estimate

$$\|\nabla |\nabla u|^{\frac{q}{2}}\|_{2} \le K\|f\|_{q}, \quad q \in](d-2) \lor 2, \frac{2}{\sqrt{\delta}}[$$
 (5.8)

for solution u to elliptic equation $(\mu - \Delta - b \cdot \nabla)u = f$. This estimate was used in [41] to construct the corresponding to $-\Delta - b \cdot \nabla$ Feller semigroup via a Moser-type iteration procedure. The norm $||f||_q$ in the right-hand side of (5.8) does not allow to obtain the uniqueness result in Theorem 4 from (5.8), unless b satisfies additional assumption $|b| \in L^{(d-2)\vee 2}$. Still, the argument of [41] can be modified to include a weaker norm of f, and this is what we do in the proof of Theorem 7. In particular, we use the test function

$$\phi = -\nabla \cdot (\nabla u |\nabla u|^{q-2}) \tag{5.9}$$

of [41]. In more recent literature one can find other test functions that give gradient bounds on u of the same type as in Theorem 7 (moreover, these test functions work for larger classes of equations). However, importantly, test function (5.9) yields the least restrictive assumptions on form-bounds δ and ν , which are in the focus of the present paper. In fact, one can argue that by multiplying the elliptic equation by test function (5.9) and integrating by parts, one differentiates the equation in the optimal direction $\frac{\nabla u}{|\nabla u|}$. We refer to [27] for more detailed discussion and references.

6. Smooth approximation of form-bounded vector fields

Let $b \in [L^1_{loc}(\mathbb{R}^d)]^d$. Define

$$b_{\varepsilon} := E_{\varepsilon}b, \quad \varepsilon > 0,$$

where, recall, $E_{\varepsilon}h$ denotes the Friedrichs mollifier of function (or vector field) h, see Section 1.3 for the definition.

Lemma 3. If $b \in \mathbf{F}_{\delta}$, then the following is true:

- 1. $b_{\varepsilon} \in [L^{\infty}(\mathbb{R}^d) \cap C^{\infty}(\mathbb{R}^d)]^d$, $b_{\varepsilon} \to b$ in $[L^2_{loc}(\mathbb{R}^d)]^d$ as $\varepsilon \downarrow 0$.
- 2. $b_{\varepsilon} \in \mathbf{F}_{\delta}$ with the same constant c_{δ} (thus, independent of ε).

Proof. 1. The smoothness of b_{ε} and the convergence follow from the standard properties of Friedrichs mollifiers, so it remains to prove that $|b_{\varepsilon}| \in L^{\infty}$. By Hölder's inequality,

$$|b_{\varepsilon}(x)| \leq \sqrt{E_{\varepsilon}|b|^2(x)} = \sqrt{\langle \gamma_{\varepsilon}(x-\cdot)|b(\cdot)|^2 \rangle},$$

so

$$|b_{\varepsilon}(x)| \leq \langle |b(\cdot)|^{2} \gamma_{\varepsilon}(x - \cdot) \rangle^{\frac{1}{2}}$$
(we apply the hypothesis $b \in \mathbf{F}_{\delta}$)
$$\leq (\delta \langle |\nabla \sqrt{\gamma_{\varepsilon}(x - \cdot)}|^{2} \rangle + c_{\delta})^{\frac{1}{2}} = (C\varepsilon^{-2} + c_{\delta})^{\frac{1}{2}}.$$

Hence $|b_{\varepsilon}| \in L^{\infty}$ for each $\varepsilon > 0$.

2. Put $\varphi_{\varepsilon} = \sqrt{E_{\varepsilon}\varphi^2}$, $\varphi \in W^{1,2}$. Then

$$||b_{\varepsilon}\varphi||_2^2 \le \langle E_{\varepsilon}|b|^2, \varphi^2\rangle = ||b\varphi_{\varepsilon}||_2^2 \le \delta ||\nabla \varphi_{\varepsilon}||_2^2 + c_{\delta} ||\varphi_{\varepsilon}||_2^2,$$

where

$$\|\nabla \varphi_{\varepsilon}\|_{2} = \left\| \frac{E_{\varepsilon}(\varphi \nabla \varphi)}{\sqrt{E_{\varepsilon} \varphi^{2}}} \right\|_{2}$$
(we apply Cauchy-Schwartz' inequality)
$$\leq \|\sqrt{E_{\varepsilon}|\nabla \varphi|^{2}}\|_{2} = \|E_{\varepsilon}|\nabla \varphi|^{2}\|_{1}^{\frac{1}{2}} \leq \|\nabla \varphi\|_{2}$$
(6.1)

and, clearly, $\|\varphi_{\varepsilon}\|_{2} \leq \|\varphi\|_{2}$.

Lemma 4. If $b \in \mathbf{MF}_{\delta}$, then the following is true:

- 1. $b_{\varepsilon} \in [L^{\infty}(\mathbb{R}^d) \cap C^{\infty}(\mathbb{R}^d)]^d$, $b_{\varepsilon} \to b$ in $[L^1_{loc}(\mathbb{R}^d)]^d$.
- 2. $b_{\varepsilon} \in \mathbf{MF}_{\delta}$ with the same c_{δ} .

Proof. 1. We only need to prove $|b_{\varepsilon}| \in L^{\infty}$. By $b \in \mathbf{MF}_{\delta}$, for all $x \in \mathbb{R}^d$,

$$|b_{\varepsilon}(x)| \le \langle |b(\cdot)|\gamma_{\varepsilon}(x-\cdot)\rangle \le \delta \langle |\nabla \sqrt{\gamma_{\varepsilon}(x-\cdot)}|^2 \rangle^{\frac{1}{2}} + c_{\delta} = C\varepsilon^{-1} + c_{\delta}.$$

2. Let $\varphi_{\varepsilon} = \sqrt{E_{\varepsilon}\varphi^2}$, $\varphi \in W^{1,2}$. We have

$$\langle |b_{\varepsilon}|\varphi,\varphi\rangle = \langle |b|E_{\varepsilon}\varphi^{2}\rangle = \langle |b|\varphi_{\varepsilon}^{2}\rangle \leq \delta \|\nabla\varphi_{\varepsilon}\|_{2} \|\varphi_{\varepsilon}\|_{2} + c_{\delta} \|\varphi_{\varepsilon}\|_{2}^{2},$$

where, repeating the previous proof, $\|\nabla \varphi_{\varepsilon}\|_{2} \leq \|\nabla \varphi\|_{2}$, $\|\varphi_{\varepsilon}\|_{2} \leq \|\varphi\|_{2}$.

Assume that div $b \in L^1_{loc}$. We can represent div $b_{\varepsilon} = E_{\varepsilon} \text{div } b$ as

$$\operatorname{div} b_{\varepsilon} = \operatorname{div} b_{\varepsilon,+} - \operatorname{div} b_{\varepsilon,-},$$

where

$$\operatorname{div} b_{\varepsilon,+} := E_{\varepsilon}(\operatorname{div} b)_{+}, \quad \operatorname{div} b_{\varepsilon,-} := E_{\varepsilon}(\operatorname{div} b)_{-}.$$

Note that smooth functions div $b_{\varepsilon,\pm} \geq 0$ are in general greater than the positive and the negative parts $(\operatorname{div} b_{\varepsilon})_{+} := \operatorname{div} b_{\varepsilon} \vee 0$, $(\operatorname{div} b_{\varepsilon})_{-} := -(\operatorname{div} b_{\varepsilon} \wedge 0)$ of $\operatorname{div} b_{\varepsilon}$.

Lemma 5. If $(\operatorname{div} b)_+ \in \mathbf{F}_{\delta_+}$, $(\operatorname{div} b)_- \in L^1 + L^{\infty}$, then the following is true:

- 1. div $b_{\varepsilon,+} \in L^{\infty} \cap C^{\infty}$, div $b_{\varepsilon,+} \to (\operatorname{div} b)_{+}$ in $L^{1}_{\operatorname{loc}}$ as $\varepsilon \downarrow 0$.
- 2. div $b_{\varepsilon,+} \in \mathbf{F}_{\delta_+}$ with the same c_{δ_+} as the one for b.

Proof. The first statement follows from the properties of Friedrichs mollifiers and the following estimate (we use notations from the previous proof): for every $x \in \mathbb{R}^d$,

$$\operatorname{div} b_{\varepsilon,+}(x) \leq \left\langle (\operatorname{div} b)_{+}(\cdot) \gamma_{\varepsilon}(x-\cdot) \right\rangle \leq \delta_{+} \left\langle \left| \nabla \sqrt{\gamma_{\varepsilon}(x-\cdot)} \right|^{2} \right\rangle + c_{\delta_{+}} = C \varepsilon^{-2} + c_{\delta_{+}}$$

Let us prove the second statement:

$$\langle \operatorname{div} b_{\varepsilon,+} \varphi, \varphi \rangle = \langle (\operatorname{div} b)_{+} \varphi_{\varepsilon}^{2} \rangle \leq \delta_{+} \|\nabla \varphi_{\varepsilon}\|_{2}^{2} + c_{\delta_{+}} \|\varphi_{\varepsilon}\|_{2}^{2} \leq \delta_{+} \|\nabla \varphi\|_{2}^{2} + c_{\delta_{+}} \|\varphi\|_{2}^{2}.$$

Finally, we will need

Lemma 6. If
$$|\mathsf{h}|^{\frac{1+\gamma}{2}} \in \mathbf{F}_{\chi}$$
 $(\gamma > 0)$, then the following is true:
1. $\mathsf{h}_{\varepsilon} := E_{\varepsilon} \mathsf{h} \in [L^{\infty}(\mathbb{R}^d) \cap C^{\infty}(\mathbb{R}^d)]^d$, $\mathsf{h}_{\varepsilon} \to \mathsf{h}$ in $[L^1_{\mathrm{loc}}(\mathbb{R}^d)]^d$ as $\varepsilon \downarrow 0$,

2.
$$|\mathsf{h}_{\varepsilon}|^{\frac{1+\gamma}{2}} \in \mathbf{F}_{\chi}$$
 with the same c_{χ} .

Proof. By Hölder's inequality, $|\mathsf{h}_{\varepsilon}|^{1+\gamma} \leq E_{\varepsilon} |\mathsf{h}|^{1+\gamma}$, so $\langle |\mathsf{h}_{\varepsilon}|^{1+\gamma} \varphi^2 \rangle \leq \langle |\mathsf{h}|^{1+\gamma}, \varphi_{\varepsilon}^2 \rangle$, where, recall, $\varphi_{\varepsilon} = \sqrt{E_{\varepsilon} \varphi^2}, \varphi \in W^{1,2}$. Now we apply $|\mathsf{h}|^{\frac{1+\gamma}{2}} \in \mathbf{F}_{\chi}$ and use $\|\nabla \varphi_{\varepsilon}\|_{2} \leq \|\nabla \varphi\|_{2}, \|\varphi_{\varepsilon}\|_{2} \leq \|\varphi\|_{2}$. \square

7. Proofs of Lemmas 1 and 2

Recall: $b = (b_1, \ldots, b_N) : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ is defined by

$$b_i(x) := \frac{1}{N} \sum_{j=1, j \neq i}^{N} K_{ij}(x_i - x_j), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^{Nd}, \quad 1 \le i \le N.$$

Below $|\cdot|$ denotes, depending on the context, the Euclidean norm in \mathbb{R}^{Nd} or \mathbb{R}^d . In this section, $\langle \, , \, \rangle$ is the integration over \mathbb{R}^{Nd} .

Proof of Lemma 1. We have

$$|b(x)|^{2} \leq \sum_{i=1}^{N} |b_{i}(x)|^{2} \leq \sum_{i=1}^{N} \left(\frac{1}{N} \sum_{j=1, j \neq i}^{N} |K_{ij}(x_{i} - x_{j})|\right)^{2}$$

$$\leq \sum_{i=1}^{N} \frac{N-1}{N^{2}} \sum_{j=1, j \neq i}^{N} |K_{ij}(x_{i} - x_{j})|^{2}.$$

Therefore, $\langle |b|^2 \varphi^2 \rangle \leq \sum_{i=1}^N \frac{N-1}{N^2} \sum_{j=1, j \neq i}^N \langle |K_{ij}(x_i - x_j)|^2 \varphi^2 \rangle$, where, denoting by \bar{x} vector x with component x_i removed, we estimate

$$\langle |K_{ij}(x_i - x_j)|^2 \varphi^2 \rangle = \int_{\mathbb{R}^{(N-1)d}} \int_{\mathbb{R}^d} |K_{ij}(x_i - x_j)|^2 \varphi^2(x_i, \bar{x}) dx_i d\bar{x}$$
(we use $K_{ij} \in \mathbf{F}_{\kappa}(\mathbb{R}^d)$ in x_i variable)
$$\leq \kappa \int_{\mathbb{R}^{(N-1)d}} \int_{\mathbb{R}^d} |\nabla_{x_i} \varphi(x_i, \bar{x})|^2 dx_i d\bar{x} + c_{\kappa} \int_{\mathbb{R}^{Nd}} \varphi^2 dx$$

$$= \kappa \langle |\nabla_{x_i} \varphi|^2 \rangle + c_{\kappa} \langle \varphi^2 \rangle.$$

Hence
$$\langle |b|^2 \varphi^2 \rangle \leq \frac{(N-1)^2}{N^2} \kappa \langle |\nabla \varphi|^2 \rangle + \frac{(N-1)^2}{N} c_{\kappa} \langle \varphi^2 \rangle$$
, as claimed.

Proof of Lemma 2. Let us first prove (2.20). We have

$$\langle |b|\varphi^2\rangle \le \sum_{i=1}^N \langle |b_i|\varphi^2\rangle \le \sum_{i=1}^N \frac{1}{N} \sum_{j=1, j \ne i}^N \langle |K_{ij}(x_i - x_j)|\varphi^2\rangle. \tag{7.1}$$

Denoting by \bar{x} the variable x with component x_i removed, we estimate

$$\langle |K_{ij}(x_i - x_j)|\varphi^2 \rangle = \int_{\mathbb{R}^{(N-1)d}} \int_{\mathbb{R}^d} |K_{ij}(x_i - x_j)|\varphi^2(x_i, \bar{x}) dx_i d\bar{x}$$
(apply $K_{ij} \in \mathbf{MF}_{\kappa}(\mathbb{R}^d)$ in x_i variable)
$$\leq \int_{\mathbb{R}^{(N-1)d}} \left[\left(\int_{\mathbb{R}^d} |\nabla_{x_i} \varphi(x_i, \bar{x})|^2 dx_i \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \varphi^2(x_i, \bar{x}) dx_i \right)^{\frac{1}{2}} + c_{\kappa} \int_{\mathbb{R}^d} \varphi^2(x_i, \bar{x}) dx_i \right] d\bar{x}$$

$$\leq \kappa \langle |\nabla_{x_i} \varphi|^2 \rangle^{\frac{1}{2}} \langle \varphi^2 \rangle^{\frac{1}{2}} + c_{\kappa} \langle \varphi^2 \rangle.$$

Therefore,

$$\sum_{i=1}^{N} \frac{1}{N} \sum_{j=1, j \neq i}^{N} \langle |K_{ij}(x_i - x_j)|\varphi^2 \rangle \leq \sum_{i=1}^{N} \frac{N-1}{N} \left[\kappa \langle |\nabla_{x_i} \varphi|^2 \rangle^{\frac{1}{2}} \langle \varphi^2 \rangle^{\frac{1}{2}} + c_{\kappa} \langle \varphi^2 \rangle \right] \\
\leq \frac{N-1}{N} \sqrt{N} \kappa \langle |\nabla \varphi|^2 \rangle^{\frac{1}{2}} \langle \varphi^2 \rangle^{\frac{1}{2}} + (N-1) c_{\kappa} \langle \varphi^2 \rangle.$$

Applying these estimates in (7.1), we obtain (2.20).

Next, we prove (2.21). We have div $b(x) = \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1, j \neq i}^{N} (\text{div } K_{ij})(x_i - x_j)$. So,

$$(\operatorname{div} b)_{+} = \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1, j \neq i}^{N} (\operatorname{div} K_{ij})_{+} (x_{i} - x_{j}).$$

Hence, by $(\operatorname{div} K_{ij})_+^{\frac{1}{2}} \in \mathbf{F}_{\kappa_+}(\mathbb{R}^d)$ (note that this condition is linear in $(\operatorname{div} K_{ij})_+$),

$$\langle (\operatorname{div} b)_{+}, \varphi^{2} \rangle \leq \frac{N-1}{N} \kappa_{+} \langle |\nabla \varphi|^{2} \rangle + (N-1) c_{\kappa_{+}} \langle \varphi^{2} \rangle,$$

i.e. we have proved (2.21) for $(\operatorname{div} b)_+$.

Now, we prove (2.22). Recall that $\alpha \in [0,1]$. We have, using Jensen's inequality,

$$|b|^{1+\alpha} \le \sum_{i=1}^{N} |b_i(x)|^{1+\alpha} \le \sum_{i=1}^{N} \left(\frac{1}{N} \sum_{j=1, j \neq i}^{N} |K_{ij}(x_i - x_j)| \right)^{1+\alpha}$$

$$\le \sum_{i=1}^{N} \frac{(N-1)^{\alpha}}{N^{1+\alpha}} \sum_{j=1, j \neq i}^{N} |K_{ij}(x_i - x_j)|^{1+\alpha}.$$

Therefore, applying $|K_{ij}|^{\frac{1+\alpha}{2}} \in \mathbf{F}_{\sigma}(\mathbb{R}^d)$, we obtain

$$\langle |b|^{1+\alpha} \varphi^2 \rangle \le \frac{(N-1)^{1+\alpha}}{N^{1+\alpha}} \sigma \langle |\nabla \varphi|^2 \rangle + N \frac{(N-1)^{1+\alpha}}{N^{1+\alpha}} c_\sigma \langle \varphi^2 \rangle,$$

which gives us (2.22).

Remark 11. In Remark 6 we promised to prove that vector field $b^M : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ defined by (3.2) is in $\mathbf{F}_{\delta^M}(\mathbb{R}^{Nd})$ with $\delta^M = \mu$, $c_{\delta^M} = Nc_{\mu}$. Here is the proof:

$$|b^{M}(x)|^{2} = \sum_{i=1}^{N} |M_{i}(x_{i})|^{2},$$

where (recall that $\langle \,,\, \rangle$ is the integration over \mathbb{R}^{Nd} , \bar{x} is vector $x \in \mathbb{R}^{Nd}$ with component $x_i \in \mathbb{R}^d$)

$$\langle |M_i(x_i)|\varphi^2\rangle = \int_{\mathbb{R}^{(N-1)d}} \int_{\mathbb{R}^d} |M_i(x_i)|\varphi^2(x_i,\bar{x})dx_i d\bar{x}$$

(we use $M_i \in \mathbf{F}_{\mu}(\mathbb{R}^d)$ in x_i variable)

$$\leq \mu \int_{\mathbb{R}^{(N-1)d}} \int_{\mathbb{R}^d} |\nabla_{x_i} \varphi(x_i, \bar{x})|^2 dx_i d\bar{x} + c_\mu \int_{\mathbb{R}^{Nd}} \varphi^2 dx = \mu \langle |\nabla_{x_i} \varphi|^2 \rangle + c_\mu \langle \varphi^2 \rangle.$$

So,

$$\langle |b^{M}(x)|^{2} \varphi^{2} \rangle = \sum_{i=1}^{N} \langle |M_{i}(x_{i})|^{2} \varphi^{2} \rangle \leq \mu \langle |\nabla \varphi|^{2} \rangle + N c_{\mu} \langle \varphi^{2} \rangle,$$

as claimed.

8. Proof of Theorem 5

With the exception of our proof of Proposition 1 which is, modulo its homogeneous L^2 version in [39], is new, we follow closely De Giorgi's method as it is presented in [17, Ch. 7] with appropriate modifications to account for our somewhat different definition of L^p De Giorgi's classes (\equiv functions satisfying the inequality in Proposition 1, see discussion in Section 2.3). If we were to take p=2 (obviously, at the cost of imposing very sub-optimal constrains on the form-bounds), then, once Proposition 1 is established, we could simply refer the reader to [17, Ch. 7].

If b satisfies $(\mathbb{A}_1) \Rightarrow$ fix throughout this proof $p > \frac{2}{2-\sqrt{\delta}}, p \geq 2$.

If b satisfies $(\bar{\mathbb{A}}_2) \Rightarrow \text{fix } p > \frac{2}{4-\delta_+}, p \geq 2$.

Fix some $R_0 \leq 1$. Recall that u is a classical solution of equation (5.1) in \mathbb{R}^d , i.e.

$$(\lambda - \Delta + b \cdot \nabla)u = f,$$

where $f \in C_c^{\infty}$, $\lambda \geq 0$.

Proposition 1 (Caccioppoli's inequality No1). Let $v := (u - k)_+, k \in \mathbb{R}$. For all $0 < r < R \le R_0$,

$$\|(\nabla v^{\frac{p}{2}})\mathbf{1}_{B_r}\|_2^2 \le \frac{K_1}{(R-r)^2} \|v^{\frac{p}{2}}\mathbf{1}_{B_R}\|_2^2 + K_2 \||f - \lambda u|^{\frac{p}{2}}\mathbf{1}_{u>k}\mathbf{1}_{B_R}\|_2^2$$

for generic constants K_1 , K_2 (in particular, independent of k or r, R).

Proof. Let us first carry out the proof in the more difficult case when b satisfies condition $(\bar{\mathbb{A}}_2)$. We will attend to the case when b satisfies (\mathbb{A}_1) in Remark 12.

We fix a family of [0,1]-valued smooth cut-off functions $\{\eta = \eta_{r_1,r_2}\}_{0 < r_1 < r_2 < R}$ on \mathbb{R}^d satisfying

$$\eta = \begin{cases} 1 & \text{in } B_{r_1}, \\ 0 & \text{in } \mathbb{R}^d - \bar{B}_{r_2}, \end{cases}$$

and

$$\frac{|\nabla \eta|^2}{\eta} \le \frac{c}{(r_2 - r_1)^2} \mathbf{1}_{B_{r_2}},\tag{8.1}$$

$$\sqrt{|\nabla \eta|} \le \frac{c}{\sqrt{r_2 - r_1}} \mathbf{1}_{B_{r_2}},\tag{8.2}$$

$$|\nabla \sqrt{|\nabla \eta|}| \le \frac{c}{(r_2 - r_1)^{\frac{3}{2}}} \mathbf{1}_{B_{r_2}} \tag{8.3}$$

for some constant c. For instance, one can take, for $r_1 \leq |y| \leq r_2$,

$$\eta(y) := 1 - \int_{1}^{1 + \frac{|y| - r_1}{r_2 - r_1}} \varphi(s) ds, \quad \text{where } \varphi(s) := Ce^{-\frac{1}{\frac{1}{4} - (s - \frac{3}{2})^2}}, \quad \text{sprt } \varphi = [1, 2],$$

with constant C adjusted to $\int_1^2 \varphi(s)ds = 1$.

We put equation (5.1) in the form

$$(-\Delta + b \cdot \nabla)(u - k) = f - \lambda u$$

(keeping in mind that even if $\lambda > 0$ solution u of (5.1) satisfies a priori estimate $||u||_{\infty} \leq \lambda^{-1}||f||_{\infty}$, so the $||\cdot||_{\infty}$ norm of the right-hand side of the previous identity is bounded by $2||f||_{\infty}$), multiply it by $v^{p-1}\eta$ and integrate, obtaining

$$\begin{split} \frac{4(p-1)}{p^2} \langle \nabla v^{\frac{p}{2}}, (\nabla v^{\frac{p}{2}}) \eta \rangle + \frac{2}{p} \langle \nabla v^{\frac{p}{2}}, v^{\frac{p}{2}} \nabla \eta \rangle \\ + \frac{2}{n} \langle b \cdot \nabla v^{\frac{p}{2}}, v^{\frac{p}{2}} \eta \rangle \leq \langle |f - \lambda u|, v^{p-1} \eta \rangle. \end{split}$$

Then, applying quadratic inequality (fix some $\epsilon > 0$), we have

$$\left(\frac{4(p-1)}{p} - \frac{4}{p}\epsilon\right) \langle |\nabla v^{\frac{p}{2}}|^2 \eta \rangle \le \frac{p}{4\epsilon} \langle v^p \frac{|\nabla \eta|^2}{\eta} \rangle - 2\langle b \cdot \nabla v^{\frac{p}{2}}, v^{\frac{p}{2}} \eta \rangle + p\langle |f - \lambda u|, v^{p-1} \eta \rangle \tag{8.4}$$

(we are integrating by parts in the second term)

$$\leq \frac{p}{4\epsilon} \left\langle v^p \frac{|\nabla \eta|^2}{\eta} \right\rangle + \left\langle bv^{\frac{p}{2}}, v^{\frac{p}{2}} \nabla \eta \right\rangle + \left\langle \operatorname{div} b, v^p \eta \right\rangle + p \left\langle |f - \lambda u|, v^{p-1} \eta \right\rangle$$

=: $I_1 + I_2 + I_3 + I_4$.

By (8.1),

$$I_1 \le \frac{cp}{4\epsilon(r_2 - r_1)^2} \|v^{\frac{p}{2}} \mathbf{1}_{B_{r_2}}\|_2^2$$

By
$$(\bar{\mathbb{A}}_2)$$
,

$$\begin{split} I_{2} &\leq \langle |b|v^{\frac{p}{2}}, v^{\frac{p}{2}}|\nabla \eta| \rangle \\ &\leq \delta \|\nabla(v^{\frac{p}{2}}\sqrt{|\nabla \eta|})\|_{2} \|v^{\frac{p}{2}}\sqrt{|\nabla \eta|}\|_{2} + c_{\delta} \|v^{\frac{p}{2}}\sqrt{|\nabla \eta|}\|_{2}^{2} \\ &\leq \delta \bigg(\|(\nabla v^{\frac{p}{2}})\sqrt{|\nabla \eta|}\|_{2} + \|v^{\frac{p}{2}}\nabla\sqrt{|\nabla \eta|}\|_{2} \bigg) \|v^{\frac{p}{2}}\sqrt{|\nabla \eta|}\|_{2} + c_{\delta} \|v^{\frac{p}{2}}\sqrt{|\nabla \eta|}\|_{2}^{2}. \end{split}$$

Hence, using (8.2), (8.3), we obtain

$$I_{2} \leq \delta c \left(\frac{1}{\sqrt{r_{2} - r_{1}}} \| (\nabla v^{\frac{p}{2}}) \mathbf{1}_{B_{r_{2}}} \|_{2} + \frac{1}{(r_{2} - r_{1})^{\frac{3}{2}}} \| v^{\frac{p}{2}} \mathbf{1}_{B_{r_{2}}} \|_{2} \right) \frac{1}{\sqrt{r_{2} - r_{1}}} \| v^{\frac{p}{2}} \mathbf{1}_{B_{r_{2}}} \|_{2} + \frac{c_{\delta} c}{r_{2} - r_{1}} \| v^{\frac{p}{2}} \mathbf{1}_{B_{r_{2}}} \|_{2}^{2}.$$

Thus, since $r_2 - r_1 < 1$,

$$I_{2} \leq \frac{C_{1}}{r_{2} - r_{1}} \| (\nabla v^{\frac{p}{2}}) \mathbf{1}_{B_{r_{2}}} \|_{2} \| v^{\frac{p}{2}} \mathbf{1}_{B_{r_{2}}} \|_{2} + C_{1} \left(1 + \frac{1}{(r_{2} - r_{1})^{2}} \right) \| v^{\frac{p}{2}} \mathbf{1}_{B_{r_{2}}} \|_{2}^{2}$$

for appropriate constant C_1 .

Next, by (\mathbb{A}_2) ,

$$I_{3} \leq \langle \operatorname{div} b_{+}, v^{p} \eta \rangle \leq \delta_{+} \|\nabla(v^{\frac{p}{2}} \sqrt{\eta})\|_{2}^{2} + c_{\delta_{+}} \|v^{\frac{p}{2}} \sqrt{\eta}\|_{2}^{2}$$

$$= \delta_{+} \|(\nabla v^{\frac{p}{2}}) \sqrt{\eta} + v^{\frac{p}{2}} \frac{\nabla \eta}{\sqrt{\eta}}\|_{2}^{2} + c_{\delta_{+}} \|v^{\frac{p}{2}} \sqrt{\eta}\|_{2}^{2}$$

$$\leq \delta_{+} \left((1 + \epsilon_{1}) \|(\nabla v^{\frac{p}{2}}) \sqrt{\eta}\|_{2}^{2} + \left(1 + \frac{1}{\epsilon_{1}} \right) \|v^{\frac{p}{2}} \frac{\nabla \eta}{\sqrt{\eta}}\|_{2}^{2} \right) + c_{\delta_{+}} \|v^{\frac{p}{2}} \sqrt{\eta}\|_{2}^{2} \qquad (\epsilon_{1} > 0)$$

$$\leq \delta_{+} (1 + \epsilon_{1}) \|(\nabla v^{\frac{p}{2}}) \sqrt{\eta}\|_{2}^{2} + \frac{c_{1}}{(r_{2} - r_{1})^{2}} \|v^{\frac{p}{2}} \mathbf{1}_{B_{r_{2}}}\|_{2}^{2}, \qquad c_{1} := \delta_{+} \left(1 + \frac{1}{\epsilon_{1}} \right) c + c_{\delta_{+}}.$$

Finally, we estimate using Young's inequality $(p' = \frac{p}{p-1})$:

$$\frac{1}{p}I_4 \le \frac{\varepsilon_2^{p'}}{p'} \langle v^p \eta \rangle + \frac{1}{p\varepsilon_2^p} \langle |f - \lambda u|^p \mathbf{1}_{v>0} \eta \rangle \qquad (\varepsilon_2 > 0)$$

Applying the estimates on I_1 - I_4 in (8.4), we obtain

$$\||\nabla v^{\frac{p}{2}}|\mathbf{1}_{B_{r_{1}}}\|_{2}^{2} \leq \frac{C_{1}}{r_{2} - r_{1}} \|(\nabla v^{\frac{p}{2}})\mathbf{1}_{B_{r_{2}}}\|_{2} \|v^{\frac{p}{2}}\mathbf{1}_{B_{R}}\|_{2} + C_{2} \left(1 + \frac{1}{(r_{2} - r_{1})^{2}}\right) \|v^{\frac{p}{2}}\mathbf{1}_{B_{R}}\|_{2}^{2} + C_{3} \||f - \lambda u|^{\frac{p}{2}}\mathbf{1}_{v>0}\mathbf{1}_{B_{R}}\|_{2}^{2}.$$
(8.5)

Divide (8.5) by $||v^{\frac{p}{2}}\mathbf{1}_{B_P}||_2^2$:

$$\frac{\|(\nabla v^{\frac{p}{2}})\mathbf{1}_{B_{r_1}}\|_2^2}{\|v^{\frac{p}{2}}\mathbf{1}_{B_R}\|_2^2} \le \frac{C_1}{r_2 - r_1} \frac{\|(\nabla v^{\frac{p}{2}})\mathbf{1}_{B_{r_2}}\|_2}{\|v^{\frac{p}{2}}\mathbf{1}_{B_R}\|_2} + C_2\left(1 + \frac{1}{(r_2 - r_1)^2}\right) + C_3S^2,\tag{8.6}$$

where

$$S^2 := \frac{\||f - \lambda u|^{\frac{p}{2}} \mathbf{1}_{v>0} \mathbf{1}_{B_R}\|_2^2}{\|v^{\frac{p}{2}} \mathbf{1}_{v>0} \mathbf{1}_{B_R}\|_2^2}.$$

Inequality (8.6) is the pre-Caccioppoli inequality that we will now iterate.

Put

$$a_n^2 := \frac{\|(\nabla v^{\frac{p}{2}}) \mathbf{1}_{B_{R - \frac{R - r}{2^{n - 1}}}}\|_2^2}{\|v^{\frac{p}{2}} \mathbf{1}_{v > 0} \mathbf{1}_{B_R}\|_2^2},$$

the inequality (8.6) yields

$$a_n^2 \le C(R-r)^{-1}2^n a_{n+1} + C^2(R-r)^{-2}2^{2n} + C^2S^2$$

with constant C independent of n. We multiply this inequality by $(R-r)^2$ and divide by $C^2 2^{2n}$. Then, setting $y_n := \frac{(R-r)a_n}{C^2}$, we obtain

$$y_n^2 \le y_{n+1} + 1 + (R - r)^2 S^2 \tag{8.7}$$

for all $n=1,2,\ldots$ A priori, all a_n 's are bounded by a non-generic constant $\|(\nabla v^{\frac{p}{2}})\mathbf{1}_{B_R}\|_2/\|v^{\frac{p}{2}}\mathbf{1}_{B_R}\|_2 < \infty$, so $\sup_n y_n < \infty$. Therefore, we can iterate (8.7) and thus estimate all y_n , $n=1,2,\ldots$, via nested square roots $1+(R-r)^2S^2+\sqrt{1+(R-r)^2S^2+\sqrt{\ldots}}$, obtaining

$$y_n^2 \le 3 + 2(R - r)^2 S^2$$
, $n = 1, 2, \dots$

Taking n = 1, we arrive at $a_1 \leq K_1(R - r)^{-2} + K_2S^2$ for appropriate constants K_1 and K_2 , i.e.

$$\frac{\|(\nabla v^{\frac{p}{2}})\mathbf{1}_{B_r}\|_2^2}{\|v^{\frac{p}{2}}\mathbf{1}_{B_R}\|_2^2} \le K_1(R-r)^{-2} + K_2 \frac{\||f|^{\frac{p}{2}}\mathbf{1}_{v>0}\mathbf{1}_{B_R}\|_2^2}{\|v^{\frac{p}{2}}\mathbf{1}_{B_R}\|_2^2},$$

as claimed.

Remark 12. If b satisfies condition (A₁), then we can work with somewhat simpler cutoff functions $\eta \in C_c^{\infty}$, $\eta = 1$ in B_{r_1} , $\eta = 0$ in $\mathbb{R}^d \setminus B_{r_2}$, i.e. $|\nabla \eta| \le c_1(r_2 - r_1)^{-1}$, $|\Delta \eta| \le c_2(r_2 - r_1)^{-2}$, and we do not need to integrate by parts in order to estimate the second term in the RHS of (8.4). Instead, we only need to apply quadratic inequality:

$$2|\langle b \cdot \nabla v^{\frac{p}{2}}, v^{\frac{p}{2}} \eta \rangle| \le \alpha \langle |\nabla v|^{\frac{p}{2}} \eta \rangle + \frac{1}{4\alpha} \langle |b|^2, v^p \eta \rangle, \quad \alpha = \frac{2}{\sqrt{\delta}}.$$

The proof of Proposition 1 is completed.

Lemma 7 ([17, Lemma 7.1]). If $\{z_i\}_{i=0}^{\infty} \subset \mathbb{R}_+$ is a sequence of positive real numbers such that

$$z_{i+1} \le NC_0^i z_i^{1+\alpha}$$

for some $C_0 > 1$, $\alpha > 0$, and

$$z_0 \le N^{-\frac{1}{\alpha}} C_0^{-\frac{1}{\alpha^2}}$$

Then $\lim_{i} z_i = 0$.

Lemma 8 ([17, Lemma 7.3]). Let $\varphi(t)$ be a positive function, and assume that there exists a constant q and a number $0 < \tau < 1$ such that for every $0 < R < R_0$

$$\varphi(\tau R) \le \tau^{\delta} \varphi(R) + BR^{\beta}$$

with $0 < \beta < \delta$, and

$$\varphi(t) \le q\varphi(\tau^k R)$$

for every t in the interval $(\tau^{k+1}R, \tau^k R)$. Then, for every $0 < \rho < R < R_0$, we have

$$\varphi(\rho) \leq C \bigg(\bigg(\frac{\rho}{R} \bigg)^{\beta} \varphi(R) + B \rho^{\beta} \bigg)$$

with constant C that depends only on q, τ , δ and β .

Proposition 2. For all $0 < r < R \le R_0$,

$$\sup_{B_{\frac{R}{2}}} u \le C_1 \left(\frac{1}{|B_R|} \langle u^p \mathbf{1}_{B_R \cap \{u > 0\}} \rangle \right)^{\frac{1}{p}} \left(\frac{|B_R \cap \{u > 0\}|}{|B_R|} \right)^{\frac{\alpha}{p}} + C_2 R^{\frac{2}{p}}$$

for generic constants C_1 , C_2 that also depend on $||f - \lambda u||_{\infty}$ ($\leq 2||f||_{\infty}$), where $\alpha > 0$ is fixed by $\alpha(\alpha + 1) = \frac{2}{d}$.

Proof. Without loss of generality, $R_0 = 1$. Let $\frac{1}{2} < r < \rho \le 1$. Fix $\eta \in C_c^{\infty}$, $\eta = 1$ on B_r , $\eta = 0$ on $\mathbb{R}^d \setminus \bar{B}_{\frac{r+\rho}{2}}$, $|\nabla \eta| \le \frac{4}{\rho-r}$. Set $\zeta := \eta v = \eta(u-k)_+$, $k \in \mathbb{R}$. Using Hölder's inequality and Sobolev's embedding theorem, we obtain

$$\begin{aligned} \|v^{\frac{p}{2}}\mathbf{1}_{B_r}\|_2^2 &\leq \|\zeta^{\frac{p}{2}}\mathbf{1}_{B_r}\|_2^2 \leq \langle \mathbf{1}_{B_r \cap \{u > k\}} \rangle^{\frac{2}{d}} \langle \zeta^{\frac{pd}{d-2}}\mathbf{1}_{B_{\frac{r+\rho}{2}}} \rangle^{\frac{d-2}{d}} \\ &\leq c_1 |B_r \cap \{u > k\}|^{\frac{2}{d}} \langle |\nabla \zeta^{\frac{p}{2}}|^2 \mathbf{1}_{B_{\frac{r+\rho}{2}}} \rangle \\ &= c_1 |B_r \cap \{u > k\}|^{\frac{2}{d}} \langle |(\nabla \eta^{\frac{p}{2}}) v^{\frac{p}{2}} + \eta^{\frac{p}{2}} \nabla v^{\frac{p}{2}}|^2 \mathbf{1}_{B_{\frac{r+\rho}{2}}} \rangle \end{aligned}$$

Hence

$$\|v^{\frac{p}{2}}\mathbf{1}_{B_r}\|_2^2 \le c_2|B_r \cap \{u > k\}|^{\frac{2}{d}} \left(\frac{1}{(\rho - r)^2} \|v^{\frac{p}{2}}\mathbf{1}_{B_{\frac{r+\rho}{2}}}\|_2^2 + \|(\nabla v^{\frac{p}{2}})\mathbf{1}_{B_{\frac{r+\rho}{2}}}\|_2^2\right).$$

On the other hand, Proposition 1 yields:

$$\|(\nabla v^{\frac{p}{2}})\mathbf{1}_{B_{\frac{r+\rho}{2}}}\|_{2}^{2} \leq \frac{K_{1}}{(\rho-r)^{2}}\|v^{\frac{p}{2}}\mathbf{1}_{B_{\rho}}\|_{2}^{2} + K_{2}\|f-\lambda u\|_{\infty}^{p}\left|B_{\rho}\cap\{u>k\}\right|,\tag{8.8}$$

SO

$$\|v^{\frac{p}{2}}\mathbf{1}_{B_{r}}\|_{2}^{2} \leq C|B_{r} \cap \{u > k\}|^{\frac{2}{d}} \left(\frac{1}{(\rho - r)^{2}} \|v^{\frac{p}{2}}\mathbf{1}_{B_{\rho}}\|_{2}^{2} + \|f - \lambda u\|_{\infty}^{p} \left|B_{\rho} \cap \{u > k\}\right|\right)$$

$$\leq \frac{C|B_{\rho} \cap \{u > k\}|^{\frac{2}{d}}}{(\rho - r)^{2}} \|v^{\frac{p}{2}}\mathbf{1}_{B_{\rho}}\|_{2}^{2} + C\|f - \lambda u\|_{\infty}^{p} |B_{\rho} \cap \{u > k\}|^{1 + \frac{2}{d}}.$$
(8.9)

Now, returning from notation v to $(u-k)_+$, we note that if h < k, then $\|(u-k)^{\frac{p}{2}} \mathbf{1}_{B_{\rho} \cap \{u>k\}}\|_2 \le \|(u-h)^{\frac{p}{2}} \mathbf{1}_{B_{\rho} \cap \{u>h\}}\|_2$ and $\|(u-h)^{\frac{p}{2}} \mathbf{1}_{B_{\rho} \cap \{u>h\}}\|_2^2 \ge (k-h)^p |B_{\rho} \cap \{u>k\}|$. Therefore, we obtain from (8.9)

$$\|(u-k)_{+}^{\frac{p}{2}}\mathbf{1}_{B_{r}}\|_{2}^{2} \leq \frac{C}{(\rho-r)^{2}}\|(u-h)_{+}^{\frac{p}{2}}\mathbf{1}_{B_{\rho}}\|_{2}^{2}|B_{\rho}\cap\{u>h\}|^{\frac{2}{d}} + \frac{C\|f-\lambda u\|_{\infty}^{p}}{(k-h)^{p}}\|(u-h)_{+}^{\frac{p}{2}}\mathbf{1}_{B_{\rho}}\|_{2}^{2}|B_{\rho}\cap\{u>h\}|^{\frac{2}{d}}.$$

Multiplying this inequality by $|B_r \cap \{u > k\}|^{\alpha}$ $\left(\leq \frac{1}{(k-h)^{p\alpha}} \|(u-h)_+^{\frac{p}{2}} \mathbf{1}_{B_\rho}\|_2^{2\alpha} \right)$ and using $\alpha^2 + \alpha = \frac{2}{d}$, we obtain

$$\begin{aligned} &\|(u-k)_{+}^{\frac{p}{2}}\mathbf{1}_{B_{r}}\|_{2}^{2}|B_{r}\cap\{u>k\}|^{\alpha} \\ &\leq C\left[\frac{1}{(\rho-r)^{2}}+\frac{\|f-\lambda u\|_{\infty}^{p}}{(k-h)^{p}}\right]\frac{1}{(k-h)^{p\alpha}}\left(\|(u-h)_{+}^{\frac{p}{2}}\mathbf{1}_{B_{\rho}}\|_{2}^{2}|B_{\rho}\cap\{u>h\}|^{\alpha}\right)^{1+\alpha}. \end{aligned}$$

Now, take $r:=r_{i+1}$, $\rho:=r_i$, where $r_i:=\frac{R}{2}(1+\frac{1}{2^i})$ and $k:=k_{i+1}$, $h:=k_i$, where $k_i:=\xi(1-2^{-i})$, with constant $\xi\geq R^{\frac{2}{p}}$ to be chosen later. Then, setting

$$z_i = z(k_i, r_i) := \|(u - k_i)_+^{\frac{p}{2}} \mathbf{1}_{B_{r_i}} \|_2^2 |B_{r_i} \cap \{u > k_i\}|^{\alpha},$$

we have

$$z_{i+1} \leq K \left[2^{2i} + \frac{2^{pi}R^2}{\xi^p} \right] \frac{1}{R^2} \frac{2^{pi\alpha}}{\xi^{p\alpha}} z_i^{1+\alpha}$$

hence (using $\xi \ge R^{\frac{2}{p}}$)

$$z_{i+1} \le 2^{p(1+\alpha)i} \frac{2K}{R^2} \frac{1}{\xi^{p\alpha}} z_i^{1+\alpha}.$$

We apply Lemma 7. In the notation of this lemma, $C_0 = 2^{p(1+\alpha)}$ and $N = \frac{2K}{R^2} \frac{1}{\xi^{p\alpha}}$. We need

$$z_0 \le N^{-\frac{1}{\alpha}} C_0^{-\frac{1}{\alpha^2}}$$

where, recall, $z_0 = \langle u^p \mathbf{1}_{B_R \cap \{u > 0\}} \rangle |B_R \cap \{u > 0\}|^{\alpha}$. The latter amounts to requiring

$$\xi \ge C_1 R^{-\frac{2}{p\alpha}} z_0^{\frac{1}{p}}$$

Take $\xi := R^{\frac{2}{p}} + C_1 R^{-\frac{2}{p\alpha}} z_0^{\frac{1}{p}}$. By Lemma 7, $z(\xi, \frac{R}{2}) = 0$, i.e. $\sup_{\underline{R}} u \leq \xi$. The claimed inequality follows.

Set

$$\operatorname{osc}(u, R) := \sup_{u, u' \in B_R} |u(y) - u(y')|.$$

Proposition 3. Fix k_0 by

$$2k_0 = M(2R) + m(2R) := \sup_{B_{2R}} u + \inf_{B_{2R}} u.$$

Assume that $|B_R \cap \{u > k_0\}| \le \gamma |B_R|$ for some $\gamma < 1$. If

$$\operatorname{osc}(u, 2R) \ge 2^{n+1} C R^{\frac{2}{p}},$$
(8.10)

then, for $k_n := M(2R) - 2^{-n-1}\operatorname{osc}(u, 2R)$,

$$|B_R \cap \{u > k_n\}| \le cn^{-\frac{d}{2(d-1)}}|B_R|.$$

Proof. 1. For $h \in]k_0, k[$, set $w := (u - h)^{\frac{p}{2}}$ if h < u < k, set $w := (k - h)^{\frac{p}{2}}$ if $u \ge k$, and w := 0 if $u \le h$. Note that w = 0 in $B_R \setminus (B_R \cap \{u > k_0\})$. The measure of this set is greater than $\gamma |B_R|$, so the Sobolev embedding theorem applies and yields

$$(k-h)^{\frac{p}{2}}|B_R \cap \{u > k\}|^{\frac{d-1}{d}} \le c_1 \langle w^{\frac{d}{d-1}} \mathbf{1}_{B_R} \rangle^{\frac{d-1}{d}} \le c_2 \langle |\nabla w| \mathbf{1}_{\Delta} \rangle$$

$$\le c_2 |\Delta|^{\frac{1}{2}} \langle |\nabla (u-h)^{\frac{p}{2}}|^2 \mathbf{1}_{B_R \cap \{u > h\}} \rangle^{\frac{1}{2}},$$

where

$$\Delta := B_R \cap \{u > h\} \setminus (B_R \cap \{u > k\}).$$

Now, it follows from Proposition 1 that

$$\langle |\nabla (u-h)^{\frac{p}{2}}|^2 \mathbf{1}_{B_R \cap \{u>h\}} \rangle \leq \frac{C_3}{R^2} \langle (u-h)^p \mathbf{1}_{B_{2R} \cap \{u>h\}} \rangle + C_4 |B_{2R} \cap \{u>h\}|$$

$$\leq C_3 R^{d-2} (M(2R) - h)^p + C_5 R^d.$$

For $h \leq k_n$ we have $M(2R) - h \geq M(2R) - k_n \geq CR^{\frac{2}{p}}$, where we have used (8.10). Therefore, summarizing what was written above, we have

$$(k-h)^{\frac{p}{2}}|B_R \cap \{u > k\}|^{\frac{d-1}{d}} \le c|\Delta|^{\frac{1}{2}}R^{\frac{d-2}{2}}(M(2R) - h)^{\frac{p}{2}}.$$

2. Select $k = k_i := M(2R) - 2^{-i-1} \operatorname{osc}(u, 2R), h = k_{i-1}$. Then

$$M(2R) - h = 2^{-i}\operatorname{osc}(u, 2R), \quad |k - h| = 2^{-i-1}\operatorname{osc}(u, 2R),$$

SO

$$|B_R \cap \{u > k_n\}|^{\frac{2(d-1)}{d}} \le |B_R \cap \{u > k_i\}|^{\frac{2(d-1)}{d}} \le C|\Delta_i|R^{d-2}$$

where $\Delta_i := B_R \cap \{u > k_i\} \setminus (B_R \cap \{u > k_{i-1}\})$. Summing up in i from 1 to n, we obtain

$$n|B_R \cap \{u > k_n\}|^{\frac{2(d-1)}{d}} \le CR^{d-2}|B_R \cap \{u > k_0\}| \le C'R^{2(d-1)},$$

and the claimed inequality follows.

Proof of Theorem 5, completed. Fix k_0 by $2k_0 = M(2R) + m(2R)$. Without loss of generality, $|B_R \cap \{u > k_0\}| \le \frac{1}{2}|B_R|$ (otherwise we replace u by -u). Set $k_n := M(2R) - 2^{-n-1}\operatorname{osc}(u, 2R) > k_0$. By Proposition 2,

$$\sup_{B_{\frac{R}{2}}} (u - k_n) \le C_1 \left(\frac{1}{|B_R|} \langle (u - k_n)^p \mathbf{1}_{B_R \cap \{u > k_n\}} \rangle \right)^{\frac{1}{p}} \left(\frac{|B_R \cap \{u > k_n\}|}{|B_R|} \right)^{\frac{\alpha}{p}} + C_2 R^{\frac{2}{p}}$$

$$\le C_1 \sup_{B_R} (u - k_n) \left(\frac{|B_R \cap \{u > k_n\}|}{|B_R|} \right)^{\frac{1+\alpha}{p}} + C_2 R^{\frac{2}{p}}$$

We now apply Proposition 3 (with, say, C = 1). Fix n by

$$cn^{-\frac{d}{2(d-1)}} \le \left(\frac{1}{2C_1}\right)^{\frac{p}{1+\alpha}}.$$

Then, if $\operatorname{osc}(u, 2R) \geq 2^{n+1} R^{\frac{2}{p}}$, we obtain from Proposition 3

$$M\left(\frac{R}{2}\right) - k_n \le \frac{1}{2}(M(2R) - k_n) + C_2 R^{\frac{2}{p}},$$

so,

$$M\left(\frac{R}{2}\right) \le M(2R) - \frac{1}{2^{n+1}}\operatorname{osc}(u, 2R) + \frac{1}{2}\frac{1}{2^{n+1}}\operatorname{osc}(u, 2R) + C_2R^{\frac{2}{p}},$$

$$M\left(\frac{R}{2}\right) - m\left(\frac{R}{2}\right) \le M(2R) - m(2R) - \frac{1}{2}\frac{1}{2^{n+1}}\operatorname{osc}(u, 2R) + C_2R^{\frac{2}{p}}.$$

Hence, since $\operatorname{osc}(u, 2R) = M(2R) - m(2R)$,

$$\operatorname{osc}\left(u, \frac{R}{2}\right) \le \left(1 - \frac{1}{2^{n+2}}\right) \operatorname{osc}\left(u, 2R\right) + C_2 R^{\frac{2}{p}}.$$
 (8.11)

If $\operatorname{osc}(u, 2R) \leq 2^{n+1} R^{\frac{2}{p}}$, then, clearly,

osc
$$\left(u, \frac{R}{2}\right) \le \left(1 - \frac{1}{2^{n+2}}\right) \operatorname{osc}\left(u, 2R\right) + \frac{1}{2}R^{\frac{2}{p}}.$$
 (8.12)

This yields the sought Hölder continuity of u via Lemma 8 with $\tau = \frac{1}{4}$, $\delta = \log_{\tau}(1 - 2^{-n-2})$ and $0 < \beta < \frac{2}{p} \wedge \delta$. (Note that the second inequality in the conditions of Lemma 8 holds if q = 1 and φ is non-decreasing, which is our case.)

9. Proof of Theorem 6

If b satisfies (5.4), then we fix throughout the proof $p > \frac{2}{2-\sqrt{\delta}}$, $p \ge 2$. If b satisfies (5.5), then we fix $p > \frac{2}{4-\delta_+}$, $p \ge 2$, $p' \le 1 + \gamma$, where, recall $p' := \frac{p}{p-1}$.

We are going to modify some parts of the proof of Theorem 5. But there are some important differences:

- (1) To prove Theorem 6, we need to obtain an L^{∞} bound on solution u of nonhomogeneous equation (5.6), i.e. estimate (5.7). At the first sight, establishing an L^{∞} bound seems to be easier than what we did in Theorem 5, i.e. proved Hölder continuity of solution. However, equation (5.6) is more sophisticated than the elliptic equation in Theorem 5: its right-hand side contains a posteriori unbounded function $|\mathbf{h}|$. (To add more details: in the proof of Theorem 3 we will need to take vector field $\mathbf{h} = b_n$ and $\mathbf{h} = b_n b_m$, but, crucially, the sought L^{∞} bound on solution u should not depend on n or m. In other words, L^{∞} bound on solution u should not depend on the boundedness or smoothness of b and b.)
- (2) Since there is now a posteriori unbounded factor |h| in the right-hand side of (5.6), we will need a different Caccioppoli's inequality, i.e. the one in Proposition 4.
- (3) Although the sought L^{∞} bound on solution u of (5.6) is a global bound on \mathbb{R}^d , we will still have to argue locally, i.e. work with cutoff functions. There are important reasons for this, see explanation Section 1.1.

Proposition 4 (Caccioppoli's inequality No.2). Fix $R_0 \le 1$. For all $0 < r < R \le R_0$ and every $k \ge 0$, the positive part $v := (u - k)_+$ of u - k satisfies

$$\begin{split} \lambda \|v^{\frac{p}{2}}\mathbf{1}_{B_r}\|_2^2 + \|(\nabla v^{\frac{p}{2}})\mathbf{1}_{B_r}\|_2^2 &\leq \frac{K_1}{(R-r)^2} \|v^{\frac{p}{2}}\mathbf{1}_{B_R}\|_2^2 \\ &+ K_2 \|(\mathbf{1}_{|\mathsf{h}|>1} + |\mathsf{h}|^{\frac{p}{2}}\mathbf{1}_{|\mathsf{h}|<1})|f|^{\frac{p}{2}}\mathbf{1}_{u>k}\mathbf{1}_{B_R}\|_2^2 \end{split}$$

for generic constants K_1 , K_2 (so, independent of r, R and k).

Remark 13. Comparing the Caccioppoli inequality of Proposition 4 with the one in Proposition 1, one notices that we kept bounded function $\mathbf{1}_{|\mathsf{h}|>1} + |\mathsf{h}|^{\frac{p}{2}} \mathbf{1}_{|\mathsf{h}|\leq 1}$ in the right-hand side. We will use this function as follows. In the proof of Theorem 3 we will consider, in particular, $\mathsf{h} = b_n - b_m$. We will need to show that the corresponding solution u goes to zero locally uniformly on \mathbb{R}^d as $n, m \to \infty$, which will be possible to do using (5.7) precisely because we kept $\mathbf{1}_{|\mathsf{h}|>1} + |\mathsf{h}|^{\frac{p}{2}} \mathbf{1}_{|\mathsf{h}|<1}$.

Proof. We extend the proof of Proposition 1 to the setting of Proposition 4. Once again, first we carry out the proof in the more difficult case when b and h satisfy condition (5.5), and then attend to the case when b and h satisfy (5.4) in Remark 14.

Let $\{\eta = \eta_{r_1,r_2}\}_{0 < r_1 < r_2 < R}$ be a family of [0, 1]-valued smooth cut-off functions satisfying (8.1)-(8.3).

From equation (5.6) we obtain, since both λ and k are non-negative,

$$(\lambda - \Delta + b \cdot \nabla)(u - k) < |\mathbf{h}f|.$$

After multiplying by $v^{p-1}\eta \geq 0$ and integrating, we obtain

$$\begin{split} \lambda \langle v^p \eta \rangle + \frac{4(p-1)}{p^2} \langle \nabla v^{\frac{p}{2}}, (\nabla v^{\frac{p}{2}}) \eta \rangle + \frac{2}{p} \langle \nabla v^{\frac{p}{2}}, v^{\frac{p}{2}} \nabla \eta \rangle \\ + \frac{2}{p} \langle b \cdot \nabla v^{\frac{p}{2}}, v^{\frac{p}{2}} \eta \rangle \leq \langle |\mathsf{h} f|, v^{p-1} \eta \rangle. \end{split}$$

Then, applying quadratic inequality (fix some $\epsilon > 0$), we have

$$p\lambda\langle v^{p}\eta\rangle + \left(\frac{4(p-1)}{p} - \frac{4}{p}\epsilon\right)\langle|\nabla v^{\frac{p}{2}}|^{2}\eta\rangle \leq \frac{p}{4\epsilon}\langle v^{p}\frac{|\nabla\eta|^{2}}{\eta}\rangle - 2\langle b\cdot\nabla v^{\frac{p}{2}}, v^{\frac{p}{2}}\eta\rangle + p\langle|\mathsf{h}f|, v^{p-1}\eta\rangle \quad (9.1)$$
(we are integrating by parts)
$$\leq \frac{p}{4\epsilon}\langle v^{p}\frac{|\nabla\eta|^{2}}{\eta}\rangle + \langle bv^{\frac{p}{2}}, v^{\frac{p}{2}}\nabla\eta\rangle + \langle \operatorname{div}b, v^{p}\eta\rangle + p\langle|\mathsf{h}f|, v^{p-1}\eta\rangle =: I_{1} + I_{2} + I_{3} + I_{4}.$$

Terms I_1 - I_3 are estimated in the same way as in the proof of Proposition 1. Term I_4 is different, so we argue as follows:

$$\frac{1}{p}I_4 \leq \langle (|\mathbf{h}|\mathbf{1}_{|\mathbf{h}|>1} + |\mathbf{h}|\mathbf{1}_{|\mathbf{h}|\leq 1})|f|, v^{p-1}\eta\rangle$$

(we open the brackets and apply Young's inequality)

$$\leq \frac{\varepsilon_{2}^{p'}}{p'} \langle |\mathbf{h}|^{p'} \mathbf{1}_{|\mathbf{h}|>1} v^{p} \eta \rangle + \frac{1}{p \varepsilon_{2}^{p}} \langle \mathbf{1}_{|\mathbf{h}|>1} |f|^{p} \eta \rangle \qquad (\varepsilon_{2} > 0)$$

$$+ \frac{\varepsilon_{2}^{p'}}{p'} \langle v^{p} \eta \rangle + \frac{1}{p \varepsilon_{2}^{p}} \langle |\mathbf{h}|^{p} \mathbf{1}_{|\mathbf{h}|\leq 1} |f|^{p} \eta \rangle$$

$$\left(\text{using } 1 + \gamma \geq p', \text{ we apply } |\mathbf{h}|^{\frac{1+\gamma}{2}} \in \mathbf{F}_{\chi} \text{ in the first term,} \right)$$

$$\text{i.e. } \langle |\mathbf{h}|^{p'} \mathbf{1}_{|\mathbf{h}|>1} v^{p} \eta \rangle \leq \langle (|\mathbf{h}|^{\frac{1+\gamma}{2}})^{2} (v^{\frac{p}{2}} \sqrt{\eta})^{2} \rangle \leq \chi \langle |(\nabla v^{\frac{p}{2}})^{2} \sqrt{\eta} + v^{\frac{p}{2}} \frac{1}{2} \frac{\nabla \eta}{\sqrt{\eta}} |^{2} \rangle + c_{\chi} \langle v^{p} \eta \rangle, \text{ so:}$$

$$\leq 2 \frac{\varepsilon_{2}^{p'}}{n'} \chi \langle |\nabla v^{\frac{p}{2}}|^{2} \eta \rangle + \frac{1}{2} \frac{\varepsilon_{2}^{p'}}{n'} \chi \langle v^{p} \frac{|\nabla \eta|^{2}}{\eta} \rangle + \frac{\varepsilon_{2}^{p'}}{n'} (c_{\chi} + 1) \langle v^{p} \eta \rangle + \frac{1}{n \varepsilon_{2}^{p}} \langle \Theta |f|^{p} \mathbf{1}_{v>0} \eta \rangle,$$

where $\Theta := \mathbf{1}_{|\mathsf{h}|>1} + |\mathsf{h}|^p \mathbf{1}_{|\mathsf{h}|\leq 1}$. Selecting ε_2 sufficiently small and applying the estimates on I_1 - I_4 in (9.1), we obtain

$$\lambda \|v^{\frac{p}{2}} \mathbf{1}_{B_{r_1}}\|_{2}^{2} + \||\nabla v^{\frac{p}{2}}| \mathbf{1}_{B_{r_1}}\|_{2}^{2} \leq \frac{C_{1}}{r_{2} - r_{1}} \|(\nabla v^{\frac{p}{2}}) \mathbf{1}_{B_{r_2}}\|_{2} \|v^{\frac{p}{2}} \mathbf{1}_{B_{R}}\|_{2}$$

$$+ C_{2} \left(1 + \frac{1}{(r_{2} - r_{1})^{2}}\right) \|v^{\frac{p}{2}} \mathbf{1}_{B_{R}}\|_{2}^{2} + C_{3} \|\Theta^{\frac{1}{2}}|f|^{\frac{p}{2}} \mathbf{1}_{v>0} \mathbf{1}_{B_{R}}\|_{2}^{2},$$

so, dividing by $||v^{\frac{p}{2}}\mathbf{1}_{B_R}||_2^2$, we arrive at

$$\frac{\lambda \|v^{\frac{p}{2}} \mathbf{1}_{B_{r_1}}\|_{2}^{2} + \|(\nabla v^{\frac{p}{2}}) \mathbf{1}_{B_{r_1}}\|_{2}^{2}}{\|v^{\frac{p}{2}} \mathbf{1}_{B_{R}}\|_{2}^{2}} \leq \frac{C_{1}}{r_{2} - r_{1}} \frac{\lambda \|v^{\frac{p}{2}} \mathbf{1}_{B_{r_{2}}}\|_{2}^{2} + \|(\nabla v^{\frac{p}{2}}) \mathbf{1}_{B_{r_{2}}}\|_{2}}{\|v^{\frac{p}{2}} \mathbf{1}_{B_{R}}\|_{2}} + C_{2} \left(1 + \frac{1}{(r_{2} - r_{1})^{2}}\right) + C_{3} S^{2}, \tag{9.2}$$

where $S^2 := \frac{\|\Theta^{\frac{1}{2}}|f|^{\frac{p}{2}}\mathbf{1}_{v>0}\mathbf{1}_{B_R}\|_2^2}{\|v^{\frac{p}{2}}\mathbf{1}_{v>0}\mathbf{1}_{B_R}\|_2^2}$. This is the pre-Caccioppoli inequality that we will iterate. Put

$$a_n^2 := \frac{\lambda \|v^{\frac{p}{2}} \mathbf{1}_{B_{R-\frac{R-r}{2^{n-1}}}}\|_2^2 + \|(\nabla v^{\frac{p}{2}}) \mathbf{1}_{B_{R-\frac{R-r}{2^{n-1}}}}\|_2^2}{\|v^{\frac{p}{2}} \mathbf{1}_{v>0} \mathbf{1}_{B_R}\|_2^2},$$

so the inequality (9.2) yields

$$a_n^2 \le C(R-r)^{-1}2^n a_{n+1} + C^2(R-r)^{-2}2^{2n} + C^2S^2$$

with constant C independent of n. We multiply this inequality by $(R-r)^2$ and divide by $C^2 2^{2n}$. Setting $y_n := \frac{(R-r)a_n}{C2^n}$, we obtain

$$y_n^2 \le y_{n+1} + 1 + (R - r)^2 S^2 \tag{9.3}$$

for all $n=1,2,\ldots$ We note that all a_n 's are bounded by a non-generic constant $(\lambda \|v^{\frac{p}{2}}\mathbf{1}_B\|_2^2 + \|(\nabla v^{\frac{p}{2}})\mathbf{1}_{B_R}\|_2)/\|v^{\frac{p}{2}}\mathbf{1}_{B_R}\|_2 < \infty$, so $\sup_n y_n < \infty$. Therefore, we can iterate (9.3) and hence

estimate all y_n , n = 1, 2, ..., via nested square roots $1 + (R - r)^2 S^2 + \sqrt{1 + (R - r)^2 S^2 + \sqrt{...}}$, obtaining

$$y_n^2 \le 3 + 2(R - r)^2 S^2$$
, $n = 1, 2, \dots$

Taking n = 1, we arrive at $a_1 \leq K_1(R - r)^{-2} + K_2S^2$ for appropriate constants K_1 and K_2 , i.e.

$$\frac{\lambda \|v^{\frac{p}{2}} \mathbf{1}_{B_r}\|_2^2 + \|(\nabla v^{\frac{p}{2}}) \mathbf{1}_{B_r}\|_2^2}{\|v^{\frac{p}{2}} \mathbf{1}_{B_R}\|_2^2} \le K_1 (R - r)^{-2} + K_2 \frac{\|\Theta^{\frac{1}{2}} |f|^{\frac{p}{2}} \mathbf{1}_{v > 0} \mathbf{1}_{B_R}\|_2^2}{\|v^{\frac{p}{2}} \mathbf{1}_{B_R}\|_2^2},$$

as claimed.

Remark 14. If b and h satisfy condition (5.4), then, again, we can work with cutoff functions $\eta \in C_c^{\infty}$, $\eta = 1$ in B_{r_1} , $\eta = 0$ in $\mathbb{R}^d \setminus B_{r_2}$, i.e. $|\nabla \eta| \le c_1(r_2 - r_1)^{-1}$, $|\Delta \eta| \le c_2(r_2 - r_1)^{-2}$, and we estimate the second term in the RHS of (9.1) right away using quadratic inequality:

$$2|\langle b \cdot \nabla v^{\frac{p}{2}}, v^{\frac{p}{2}} \eta \rangle| \le \alpha \langle |\nabla v|^{\frac{p}{2}} \eta \rangle + \frac{1}{4\alpha} \langle |b|^2, v^p \eta \rangle, \quad \alpha = \frac{2}{\sqrt{\delta}}.$$

Regarding the terms containing h, we simply take $\gamma = 1$, which transforms condition $|h|^{\frac{1+\gamma}{2}} \in \mathbf{F}_{\chi}$, $\chi < \infty$ from (5.5) into condition $h \in \mathbf{F}_{\chi}$ in (5.4), and argue as in the estimate on I_4 above.

This ends the proof of Proposition 4.

Recall that we have fixed $1 < \theta < \frac{d}{d-2}$ in the statement of the theorem.

Proposition 5. There exists generic constants K and $\beta \in]0,1[$ such that, for all $\lambda \geq 1$, the positive part u_+ of solution u of non-homogeneous equation (5.6) satisfies

$$\sup_{B_{\frac{1}{2}}} u_{+} \leq K \left(\langle u_{+}^{p\theta} \mathbf{1}_{B_{1}} \rangle^{\frac{1}{p\theta}} + \lambda^{-\frac{\beta}{p}} \left\langle (\mathbf{1}_{|\mathsf{h}|>1} + |\mathsf{h}|^{p} \mathbf{1}_{|\mathsf{h}|\leq 1})^{\theta'} |f|^{p\theta'} \mathbf{1}_{B_{1}} \right\rangle^{\frac{1}{p\theta'}} \right). \tag{9.4}$$

Remark 15. In the proof of Proposition 5 we iterate simultaneously over (a) balls of decreasing radius (b) super-level sets of solution u. To get an overview of the proof, one can first formally take $r = R = \infty$ to see that the iterations over super-level sets indeed work as intended.

Proof. Proposition 4 yields

$$\begin{split} \lambda \|v^{\frac{p}{2}}\mathbf{1}_{B_r}\|_2^2 + \|v^{\frac{p}{2}}\|_{W^{1,2}(B_r)}^2 &\leq \tilde{K}_1(R-r)^{-2}\|v\|_{L^p(B_R)}^p \\ &+ K_2 \|\Theta^{\frac{1}{p}}f\mathbf{1}_{u>k}\|_{L^p(B_R)}^p, \qquad v := (u-k)_+, \ k \geq 0, \end{split}$$

where $\Theta := \mathbf{1}_{|\mathsf{h}|>1} + |\mathsf{h}|^p \mathbf{1}_{|\mathsf{h}|\leq 1}$ and \tilde{K}_1 , K_2 are generic constants. By the Sobolev embedding theorem,

$$\lambda \|v\|_{L^p(B_r)}^p + \|v\|_{L^{\frac{pd}{d-2}}(B_r)}^p \le C_1(R-r)^{-2} \|v\|_{L^p(B_R)}^p + C_2 \|\Theta^{\frac{1}{p}} f \mathbf{1}_{u>k}\|_{L^p(B_R)}^p.$$

Next, we estimate the left-hand side from below using interpolation inequality:

$$\lambda^{\beta} \|v\|_{L^{q}(B_{r})}^{p} \leq \beta \lambda \|v\|_{L^{p}(B_{r})}^{p} + (1-\beta) \|v\|_{L^{\frac{pd}{d-2}}(B_{r})}^{p}, \quad 0 < \beta < 1, \quad \frac{1}{q} = \beta \frac{1}{p} + (1-\beta) \frac{d-2}{pd}.$$

Remark 16. 1. To prove the previous inequality, we put for brevity $r = \frac{pd}{d-2}$, $\mu = \lambda^{\frac{q\beta}{p}}$ and write $\mu |v|^q = (\mu^{\frac{1}{\beta q}} v)^{p\frac{\beta q}{p}} |v|^{r\frac{(1-\beta)q}{r}}$, so, denoting here the integration over B_r by $\langle \cdot \rangle$, we have

$$\langle \mu | v |^q \rangle \le \langle (\mu^{\frac{1}{\beta q}} v)^p \rangle^{\frac{q\beta}{p}} \langle |v|^r \rangle^{\frac{(1-\beta)q}{r}}.$$

Hence

$$\mu^{\frac{p}{q}} \|v\|_q^p \le \langle (\mu^{\frac{1}{\beta q}} v)^p \rangle^{\beta} \langle |v|^r \rangle^{(1-\beta)\frac{p}{r}},$$

and it remains to apply in the right-hand side Young's inequality $ab \leq \beta a^{\frac{1}{\beta}} + (1-\beta)b^{\frac{1}{1-\beta}}$.

2. We could take $\beta = 0$, in which case

$$q = \frac{pd}{d-2}, \quad \theta_0 = \frac{d}{d-2},$$

but then we lose the dependence on λ in the second term in the right-hand side of (9.4). Strictly speaking, we do not need to keep track of the dependence on λ in this work, but this is needed for some other applications of Theorem 6, e.g. to construct strongly continuous Feller semigroup in [37]. To get an overview of this proof it is worthwhile to first ignore λ and to take $\beta = 0$ and $\theta_0 = \frac{d}{d-2}$.

Put $\theta_0 := \frac{q}{p}$, so $1 < \theta_0 < \frac{d}{d-2}$. We fix $\beta \in]0,1[$ sufficiently small so that $\theta < \theta_0$.

Hence, taking into account that $q = p\theta_0$,

$$\lambda^{\beta} \|v\|_{L^{p\theta_0}(B_r)}^p \le \tilde{C}_1 (R-r)^{-2} \|v\|_{L^p(B_R)}^p + \tilde{C}_2 \|\Theta^{\frac{1}{p}} f \mathbf{1}_{u>k}\|_{L^p(B_R)}^p.$$

Applying Hölder's inequality in the RHS, we obtain

$$\lambda^{\beta} \|v\|_{L^{p\theta_0}(B_r)}^p \le \tilde{C}_1(R-r)^{-2} |B_R|^{\frac{\theta-1}{\theta}} \|v\|_{L^{p\theta}(B_R)}^p + \tilde{C}_2 \|\Theta^{\frac{1}{p}} f \mathbf{1}_{u>k}\|_{L^{p}(B_R)}^p. \tag{9.5}$$

Set

$$R_m := \frac{1}{2} + \frac{1}{2m+1}, \quad m \ge 0,$$

so $B^m \equiv B_{R_m}$ is a decreasing sequence of balls converging to the ball of radius $\frac{1}{2}$. By (9.5),

$$\lambda^{\beta} \|v\|_{L^{p\theta_0}(B^{m+1})}^p \le \hat{C}_1 2^{2m} \|v\|_{L^{p\theta}(B^m)}^p + \tilde{C}_2 \|\Theta^{\frac{1}{p}} f \mathbf{1}_{u>k}\|_{L^{p}(B^m)}^p$$

$$\le \hat{C}_1 2^{2m} \|v\|_{L^{p\theta}(B^m)}^p + \tilde{C}_2 H |B^m \cap \{v>0\}|^{\frac{1}{\theta}},$$

$$(9.6)$$

where

$$H := \langle \Theta^{\theta'} | f |^{p\theta'} \mathbf{1}_{B^0} \rangle^{\frac{1}{\theta'}}$$
 $(B^0 = B_1, \text{ i.e. ball of radius } 1)$

On the other hand, by Hölder's inequality,

$$||v||_{L^{p\theta}(B^{m+1})}^{p\theta} \le ||v||_{L^{p\theta_0}(B^{m+1})}^{p\theta} \left(|B^m \cap \{v > 0\}|\right)^{1 - \frac{\theta}{\theta_0}}.$$

Applying (9.6) in the first multiple in the RHS, we obtain

$$||v||_{L^{p\theta}(B^{m+1})}^{p\theta} \leq \tilde{C}\lambda^{-\beta\theta} \left(2^{2\theta m} ||v||_{L^{p\theta}(B^m)}^{p\theta} + H^{\theta}|B^m \cap \{v > 0\}| \right) \left(|B^m \cap \{v > 0\}| \right)^{1 - \frac{\theta}{\theta_0}}.$$

Now, put $v_m := (u - k_m)_+$ where $k_m := \xi(1 - 2^{-m}) \uparrow \xi$, where constant $\xi > 0$ will be chosen later. Then, using $2^{2\theta m} \le 2^{p\theta m}$ and dividing by $\xi^{p\theta}$,

$$\frac{1}{\xi^{p\theta}} \|v_{m+1}\|_{L^{p\theta}(B^{m+1})}^{p\theta}$$

$$\leq \tilde{C}\lambda^{-\beta\theta} \left(\frac{2^{p\theta m}}{\xi^{p\theta}} \|v_{m+1}\|_{L^{p\theta}(B^m)}^{p\theta} + \frac{1}{\xi^{p\theta}} H^{\theta} |B^m \cap \{u > k_{m+1}\}| \right) \left(|B^m \cap \{u > k_{m+1}\}| \right)^{1-\frac{\theta}{\theta_0}}.$$

Using that $\lambda \geq 1$, we further obtain

$$\frac{1}{\xi^{p\theta}} \|v_{m+1}\|_{L^{p\theta}(B^{m+1})}^{p\theta}$$

$$\leq \tilde{C}\left(\frac{2^{p\theta m}}{\xi^{p\theta}}\|v_{m+1}\|_{L^{p\theta}(B^m)}^{p\theta} + \frac{1}{\xi^{p\theta}}\lambda^{-\beta\theta}H^{\theta}|B^m \cap \{u > k_{m+1}\}|\right)\left(|B^m \cap \{u > k_{m+1}\}|\right)^{1-\frac{\theta}{\theta_0}}.$$

From now on, we require that constant ξ satisfies $\xi^p \geq \lambda^{-\beta} H$, so

$$\frac{1}{\xi^{p\theta}} \|v_{m+1}\|_{L^{p\theta}(B^{m+1})}^{p\theta}$$

$$\leq \tilde{C} \left(\frac{2^{p\theta m}}{\xi^{p\theta}} \|v_{m+1}\|_{L^{p\theta}(B^m)}^{p\theta} + |B^m \cap \{u > k_{m+1}\}| \right) \left(|B^m \cap \{u > k_{m+1}\}| \right)^{1 - \frac{\theta}{\theta_0}}.$$
(9.7)

Now,

$$|B^{m} \cap \{u > k_{m+1}\}| = \left| B^{m} \cap \left\{ \left(\frac{u - k_{m}}{k_{m+1} - k_{m}} \right)^{2\theta} > 1 \right\} \right|$$

$$\leq (k_{m+1} - k_{m})^{-p\theta} \langle v_{m}^{p\theta} \mathbf{1}_{B^{m}} \rangle = \xi^{-p\theta} 2^{p\theta(m+1)} ||v_{m}||_{L^{p\theta}(B^{m})}^{p\theta},$$

so using in (9.7) $||v_{m+1}||_{L^{p\theta}(B^m)} \leq ||v_m||_{L^{p\theta}(B^m)}$ and applying the previous inequality, we obtain

$$\frac{1}{\xi^{p\theta}} \|v_{m+1}\|_{L^{p\theta}(B^{m+1})}^{p\theta} \le \tilde{C} 2^{p\theta m(2-\frac{\theta}{\theta_0})} \left(\frac{1}{\xi^{p\theta}} \|v_m\|_{L^{p\theta}(B^m)}^{p\theta}\right)^{2-\frac{\theta}{\theta_0}}.$$

Denote $z_m := \frac{1}{\xi^{p\theta}} ||v_m||_{L^{p\theta}(B^m)}^{p\theta}$. Then

$$z_{m+1} \le \tilde{C} \gamma^m z_m^{1+\alpha}, \quad m = 0, 1, 2, \dots, \quad \alpha := 1 - \frac{\theta}{\theta_0}, \quad \gamma := 2^{p\theta(2 - \frac{\theta}{\theta_0})}$$

and $z_0 = \frac{1}{\xi^{p\theta}} \langle u_+^{p\theta} \mathbf{1}_{B^0} \rangle \leq \tilde{C}^{-\frac{1}{\alpha}} \gamma^{-\frac{1}{\alpha^2}}$ (recall: $B^0 := B_{R_0} \equiv B_1$) provided that we fix c by

$$\xi^{p\theta} := \tilde{C}^{\frac{1}{\alpha}} \gamma^{\frac{1}{\alpha^2}} \langle u_+^{p\theta} \mathbf{1}_{B^0} \rangle + \lambda^{-\beta\theta} H^{\theta}.$$

Hence, by Lemma 7, $z_m \to 0$ as $m \to \infty$. It follows that $\sup_{B_{1/2}} u_+ \le \xi$, and the claimed inequality follows.

To end the proof of Theorem 6, we need to estimate $\langle u_+^{p\theta} \mathbf{1}_{B_1} \rangle^{1/p\theta}$ in the RHS of (9.4) in terms of h and f. We will do it by estimating $\langle u_+^{p\theta} \rho \rangle^{1/p\theta}$, where, recall,

$$\rho(x) = (1 + k|x|^2)^{-\frac{d}{2} - 1},$$

and then applying inequality $\rho \geq c \mathbf{1}_{B_1}$ for appropriate constant $c = c_d$.

Proposition 6. There exist generic constants C, k and $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$,

$$(\lambda - \lambda_0)\langle u^p \rho \rangle + \langle |\nabla u^{\frac{p}{2}}|^2 \rho \rangle \le C \langle (\mathbf{1}_{|\mathbf{h}| > 1} + |\mathbf{h}|^p \mathbf{1}_{|\mathbf{h}| \le 1}) |f|^p \rho \rangle. \tag{9.8}$$

Proof. Let b satisfy condition (5.5). We may assume without loss of generality that $p > \frac{2}{2-\delta_+}$ is rational with odd denominator. We multiply equation (5.6) by $u^{p-1}\rho$ and integrate to obtain

$$\lambda \langle u^p \rho \rangle + \frac{4(p-1)}{p^2} \langle \nabla u^{\frac{p}{2}}, (\nabla u^{\frac{p}{2}}) \rho \rangle + \frac{2}{p} \langle \nabla u^{\frac{p}{2}}, u^{\frac{p}{2}} \nabla \rho \rangle + \frac{2}{p} \langle b \cdot \nabla u^{\frac{p}{2}}, u^{\frac{p}{2}} \rho \rangle = \langle |\mathsf{h}| f, u^{p-1} \rho \rangle.$$

Now we argue as in the proof of Proposition 4, but instead of the iterations we use a straightforward estimate $|\nabla \rho| \leq (\frac{d}{2} + 1)\sqrt{k}\rho$ in order to get rid of $\nabla \rho$ in the previous identity. We arrive at

$$\begin{split} p\lambda\langle u^{p}\rho\rangle &+ \left(\frac{4(p-1)}{p} - \frac{4}{p}\varepsilon\right)\langle |\nabla u^{\frac{p}{2}}|^{2}\rho\rangle \\ &\leq \frac{p}{4\varepsilon}(\frac{d}{2}+1)^{2}k\langle v^{p}\rho\rangle + (\frac{d}{2}+1)\sqrt{k}\langle |b|u^{p}\rho\rangle + \langle\operatorname{div}b_{+},u^{p}\rho\rangle \\ &+ p\bigg(2\varepsilon'\chi\langle |\nabla u^{\frac{p}{2}}|^{2}\rho\rangle + 2\varepsilon'\chi(\frac{d}{2}+1)^{2}k\langle u^{p}\rho\rangle \\ &+ \varepsilon'(c_{\chi}+1)\langle u^{p}\rho\rangle + \frac{1}{4\varepsilon'}\langle \left(\mathbf{1}_{|\mathsf{h}|>1} + |\mathsf{h}|^{p}\mathbf{1}_{|\mathsf{h}|\leq 1}\right)|f|^{p}\rho\rangle \bigg). \end{split}$$

The terms $\langle |b|u^p\rho\rangle$, $\langle (\operatorname{div} b)_+, u^p\rho\rangle$ are estimated by applying quadratic inequality and using condition (5.5). Selecting ε , ε' , k sufficiently small, we arrive at the sought inequality.

If b satisfies (5.4), then the proof is similar but easier (i.e. we do not need to integrate by parts, only to apply quadratic inequality to $\langle b \cdot \nabla u^{\frac{p}{2}}, u^{\frac{p}{2}} \rho \rangle$ and use form-boundedness of b).

Proof of Theorem 6, completed. By Proposition 5, for all $\lambda \geq 1$,

$$\sup_{y \in B_{\frac{1}{2}}(x)} |u(y)| \leq K \bigg(\langle |u|^{p\theta} \rho_x \rangle^{\frac{1}{p\theta}} + \lambda^{-\frac{\beta}{p}} \big\langle \big(\mathbf{1}_{|\mathbf{h}|>1} + |\mathbf{h}|^{p\theta'} \mathbf{1}_{|\mathbf{h}|\leq 1} \big) |f|^{p\theta'} \rho_x \big\rangle^{\frac{1}{p\theta'}} \bigg),$$

where $\rho_x(y) := \rho(y - x)$, and constant C is generic, so

$$||u||_{\infty} \leq K \sup_{x \in \frac{1}{2}\mathbb{Z}^d} \left(\langle |u|^{p\theta} \rho_x \rangle^{\frac{1}{p\theta}} + \lambda^{-\frac{\beta}{p}} \left\langle \left(\mathbf{1}_{|\mathsf{h}| > 1} + |\mathsf{h}|^{p\theta'} \mathbf{1}_{|\mathsf{h}| \le 1} \right) |f|^{p\theta'} \rho_x \right\rangle^{\frac{1}{p\theta'}} \right).$$

Applying Proposition 6 to the first term in the RHS (with $p\theta$ instead of p), we obtain for all $\lambda \geq \lambda_0 \vee 1$

$$||u||_{\infty} \leq C \sup_{x \in \frac{1}{2}\mathbb{Z}^d} \left((\lambda - \lambda_0)^{-\frac{1}{p\theta}} \left\langle \left(\mathbf{1}_{|\mathsf{h}| > 1} + |\mathsf{h}|^{p\theta} \mathbf{1}_{|\mathsf{h}| \le 1} \right) |f|^{p\theta} \rho_x \right\rangle^{\frac{1}{p\theta}} + \lambda^{-\frac{\beta}{p}} \left\langle \left(\mathbf{1}_{|\mathsf{h}| > 1} + |\mathsf{h}|^{p\theta'} \mathbf{1}_{|\mathsf{h}| \le 1} \right) |f|^{p\theta'} \rho_x \right\rangle^{\frac{1}{p\theta'}} \right).$$

This ends the proof of Theorem 6.

10. Proof of Theorem 3

(i) By the assumption of the theorem, the Borel measurable vector field $b: \mathbb{R}^d \to \mathbb{R}^d$ satisfies either

$$b \in \mathbf{F}_{\delta} \quad \text{with } \delta < 4$$
 (A₁)

or

$$\begin{cases} b \in \mathbf{MF}_{\delta} \text{ for some } \delta < \infty, \\ (\operatorname{div} b)_{-} \in L^{1} + L^{\infty}, \\ (\operatorname{div} b)_{+}^{\frac{1}{2}} \in \mathbf{F}_{\delta_{+}} \text{ with } \delta_{+} < 4, \\ |b|^{\frac{1+\alpha}{2}} \in \mathbf{F}_{\chi} \quad \text{for some } \alpha > 0 \text{ fixed arbitrarily small, and some } \chi < \infty. \end{cases}$$

We define a regularization of b as in Section 6:

$$b_{\varepsilon} := E_{\varepsilon}b, \quad \varepsilon \downarrow 0,$$

where E_{ε} is the Friedrichs mollifier. Then, recall, $\{b_{\varepsilon}\}$ are bounded and smooth, preserve all form-bounds in (\mathbb{A}_1) or in (\mathbb{A}_2) , and converge to b in $[L^2_{\text{loc}}]^d$ or in $[L^1_{\text{loc}}]^d$, respectively.

Step 1. By the classical theory, for every $\varepsilon > 0$, there exist unique strong solution Y_{ε} to SDE

$$Y_{\varepsilon}(t) = y - \int_0^t b_{\varepsilon}(Y_{\varepsilon}(s))ds + \sqrt{2}B(t), \quad y \in \mathbb{R}^d,$$

where $\{B(t)\}_{t\geq 0}$ is a Brownian motion in \mathbb{R}^d on a fixed complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$. Fix T>0.

Lemma 9. Let vector field $g \in [C_b(\mathbb{R}^d)]^d$ be such that:

1. If b satisfies condition (\mathbb{A}_2) , then

$$\langle |\mathbf{g}|^{1+\alpha}\varphi,\varphi\rangle \leq \chi \|\nabla\varphi\|_2^2 + c_\chi \|\varphi\|_2^2, \quad \varphi \in W^{1,2}, \tag{10.1}$$

where constants χ , c_{χ} are from condition (\mathbb{A}_2) .

2. If b satisfies condition (A₁), then (10.1) holds with $\alpha = 1$, $\chi = \delta$ and $c_{\chi} = c_{\delta}$.

Fix $\gamma > 0$ by $1 + \alpha = (1 + \gamma)^2$. Then

$$\mathbf{E} \int_{t_0}^{t_1} |\mathsf{g}(Y_{\varepsilon}(s))| ds \le C_2 (t_1 - t_0)^{\frac{\gamma}{1 + \gamma}}, \tag{10.2}$$

where constant C_2 does not depend on ε , y or t_0 , t_1 (but it depends, by Theorem 6, on constants χ , c_{χ}).

(We will be applying (10.2) with $g = b_{\varepsilon}$.)

Proof of Lemma 9. First, let $g \in [C_c(\mathbb{R}^d)]^d$. By Hölder's inequality,

$$\mathbf{E} \int_{t_0}^{t_1} |\mathsf{g}(Y_{\varepsilon}(s))| ds = \mathbf{E} \int_{t_0}^{t_1} e^{\lambda t} e^{-\lambda t} |\mathsf{g}(Y_{\varepsilon}(s))| ds$$

$$\leq e^{\lambda T} (t_1 - t_0)^{\frac{\gamma}{1+\gamma}} \left(\mathbf{E} \int_0^{\infty} e^{-(1+\gamma)\lambda t} |\mathsf{g}(Y_{\varepsilon}(s))|^{1+\gamma} ds \right)^{\frac{1}{1+\gamma}}$$

$$= e^{\lambda T} (t_1 - t_0)^{\frac{\gamma}{1+\gamma}} u_{\varepsilon}(x)^{\frac{1}{1+\gamma}}$$
(10.3)

where u_{ε} is the classical solution to non-homogeneous elliptic equation

$$[(1+\gamma)\lambda - \Delta + b_{\varepsilon} \cdot \nabla]u_{\varepsilon} = |\mathbf{g}|^{1+\gamma}.$$

Note that, in view of the results of Section 6, condition (\mathbb{A}_2) implies the second condition (\mathbb{A}_2) on b of Theorem 5 for b_{ε} . (If b satisfies condition (\mathbb{A}_1), then b_{ε} satisfy the same condition in Theorem 5.) Further, we take in Theorem 6 $h := g|g|^{\gamma}$ and f = 1 in a neighbourhood of the support of g. In view of $1 + \alpha = (1 + \gamma)^2$ and (10.1), h satisfies condition $|h|^{\frac{1+\gamma}{2}} \in \mathbf{F}_{\chi}$ of Theorem 6. Thus, Theorem 6 applies and yields

$$||u_{\varepsilon}||_{\infty} \leq C \sup_{x \in \frac{1}{2}\mathbb{Z}^{d}} \left(\left\langle \left(\mathbf{1}_{|\mathbf{g}|>1} + |\mathbf{g}|^{(1+\gamma)p\theta} \mathbf{1}_{|\mathbf{g}|\leq 1}\right) \rho_{x} \right\rangle^{\frac{1}{p\theta}} + \left\langle \left(\mathbf{1}_{|\mathbf{g}|>1} + |\mathbf{g}|^{(1+\gamma)p\theta'} \mathbf{1}_{|\mathbf{g}|\leq 1}\right) \rho_{x} \right\rangle^{\frac{1}{p\theta'}} \right), \tag{10.4}$$

where the right-hand side is finite (by our choice of ρ) and clearly does not depend on ε . It is seen now that (10.2) follows from (10.3). Using Fatou's lemma, we can replace the requirement that g has compact support by $g \in [C_b(\mathbb{R}^d)]^d$.

Inequality (10.2) yields, upon taking $g := b_{\varepsilon}$,

$$\mathbf{E} \int_{t_0}^{t_1} |b_{\varepsilon}(Y_{\varepsilon}(s))| ds \le C_2 (t_1 - t_0)^{\frac{\gamma}{1 + \gamma}} \tag{10.5}$$

(note that $|b_{\varepsilon}|^{1+\gamma}$ have independent of ε finite form-bound χ and constant c_{χ} , see Lemma 6). This gives us the next lemma. We will write Y_{ε}^{y} to emphasize the dependence of solution Y_{ε} on y.

Lemma 10. (i) For every $\beta > 0$,

$$\sup_{\varepsilon>0} \sup_{y\in\mathbb{R}^d} \mathbf{P} \left[\sup_{t\in[0,1],\sigma'\in[0,\sigma]} |Y_{\varepsilon}^y(t+\sigma') - Y_{\varepsilon}^y(t)| > \beta \right] \le \hat{C}H(\sigma), \tag{10.6}$$

where constant \hat{C} and function H are independent of ε , and $H(\sigma) \downarrow 0$ as $\sigma \downarrow 0$.

(ii) For every $y \in \mathbb{R}^d$, the family of probability measures

$$\mathbb{P}_x^{\varepsilon} := (\mathbf{P} \circ Y_{\varepsilon}^y)^{-1}, \quad \varepsilon > 0,$$

is tight on the canonical space of continuous trajectories on [0, T].

Proof of Lemma 10. The argument is standard. For reader's convenience, we include it below (we repeat more or less verbatim a part of [32]). Put for brevity T=1. We have, for a stopping time $0 \le \tau \le 1$,

$$Y_{\varepsilon}^{y}(\tau+\sigma) - Y_{\varepsilon}^{y}(\tau) = \int_{\tau}^{\tau+\sigma} b_{n}(s, Y_{\varepsilon}^{y}(s))ds + \sqrt{2}(B(\tau+\sigma) - B(\tau)), \quad 0 < \sigma < 1.$$
 (10.7)

Next, note that (10.5) yields

$$\mathbf{E} \int_{\tau}^{\tau+\sigma} |b_n(s, Y_{\varepsilon}^y(s))| ds \le C_0 \sigma^{\frac{\gamma}{\gamma+1}}, \tag{10.8}$$

see Remark 1.2 in [56] (to show that $(10.5) \Rightarrow (10.8)$, the authors of [56] use a decreasing sequence of stopping times τ_m converging to τ and taking values in $S = \{k2^{-m} \mid k \in \{0, 1, 2, ...\}\}$, and note that the proof of estimate (10.8) with τ_m in place of τ can be reduced to applying (10.5) on intervals $[t_0, t_1] := [c, c + \sigma], c \in S$.) Thus, applying (10.8) in (10.7), one obtains

$$\mathbf{E} \sup_{\sigma' \in [0,\sigma]} |Y_{\varepsilon}^{y}(\tau + \sigma') - Y_{\varepsilon}^{y}(\tau)| \le C_0 \sigma^{\frac{\gamma}{\gamma+1}} + C_1 \sigma^{\frac{1}{2}} =: H(\sigma).$$

Now, applying [57, Lemma 2.7], we obtain: there exists constant \hat{C} independent of ε such that

$$\sup_{\varepsilon} \sup_{y \in \mathbb{R}^d} \mathbf{E} \left[\sup_{t \in [0,1], \sigma' \in [0,\sigma]} |Y_{\varepsilon}^y(t + \sigma') - Y_{\varepsilon}^y(t)|^{\frac{1}{2}} \right] \le \hat{C}H(\sigma). \tag{10.9}$$

Applying Chebyshev's inequality in (10.9), since $H(\sigma) \downarrow 0$ as $\sigma \downarrow 0$, we obtain the first assertion of the lemma. The second assertion follows from the first one, see [51, Theorem 1.3.2].

Fix $y \in \mathbb{R}^d$. Let \mathbb{P}_y be a weak subsequential limit point of $\{\mathbb{P}_y^{\varepsilon}\}$,

$$\mathbb{P}_y^{\varepsilon_k} \to \mathbb{P}_y \text{ weakly} \quad \text{ for some } \varepsilon_k \downarrow 0. \tag{10.10}$$

Let us rewrite (10.2) as

$$\mathbb{E}_y^{\varepsilon} \int_{t_0}^{t_1} |\mathsf{g}(\omega_s)| ds \le C_2 (t_1 - t_0)^{\frac{\gamma}{1 + \gamma}}.$$

Taking $\mathbf{g} := b_{\varepsilon_m}$ and then applying (10.10), we obtain $\mathbb{E}_y \int_{t_0}^{t_1} |b_{\varepsilon_m}(\omega_s)| ds \leq C_2 (t_1 - t_0)^{\frac{\gamma}{1+\gamma}}$, and hence, using e.g. Fatou's lemma, $\mathbb{E}_y \int_{t_0}^{t_1} |b(\omega_s)| ds \leq C_2 (t_1 - t_0)^{\frac{\gamma}{1+\gamma}} < \infty$.

Step 2. Let us show that, for any fixed $y \in \mathbb{R}^d$, any subsequential limit point \mathbb{P}_y of $\{\mathbb{P}_y^{\varepsilon}\}$ (say, (10.10) holds) is a solution to the martingale problem for SDE (4.1). Set

$$M_t^{\varphi,\varepsilon} := \varphi(\omega_t) - \varphi(\omega_0) + \int_0^t (-\Delta \varphi + b_{\varepsilon} \cdot \nabla \varphi)(\omega_s) ds, \quad \varphi \in C_c^2.$$

It suffices to show that $\mathbb{E}_y[M_{t_1}^{\varphi}G] = \mathbb{E}_y[M_{t_0}^{\varphi}G]$ for every \mathcal{B}_{t_0} -measurable $G \in C_b(C([0,T],\mathbb{R}^d))$. We will do this by passing to the limit in k in

$$\mathbb{E}_{y}^{\varepsilon_{k}}[M_{t_{1}}^{\varphi,\varepsilon_{k}}G] = \mathbb{E}_{y}^{\varepsilon_{k}}[M_{t_{0}}^{\varphi,\varepsilon_{k}}G].$$

That is, we need to prove

$$\lim_{k} \mathbb{E}_{y}^{\varepsilon_{k}} \int_{0}^{t} (b_{\varepsilon_{k}} \cdot \nabla \varphi)(\omega_{s}) G(\omega) ds = \mathbb{E}_{y} \int_{0}^{t} (b \cdot \nabla \varphi)(\omega_{s}) G(\omega) ds, \tag{10.11}$$

Proof of (10.11). First, let us note that repeating the proof of (10.2), but this time selecting $h := g|g|^{\gamma}$, $g := b_{\varepsilon_{m_1}} - b_{\varepsilon_{m_2}}$, $f := |\nabla \varphi|$, we have

$$\begin{split} & \mathbb{E}_y^{\varepsilon} \int_{t_0}^{t_1} \left| b_{\varepsilon_{m_1}}(\omega_s) - b_{\varepsilon_{m_2}}(\omega_s) \right| |\nabla \varphi(\omega_s)| ds \\ & \leq C_3 \sup_{x \in \frac{1}{2} \mathbb{Z}^d} \bigg(\langle \left(\mathbf{1}_{|\mathbf{g}| > 1} + |\mathbf{g}|^{(1+\gamma)p\theta} \mathbf{1}_{|\mathbf{g}| \leq 1} \right) |\nabla \varphi|^{p\theta} \rho_x \rangle^{\frac{1}{p\theta}} + \langle \left(\mathbf{1}_{|\mathbf{g}| > 1} + |\mathbf{g}|^{(1+\gamma)p\theta'} \mathbf{1}_{|\mathbf{g}| \leq 1} \right) |\nabla \varphi|^{p\theta'} \rho_x \rangle^{\frac{1}{p\theta'}} \bigg)^{\frac{1}{1+\gamma}}, \end{split}$$

Since φ has compact support, the RHS converges to 0 as $m_1, m_2 \to \infty$. Now, it follows from the weak convergence (10.10) and Fatou's lemma that

$$\mathbb{E}_{y} \int_{t_{0}}^{t_{1}} |b(\omega_{s}) - b_{\varepsilon_{m}}(\omega_{s})| |\nabla\varphi(\omega_{s})| ds$$

$$\leq C_{3} \sup_{x \in \frac{1}{2}\mathbb{Z}^{d}} \left(\langle \left(\mathbf{1}_{|b-b_{\varepsilon_{m}}|>1} + |b-b_{\varepsilon_{m}}|^{(1+\gamma)p\theta} \mathbf{1}_{|b-b_{\varepsilon_{m}}|\leq 1}\right) |\nabla\varphi|^{p\theta} \rho_{x} \rangle^{\frac{1}{p\theta}} + \langle \left(\mathbf{1}_{|b-b_{\varepsilon_{m}}|>1} + |b-b_{\varepsilon_{m}}|^{(1+\gamma)p\theta'} \mathbf{1}_{|b-b_{\varepsilon_{m}}|\leq 1}\right) |\nabla\varphi|^{p\theta'} \rho_{x} \rangle^{\frac{1}{p\theta'}} \right)^{\frac{1}{1+\gamma}},$$

where the RHS converges to 0 as $m \to \infty$. We are in position to prove (10.11):

$$\begin{split} & \left| \mathbb{E}_{y}^{\varepsilon_{n_{k}}} \int_{0}^{t} (b_{\varepsilon_{n_{k}}} \cdot \nabla \varphi)(\omega_{s}) G(\omega) ds - \mathbb{E}_{y} \int_{0}^{t} (b \cdot \nabla \varphi)(\omega_{s}) G(\omega) ds \right| \\ & \leq \left| \mathbb{E}_{y}^{\varepsilon_{n_{k}}} \int_{0}^{t} |b_{\varepsilon_{n_{k}}} - b_{\varepsilon_{m}}| |\nabla \varphi|(\omega_{s})| G(\omega)| ds \right| \\ & + \left| \mathbb{E}_{y}^{\varepsilon_{n_{k}}} \int_{0}^{t} (b_{\varepsilon_{m}} \cdot \nabla \varphi)(\omega_{s}) G(\omega) ds - \mathbb{E}_{y} \int_{0}^{t} (b_{\varepsilon_{m}} \cdot \nabla \varphi)(\omega_{s}) G(\omega) ds \right| \\ & + \left| \mathbb{E}_{y} \int_{0}^{t} |b_{\varepsilon_{m}} - b| |\nabla \varphi|(\omega_{s})| G(\omega)| ds \right|, \end{split}$$

where the first and the third terms in the RHS can be made arbitrarily small using the estimates above and the boundedness of G by selecting m, and then n_k , sufficiently large. The second term can be made arbitrarily small in view of (10.10) by selecting n_k even larger. Thus, (10.11) follows.

Step 3. Let us now find a subsequence $\varepsilon_k \downarrow 0$ that works for all $y \in \mathbb{R}^d$ and yields a strong Markov family of probability measures \mathbb{P}_y , $y \in \mathbb{R}^d$, solutions to the martingale problem for SDE (4.1). Denote $R_{\lambda}^{\varepsilon}f := u_{\varepsilon}$, where u_{ε} is the classical solution of $(\lambda - \Delta + b_{\varepsilon} \cdot \nabla)u_{\varepsilon} = f$ in \mathbb{R}^d , $f \in C_c^{\infty}$, $\lambda \geq \lambda_0 \vee 1$;

$$R_{\lambda}^{\varepsilon}f(y) = \mathbb{E}_{\mathbb{P}_{y}^{\varepsilon}} \int_{0}^{\infty} e^{-\lambda s} f(\omega_{s}) ds.$$

By Theorem 6, u_{ε} are uniformly in ε bounded on \mathbb{R}^d . By Theorem 5 applied to b_{ε} , solutions u_{ε} are Hölder continuous on every compact, also uniformly in $\varepsilon > 0$. By the Arzelà-Ascoli theorem and a standard diagonal argument there exists a subsequence $\varepsilon_k \downarrow 0$ such that sequence $\{R_{\lambda}^{\varepsilon}f\}$ converges locally uniformly on \mathbb{R}^d , for every f in a fixed dense subset of C_b . Let us denote the limit by $R_{\lambda}f$. The latter, and the uniform in ε estimate $\|R_{\lambda}^{\varepsilon}f\|_{\infty} \leq \frac{1}{\lambda}\|f\|_{\infty}$ allow us to extend $R_{\lambda}f$ to all $f \in C_b$. Thus, $R_{\lambda}f \in C_b$. Now, for this subsequence $\varepsilon_k \downarrow 0$, for any $y_k \to y$, any two subsequential

limits \mathbb{P}^1 , \mathbb{P}^2 of $\{\mathbb{P}^{\varepsilon_k}_{y_k}\}$ (we use (10.10)) have the same finite-dimensional distributions (see [5] for details, if needed) and therefore coincide: $\mathbb{P}_y := \mathbb{P}^1 = \mathbb{P}^2$. Hence $\mathbb{E}_{\mathbb{P}_y} \int_0^\infty e^{-\lambda s} f(\omega_s) ds = R_{\lambda} f(y)$. By what was proved above, \mathbb{P}_y is a martingale solution of (4.1). A simple argument (see [5]) now gives that, for every t > 0, $y \mapsto \mathbb{E}_{\mathbb{P}_y} f(X_t)$ is a continuous function. The latter, in turn, yields that $\{\mathbb{P}_y\}_{y \in \mathbb{R}^d}$ is a strong Markov family (the proof can be found e.g. in [5] or [6, Sect. I.3]).

This completes the proof of assertion (i).

(ii) Let b_n be defined by (4.2), so that vector fields $\{b_n\}$ do not increase the form-bounds of b. In the end of the proof of (i) we show that there exists a subsequence b_{n_k} (for brevity, $\{b_n\}$ itself) such that, for every $f \in C_c^{\infty}(\mathbb{R}^d)$, the classical solutions $\{u_n\}$ to elliptic equations

$$(\lambda - \Delta + b_n \cdot \nabla)u_n = f$$

converge locally uniformly on \mathbb{R}^{Nd} to

$$x \mapsto \mathbb{E}_{\mathbb{P}_x} \int_0^\infty e^{-\lambda s} f(\omega_s^1, \dots, \omega_s^N) ds, \quad x \in \mathbb{R}^{Nd},$$
 (10.12)

where λ is assumed to be sufficiently large. This yields the local Hölder continuity of u. At the same time, u_n are weak solutions of (2.4) in the sense of Definitions 5 and 7. The possibility to pass to the limit $\varepsilon \downarrow 0$ in these definitions follows from the standard compactness argument (for details, if needed, see e.g. [39]).

- (iii) The proof goes by showing that v_n constitute a Cauchy sequence in $L^{\infty}([0,1], L^p(\mathbb{R}^d))$, see [25], see also [33]. At the elliptic level this was done earlier in [41] using Trotter's theorem. The proof of the (L^p, L^q) estimate is due to [50]. (Strictly speaking, these papers did not consider condition (\mathbb{A}_3) , but it is easy to modify the proofs there to cover the case (\mathbb{A}_3) as well.)
 - (iv) It suffices to show that, for all $\mu \geq \mu_0$, for every $f \in C_c^{\infty}$,

$$R_{\mu}^{\varepsilon}f \to (\mu + \Lambda_p)^{-1}f$$
 in C_{∞} as $\varepsilon \downarrow 0$, (10.13)

possibly after a modification of $(\mu + \Lambda_p)^{-1} f$ on a measure zero set. The rest follows from estimates $||R_{\mu,\varepsilon}f||_{\infty} \leq \mu^{-1}||f||_{\infty}$, $||(\mu + \Lambda_p)^{-1}f||_{\infty} \leq \mu^{-1}||f||_{\infty}$ (an immediate consequence of the fact that the corresponding semigroups are L^{∞} contractions) using a density argument.

Let us prove (10.13). Put $u_{\varepsilon} := R_{\mu,\varepsilon} f$, so u_{ε} is the classical solution to $(\mu - \Delta + b \cdot \nabla)u_{\varepsilon} = f$ on \mathbb{R}^d . Then, by Propositions 5 and 6 (with $|\mathbf{h}| = 1$), for all $\mu (\geq 1 \vee \lambda_0) + 1$

$$\sup_{y \in B_{\frac{1}{\alpha}}(x)} |u_{\varepsilon}(y)| \le C \left(\langle |f|^{p\theta} \rho_x \rangle^{\frac{1}{p\theta}} + \langle |f|^{p\theta'} \rho_x \rangle^{\frac{1}{p\theta'}} \right)$$

for constant C independent of ε . It is seen now that for a fixed $f \in C_c^{\infty}$, for a given $\varepsilon > 0$, we can find R > 0 such that

$$\sup_{y \in \mathbb{R}^d \setminus \bar{B}_R(0)} |u_{\varepsilon}(y)| < \varepsilon.$$

In turn, inside the closed ball $\bar{B}_R(0)$, the family of solutions $\{u_{\varepsilon}\}_{{\varepsilon}>0}$ is equicontinuous by Theorem 5. So, applying Arzelà-Ascoli theorem and using the convergence result for the semigroups in L^p from assertion (iii), we obtain (10.13).

(v) The proof is an application of Proposition 1 and Gehring's lemma:

Lemma 11. Assume that there exist constants $K \ge 1$, $1 < q < \infty$ such that, for given $0 \le g \in L^q$, $0 \le h \in L^q \cap L^\infty$ we have

$$\left(\frac{1}{|B_R|}\langle g^q \mathbf{1}_{B_R} \rangle\right)^{\frac{1}{q}} \le \frac{K}{|B_{2R}|}\langle g \mathbf{1}_{B_{2R}} \rangle + \left(\frac{1}{|B_{2R}|}\langle h^q \mathbf{1}_{B_{2R}} \rangle\right)^{\frac{1}{q}}$$

for all $0 < R < \frac{1}{2}$. Then $g \in L^s$ for some s > q and

$$\left(\frac{1}{|B_R|}\langle g^s \mathbf{1}_{B_R} \rangle\right)^{\frac{1}{s}} \leq C_1 \left(\frac{1}{|B_{2R}|}\langle g^q \mathbf{1}_{B_{2R}} \rangle\right)^{\frac{1}{q}} + C_2 \left(\frac{1}{|B_{2R}|}\langle h^s \mathbf{1}_{B_{2R}} \rangle\right)^{\frac{1}{s}}.$$

We are in position to prove assertion (v). Without loss of generality, $f \geq 0$, so $u_n \geq 0$.

Step 1. Set $(u_n)_{B_{2R}} := \frac{1}{|B_{2R}|} \langle u_n \mathbf{1}_{B_{2R}} \rangle$. Repeating the proof of Proposition 1 with p=2 for $u_n - (u_n)_{B_{2R}}$, we obtain

$$\langle |\nabla u_n|^2 \mathbf{1}_{B_R} \rangle \le \frac{K_1}{|B_{2R}|^{\frac{2}{d}}} \langle (u_n - (u_n)_{B_{2R}})^2 \mathbf{1}_{B_{2R}} \rangle + K_2 \langle |f - \mu u_n|^2 \mathbf{1}_{B_{2R}} \rangle, \quad 0 < R < \frac{1}{2}. \quad (10.14)$$

By the Sobolev-Poincaré inequality,

$$\left(\frac{1}{|B_{2R}|}\langle (u_n - (u_n)_{B_{2R}})^2 \mathbf{1}_{B_{2R}} \rangle\right)^{\frac{1}{2}} \le C|B_R|^{\frac{1}{d}} \left(\frac{1}{|B_{2R}|} \langle |\nabla u_n|^{\frac{2d}{d+2}} \mathbf{1}_{B_{2R}} \rangle\right)^{\frac{d+2}{2d}},$$
(10.15)

i.e.

$$\langle (u_n - (u_n)_{B_{2R}})^2 \mathbf{1}_{B_{2R}} \rangle \le C^2 |B_R|^{\frac{2}{d} + 1} \left(\frac{1}{|B_{2R}|} \langle |\nabla u_n|^{\frac{2d}{d+2}} \mathbf{1}_{B_{2R}} \rangle \right)^{\frac{d+2}{d}}$$

Then the condition of the Gehring lemma is verified with $g = |\nabla u_n|^{\frac{2d}{d+2}}$, $g^q = |\nabla u_n|^2$ (so $q = \frac{d+2}{d}$) and $h = c|f - \mu u_n|^{\frac{2d}{d+2}}$. Hence there exists $s > \frac{d+2}{d}$ such that

$$\left(\frac{1}{|B_R|}\langle |\nabla u_n|^{s\frac{2d}{d+2}}\mathbf{1}_{B_R}\rangle\right)^{\frac{1}{s}} \leq C_1 \left(\frac{1}{|B_{2R}|}\langle |\nabla u_n|^2\mathbf{1}_{B_{2R}}\rangle\right)^{\frac{d}{d+2}} + C_2 \left(\frac{1}{|B_{2R}|}\langle |f-\mu u_n|^{s\frac{2d}{d+2}}\mathbf{1}_{B_{2R}}\rangle\right)^{\frac{1}{s}},$$

where all constants are independent of n.

Now, passing in both sides of the previous inequality to the cubes (inscribed in B_R in the left-hand side and circumscribed over B_{2R} in the right-hand side), then considering an equally spaced grid in \mathbb{R}^d so that the smaller cubes centered at the nodes of the grid cover \mathbb{R}^d , applying the previous estimate on each cube, and then summing up, we obtain the global estimate

$$\|\nabla u_n\|_{s\frac{2d}{d+2}}^2 \le C_3 \|\nabla u_n\|_2^2 + C_4 \|f - \mu u_n\|_{s\frac{2d}{d+2}}^2.$$

Step 2. Let us show that $\sup_n \|\nabla u_n\|_2^2 < \infty$. To this end, we multiply $(\mu - \Delta + b_n \cdot \nabla)u_n = f$ by u_n and integrate, obtaining $\mu \|u_n\|_2^2 + \|\nabla u_n\|_2^2 + \langle b_n \cdot \nabla u_n, u_n \rangle = \langle f, u_n \rangle$, where

$$\langle b_n \cdot \nabla u_n, u_n \rangle = -\frac{1}{2} \langle \operatorname{div} b_n, u_n^2 \rangle \ge -\frac{1}{2} \langle (\operatorname{div} b_n)_+, u_n^2 \rangle.$$

Hence, by our form-boundedness assumption on $(\operatorname{div} b_n)_+$,

$$\left(\mu - \frac{c_{\delta_{+}}}{2}\right) \|u_{n}\|_{2}^{2} + \left(1 - \frac{\delta_{+}}{2}\right) \|\nabla u_{n}\|_{2}^{2} \le \langle f, u_{n} \rangle. \tag{10.16}$$

So, applying the quadratic inequality in the right-hand side, we arrive at $(\mu - \frac{c_{\delta_+}}{2} - \frac{1}{2})\|u_n\|_2^2 + (1 - \frac{\delta_+}{2})\|\nabla u_n\|_2^2 \le \frac{1}{2}\|f\|_2^2$. Since $\delta_+ < 2$, $\sup_n \|\nabla u_n\|_2^2 < \infty$ for $\mu \ge \mu_0 := \frac{c_{\delta_+}}{2} + \frac{1}{2}$.

Step 3. Next, $||u_n||_2 \le C||f||_2$ and a priori bound $||u_n||_{\infty} \le ||f||_{\infty}$ yield $\sup_n ||u_n||_{s\frac{2d}{d+2}} < \infty$. Hence $\sup_n ||f - \mu u_n||_{s\frac{2d}{d+2}}^2 < \infty$.

Steps 1-3 give us a gradient bound

$$\sup_{n} \|\nabla u_n\|_{s\frac{2d}{d+2}}^2 < \infty$$

which we are going to use at the next step.

Step 4. Put $h := u_n - u_m$. Then

$$\mu \|h\|_2^2 + \|\nabla h\|_2^2 + \langle b_n \cdot \nabla h, h \rangle + \langle (b_n - b_m) \cdot \nabla u_m, h \rangle = 0.$$

So,

$$\left(\mu - \frac{c_{\delta_{+}}}{2} - \frac{1}{2}\right) \|h\|_{2}^{2} + \left(1 - \frac{\delta_{+}}{2}\right) \|\nabla h\|_{2}^{2} \le |\langle (b_{n} - b_{m}) \cdot \nabla u_{m}, h \rangle|. \tag{10.17}$$

In turn, the right-hand side

$$|\langle (b_n - b_m) \cdot \nabla u_m, h \rangle| \le ||b_n - b_m||_{2-\varkappa} ||\nabla u_m||_{s^{\frac{2d}{d+2}}} 2||f||_{\infty}$$

where $0 < \varkappa < 1$ is defined by

$$2 - \varkappa := \left(s \frac{2d}{d+2}\right)' = \frac{s \frac{2d}{d+2}}{s \frac{2d}{d+2} - 1}$$

(recall that $s\frac{2d}{d+2} > 2$). Since $\{b_n\}$ converge to b in $L^{2-\varkappa}$, we obtain that the RHS of (10.17) converges to zero as $n, m \to \infty$, so $\{u_n\}$ is a Cauchy sequence in L^2 . (This yields the independence of the limit on a particular choice of $\{b_n\}$ since we can always combine two different approximations of b obtaining again a Cauchy sequence of the approximating solutions.)

Remark 17. We carried out the proof of assertion (v) assuming that $\delta_+ < 2$ instead of $\delta_+ < 4$ as in the other assertions. In fact, to handle $\delta_+ < 4$ we need to consider $(\mu - \Delta + b_n \cdot \nabla)u_n = f$ in L^p , $p > \frac{4}{4-\delta_+}$. However, the step where we use the Sobolev-Poincaré inequality (10.15) in (10.14) is ultimately an L^2 argument. Hence the need for a more restrictive condition $\delta_+ < 2$.

11. Proofs of Theorem 1(i),(ii) and Theorem 2(i)

Theorem 1(i),(ii) and Theorem 2(i) follows right away, in view of Lemmas 1, 2, from Theorem 3(i),(ii) where we consider the general SDE in \mathbb{R}^{Nd} with $Y=(X_1,\ldots,X_N),\,B=(B_1,\ldots,B_N),\,y=(x_1,\ldots,x_N)$ and drift $b:\mathbb{R}^{Nd}\to\mathbb{R}^{Nd}$ defined by (2.19).

12. Proof of Theorem 2(ii)

This follows right away from Theorem 5(iii) and Lemmas 1, 2.

13. Proof of Theorem 1(iii)-(v)

- (iii) follows from Theorem 3(iii) and Lemmas 1, 2.
- (iv) follows from the uniqueness result in [30], see also [28], and Lemma 1.
- (v) follows from the result in [29] upon applying Lemma 1. The assertion before, i.e. that Theorem 1(i)-(iv) is also valid for the interaction kernels of the form (2.8), follows upon applying an appropriate (straightforward) modification of Lemma 1.

14. Proof of Theorem 2(iii), (iv)

(iii) Since the sum of two form-bounded vector fields is again form-bounded, we only need to improve Lemma 2 for $K(y) = \sqrt{\kappa} \frac{d-2}{2} |y|^{-2} y$ and then simply repeat the proof of Theorem 1(i)-(iii). In Lemma 2 we have three estimates (2.20), (2.21) (2.22) for $b = (b_1, \ldots, b_N) : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$, where now

$$b_i(x) := \sqrt{\kappa} \frac{d-2}{2} \frac{1}{N} \sum_{j=1, j \neq i}^{N} \frac{x_i - x_j}{|x_i - x_j|^2}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^{Nd}, \quad 1 \le i \le N.$$
 (14.1)

We do not need to change (2.20) and (2.22) since the actual values of the form-bounds there are not important for the sake of repeating the proof of Theorem 2, only their finiteness matters. The form-bound δ_+ in (2.21), however, plays a crucial role. Let us estimate it using the many-particle Hardy's inequality (1.20):

$$(\operatorname{div} b)_{+} = \operatorname{div} b = \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1, j \neq i}^{N} \operatorname{div} K(x_{i} - x_{j})$$
$$= \sqrt{\kappa} \frac{(d-2)^{2}}{N} \sum_{1 \leq i \leq j \leq N} \frac{1}{|x_{i} - x_{j}|^{2}}.$$

Applying (1.20), we obtain that $(\operatorname{div} b)_+^{\frac{1}{2}} \in \mathbf{F}_{\delta_+}$ with $\delta_+ = \sqrt{\kappa}$. Armed with this result, i.e. a replacement of Lemma 2, we repeat the proof of Theorem 2 (i.e. we apply Theorem 3 where we still have $\delta_+ < 4$).

(iv) We apply Theorem 8 from Appendix A. There $\Omega := \mathbb{R}^{Nd}$ and μ is the Lebesgue measure on \mathbb{R}^{Nd} . The semigroup $e^{-t\Lambda}$ and thus the heat kernel $e^{-t\Lambda}(x,y)$ is from assertion (ii). The weights $\{\varphi_s\}_{s>0}$ are defined by

$$\varphi_s(x) := \prod_{1 \le i < j \le N} \eta(s^{-\frac{1}{2}} |x_i - x_j|), \quad s > 0.$$

It is easily seen that these weights φ_s satisfy conditions (S_2) and (S_3) of Theorem 8. In turn, condition (S_1) with $j = \frac{d}{d-2}$ and $r > 2(2 - \frac{N-1}{N}\sqrt{\kappa})^{-1}$ was verified in Theorem 2(ii) under hypothesis (2.15), see (2.5). Let us verify the "desingularizing $L^1 \to L^1$ bound" (S_4) for $0 < s \le t$: Step 1. Set

$$\eta_s(r) := \eta(s^{-\frac{1}{2}}r), \quad r > 0$$

and put

$$\varphi_s^{\varepsilon}(x) \equiv \varphi^{\varepsilon}(x) := \prod_{1 \le i < j \le N} \eta_s(|x_i - x_j|_{\varepsilon}), \quad |x_i - x_j|_{\varepsilon} := \sqrt{|x_i - x_j|^2 + \varepsilon}, \quad \varepsilon > 0.$$

Define

$$\psi^{\varepsilon}(x) := \prod_{1 \le i \le j \le N} (s^{-\frac{1}{2}} |x_i - x_j|_{\varepsilon})^{-\sqrt{\kappa} \frac{d-2}{2} \frac{1}{N}}.$$

and put

$$b_{\varepsilon} := -\frac{\nabla_x \psi^{\varepsilon}}{\psi^{\varepsilon}}$$
 (clearly, independent of s).

This is a vector field $\mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ such that

$$b_{\varepsilon} \cdot \nabla_{x} = \sqrt{\kappa} \frac{d-2}{2} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1, i \neq i}^{N} \frac{x_{i} - x_{j}}{|x_{i} - x_{j}|_{\varepsilon}^{2}} \cdot \nabla_{x_{i}}.$$

Without loss of generality, we discuss the (minus) first component $\mathbb{R}^{Nd} \to \mathbb{R}^d$ of b_{ε} :

$$\begin{split} \frac{\nabla_{x_1} \psi^{\varepsilon}}{\psi^{\varepsilon}} &= \frac{1}{\psi^{\varepsilon}} \sum_{2 \leq k \leq N} \prod_{1 \leq i < j \leq N, (i,j) \neq (1,k)} |x_i - x_j|_{\varepsilon}^{-\sqrt{\kappa} \frac{d-2}{2} \frac{1}{N}} \nabla_{x_1} \left(|x_1 - x_k|_{\varepsilon}^{-\sqrt{\kappa} \frac{d-2}{2} \frac{1}{N}} \right) \\ &= \sum_{2 \leq k \leq N} \frac{\nabla_{x_1} |x_1 - x_k|_{\varepsilon}^{-\sqrt{\kappa} \frac{d-2}{2} \frac{1}{N}}}{|x_1 - x_k|_{\varepsilon}^{-\sqrt{\kappa} \frac{d-2}{2} \frac{1}{N}}} \\ &= -\sqrt{\kappa} \frac{d-2}{2} \frac{1}{N} \sum_{2 \leq k \leq N} \frac{x_1 - x_k}{|x_1 - x_k|_{\varepsilon}^2}. \end{split}$$

In the same way,

$$\frac{\nabla_{x_1} \varphi^{\varepsilon}}{\varphi^{\varepsilon}} = \sum_{2 \le k \le N} \frac{\nabla_{x_1} \eta_s(|x_1 - x_k|_{\varepsilon})}{\eta_s(|x_1 - x_k|_{\varepsilon})}.$$

We now compare these quantities (this will be needed at the next step):

(a) If $|x_1 - x_k|_{\varepsilon} \leq \sqrt{s}$ for all $2 \leq k \leq N$, then, by the definition of η ,

$$-\frac{\nabla_{x_1}\psi^{\varepsilon}}{\psi^{\varepsilon}} + \frac{\nabla_{x_1}\varphi^{\varepsilon}}{\varphi^{\varepsilon}} = 0.$$

Therefore,

$$\frac{\nabla_{x_1}\varphi^{\varepsilon}}{\varphi^{\varepsilon}} \cdot \left(-\frac{\nabla_{x_1}\psi^{\varepsilon}}{\psi^{\varepsilon}} + \frac{\nabla_{x_1}\varphi^{\varepsilon}}{\varphi^{\varepsilon}} \right) = 0, \quad \operatorname{div}_{x_1} \left(-\frac{\nabla_{x_1}\psi^{\varepsilon}}{\psi^{\varepsilon}} + \frac{\nabla_{x_1}\varphi^{\varepsilon}}{\varphi^{\varepsilon}} \right) = 0.$$

(b) If there exists one k_0 such that $|x_1 - x_{k_0}|_{\varepsilon} \ge 2\sqrt{s}$, but for the other $k \ne k_0 |x_1 - x_k|_{\varepsilon} \le \sqrt{s}$, then, since $x_1 \mapsto \eta_s(|x_1 - x_{k_0}|_{\varepsilon})$ is constant and so $\nabla_{x_1} \varphi^{\varepsilon} = 0$, we have

$$\frac{\nabla_{x_1}\psi^{\varepsilon}}{\psi^{\varepsilon}} - \frac{\nabla_{x_1}\varphi^{\varepsilon}}{\varphi^{\varepsilon}} = -\sqrt{\kappa} \frac{d-2}{2} \frac{1}{N} \frac{x_1 - x_{k_0}}{|x_1 - x_{k_0}|_{\varepsilon}^2}.$$

Hence

$$\frac{\nabla_{x_1}\varphi^{\varepsilon}}{\varphi^{\varepsilon}} \cdot \left(-\frac{\nabla_{x_1}\psi^{\varepsilon}}{\psi^{\varepsilon}} + \frac{\nabla_{x_1}\varphi^{\varepsilon}}{\varphi^{\varepsilon}}\right) = 0, \quad \left|\operatorname{div}_{x_1}\left(-\frac{\nabla_{x_1}\psi^{\varepsilon}}{\psi^{\varepsilon}} + \frac{\nabla_{x_1}\varphi^{\varepsilon}}{\varphi^{\varepsilon}}\right)\right| \leq \sqrt{\kappa} \frac{(d-2)^2}{2} \frac{1}{N} 4s^{-1}.$$

(c) More generally, if there exist $2 \le M \le N-1$ indices k_0 such that $|x_1 - x_{k_0}|_{\varepsilon} \ge 2\sqrt{s}$, but for the other $k \ne k_0 |x_1 - x_k|_{\varepsilon} \le \sqrt{s}$, then we have

$$\frac{\nabla_{x_1}\varphi^{\varepsilon}}{\varphi^{\varepsilon}} \cdot \left(-\frac{\nabla_{x_1}\psi^{\varepsilon}}{\psi^{\varepsilon}} + \frac{\nabla_{x_1}\varphi^{\varepsilon}}{\varphi^{\varepsilon}} \right) = 0, \quad \left| \operatorname{div}_{x_1} \left(-\frac{\nabla_{x_1}\psi^{\varepsilon}}{\psi^{\varepsilon}} + \frac{\nabla_{x_1}\varphi^{\varepsilon}}{\varphi^{\varepsilon}} \right) \right| \le \sqrt{\kappa} \frac{(d-2)^2}{N} 4s^{-1}.$$

Over the annuli $\sqrt{s} < |x_1 - x_k|_{\varepsilon} < 2\sqrt{s}$ we make a change of variable to finally obtain, for all possible values of $|x_1 - x_k|_{\varepsilon}$, $2 \le k \le N$,

$$\left| \frac{\nabla_{x_1} \varphi^{\varepsilon}}{\varphi^{\varepsilon}} \cdot \left(- \frac{\nabla_{x_1} \psi_{\varepsilon}}{\psi_{\varepsilon}} + \frac{\nabla_{x_1} \varphi^{\varepsilon}}{\varphi^{\varepsilon}} \right) \right| \le c_1 \frac{N - 1}{N} s^{-1}, \quad \left| \operatorname{div}_{x_1} \left(- \frac{\nabla_{x_1} \psi^{\varepsilon}}{\psi^{\varepsilon}} + \frac{\nabla_{x_1} \varphi^{\varepsilon}}{\varphi^{\varepsilon}} \right) \right| \le c_2 \frac{N - 1}{N} s^{-1}$$

for constants c_1 and c_2 independent of ε and s.

The same holds for the other components of $b_{\varepsilon} = -\frac{\nabla_x \psi_{\varepsilon}}{\psi_{\varepsilon}}$. Thus,

$$\left| \frac{\nabla_x \varphi^{\varepsilon}}{\varphi^{\varepsilon}} \cdot \left(b_{\varepsilon} + \frac{\nabla_x \varphi^{\varepsilon}}{\varphi^{\varepsilon}} \right) \right| \le c_1 \frac{N - 1}{\sqrt{N}} s^{-1}, \quad \left| \operatorname{div} \left(b_{\varepsilon} + \frac{\nabla_x \varphi^{\varepsilon}}{\varphi^{\varepsilon}} \right) \right| \le c_2 \frac{N - 1}{\sqrt{N}} s^{-1}. \tag{14.2}$$

Step 2. Define the approximating operators $\Lambda_{\varepsilon} := -\Delta_x + b_{\varepsilon} \cdot \nabla_x$ having domain $\mathcal{W}^{2,1} = (1 - \Delta)^{-1}L^1$. Since φ^{ε} , $(\varphi^{\varepsilon})^{-1}$ are bounded and continuous, one sees right away that $\varphi^{\varepsilon}e^{-t\Lambda_{\varepsilon}}(\varphi^{\varepsilon})^{-1}$ is a strongly continuous semigroup in L^1 whose generator coincides with $-\varphi_{\varepsilon}\Lambda_{\varepsilon}(\varphi_{\varepsilon})^{-1}$ having domain $\mathcal{W}^{2,1}$. This generator can be computed explicitly:

$$\varphi^{\varepsilon} \Lambda^{\varepsilon} (\varphi^{\varepsilon})^{-1} = -\Delta + \nabla \cdot (b_{\varepsilon} + 2 \frac{\nabla \varphi^{\varepsilon}}{\varphi^{\varepsilon}}) + W_{\varepsilon}, \tag{14.3}$$

$$W_{\varepsilon} := -\frac{\nabla \varphi^{\varepsilon}}{\varphi^{\varepsilon}} \cdot \left(b_{\varepsilon} + \frac{\nabla \varphi^{\varepsilon}}{\varphi^{\varepsilon}}\right) - \operatorname{div}\left(b_{\varepsilon} + \frac{\nabla \varphi^{\varepsilon}}{\varphi^{\varepsilon}}\right).$$

By (14.2), potential W_{ε} is (uniformly in ε) bounded: $|W_{\varepsilon}| \leq \frac{N-1}{\sqrt{N}} \frac{c}{s}$ for a constant c independent of ε . Employing formula (14.3) and using the general fact that $e^{t(\Delta-\nabla\cdot f)}$ is an L^1 contraction, we obtain

$$\|\varphi^{\varepsilon}e^{-t\Lambda^{\varepsilon}}(\varphi^{\varepsilon})^{-1}h\|_{1} \leq e^{c\frac{N-1}{\sqrt{N}}\frac{t}{s}}\|h\|_{1}, \quad h \in L^{1}.$$

$$(14.4)$$

It remains to pass to the limit $\varepsilon \downarrow 0$ in (14.4). This is done at the next step.

Step 3. Define $b = -\frac{\nabla_x \psi}{\psi}$, where $\psi(x) = \prod_{1 \le i < j \le N} |x_i - x_j|^{-\sqrt{\kappa} \frac{d-2}{2} \frac{1}{N}}$ is a Lyapunov function of the formal adjoint of Λ (i.e. 2.17 holds). Then

$$b \cdot \nabla_x = \sqrt{\kappa} \frac{d-2}{2} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \frac{x_i - x_j}{|x_i - x_j|^2} \cdot \nabla_{x_i}$$

It is seen using e.g. the Monotone convergence theorem that $b_{\varepsilon} \to b$ in $[L^2_{\rm loc}]^{Nd}$. Moreover, the vector fields b_{ε} do not increase the form-bound $\delta = \kappa \left(\frac{N-1}{N}\right)^2$ (< 4) of b. Therefore, by Theorem

3(iii),

$$e^{-t\Lambda_{\varepsilon}} \to e^{-t\Lambda} \quad \text{in } L^r(\mathbb{R}^{Nd}),$$
 (14.5)

where $r > \frac{2}{2 - \frac{N-1}{N} \sqrt{\kappa}}$. Now, from (14.4) we have

$$\|\varphi_{\varepsilon}e^{-t\Lambda^{\varepsilon}}g\|_{1} \leq e^{c\frac{N-1}{\sqrt{N}}\frac{t}{s}}\|\varphi^{\varepsilon}g\|_{1}, \quad g \in \varphi L^{1} \cap L^{\infty}.$$

In view of (14.5) and since $\varphi^{\varepsilon} \to \varphi$ a.e., we can use Fatou's lemma to obtain $\|\varphi e^{-t\Lambda}g\|_1 \le$ $e^{c\frac{t}{s}}\|\varphi g\|_1$, which yields condition (S_4) of Theorem 8 (recall that by our assumption $s \geq t$).

Thus, Theorem 8 applies and gives assertion (iv) of Theorem 2.

15. Proof of Theorem 7

We will need the following result on the regularization of the vector field b in Theorem 7.

Lemma 12. Assume that $b \in [W_{loc}^{1,1}(\mathbb{R}^d)]^d$ has symmetric Jacobian $Db = (\nabla_k b_i)_{k,i=1}^d$ and the negative part B_{-} of matrix

$$B(b) := Db - \frac{\operatorname{div} b}{q}I, \quad \text{ for some } q > (d-2) \vee 2,$$

has normalized eigenvectors e_j and eigenvalues $\lambda_j \geq 0$ satisfying $\sqrt{\lambda_j} e_j \in \mathbf{F}_{\nu_j}$. Set $\nu := \sum_{j=1}^d \nu_j$. Set $b_{\varepsilon} := E_{\varepsilon}b$. The following are true:

1.

$$B(b_{\varepsilon}) + E_{\varepsilon}B_{-} > 0,$$

2.

$$\langle B_- h, h \rangle \le \nu \langle |\nabla |h||^2 \rangle + c_\nu \langle |h|^2 \rangle,$$
 (15.1)

and

$$\langle (E_{\varepsilon}B_{-})h, h \rangle \le \nu \langle |\nabla |h||^{2} \rangle + c_{\nu} \langle |h|^{2} \rangle, \quad \varepsilon > 0,$$

for all $h \in [C_c^{\infty}(\mathbb{R}^d)]^d$, with $c_{\nu} := \sum_{i=1}^d c_{\nu_i}$.

Proof. 1. We have, by definition, $B(b) = B_+ - B_-$, and $B(b_{\varepsilon}) = E_{\varepsilon}B_+ - E_{\varepsilon}B_-$. Clearly, $E_{\varepsilon}B_+ \ge 0$, which yields the required.

2. We have $B_- = \sum_{j=1}^d \lambda_j e_j e_j^{\top}$. Put for brevity $\lambda = \lambda_j$ and $e = e_j$. Denote the components of e by e^k , $k = 1, \ldots, d$. Then $\langle \lambda(ee^{\top})h, h \rangle = \sum_{k,i=1}^d \langle h_k \sqrt{\lambda} e^k \sqrt{\lambda} e^i h_i \rangle = \langle \lambda(h \cdot e)^2 \rangle \leq \langle \lambda |h|^2 |e|^2 \rangle$.

Therefore,

$$\langle B_{-}h, h \rangle \leq \sum_{j=1}^{d} \langle \lambda_{j} | h |^{2} | e_{j} |^{2} \rangle$$

$$(\text{we use } \sqrt{\lambda_{j}} e_{j} \in \mathbf{F}_{\nu_{j}})$$

$$\leq \sum_{j=1}^{d} \nu_{j} \langle |\nabla | h ||^{2} \rangle + \sum_{j=1}^{d} c_{\nu_{j}} \langle |h|^{2} \rangle,$$

which gives us the first inequality in assertion 2.

Let us prove the second inequality in assertion 2. Writing again $\lambda = \lambda_j$ and $e = e_j$ and denoting the k-th component of e by e^k , we have

$$\langle E_{\varepsilon}(\lambda e e^{\top})h, h \rangle = \sum_{k,i=1}^{d} \langle E_{\varepsilon}(\sqrt{\lambda} e^{k} \sqrt{\lambda} e^{i})h_{k}h_{i} \rangle = \sum_{k,i=1}^{d} \langle \sqrt{\lambda} e^{k} \sqrt{\lambda} e^{i} E_{\varepsilon}(h_{k}h_{i}) \rangle$$

$$\leq \sum_{k,i=1}^{d} \langle \sqrt{E_{\varepsilon}|h_{k}|^{2}} \sqrt{\lambda} |e^{k}| \sqrt{\lambda} |e^{i}| \sqrt{E_{\varepsilon}|h_{i}|^{2}} \rangle$$

$$\leq \langle \lambda |e|^{2}, |h_{\varepsilon}|^{2} \rangle,$$

where h_{ε} denotes the vector field with k-th component $\sqrt{E_{\varepsilon}|h_k|^2}$. Hence, using the previous estimate, we obtain

$$\langle (E_{\varepsilon}B_{-})h, h \rangle = \sum_{j=1}^{d} \langle E_{\varepsilon}(\lambda_{j}e_{j}e_{j}^{\top})h, h \rangle \leq \sum_{j=1}^{d} \langle \lambda_{j}|e_{j}|^{2}, |h_{\varepsilon}|^{2} \rangle$$

$$(\text{use } \sqrt{\lambda_{j}}e_{j} \in \mathbf{F}_{\nu_{j}})$$

$$\leq \nu \langle |\nabla|h_{\varepsilon}||^{2} \rangle + c_{\nu} \langle |h_{\varepsilon}|^{2} \rangle$$

$$(\text{note that } |h_{\varepsilon}| = \sqrt{E_{\varepsilon}|h|^{2}} \text{ and apply (6.1)})$$

$$\leq \nu \langle |\nabla|h||^{2} \rangle + c_{\nu} \langle |h|^{2} \rangle,$$

as needed.

Proof of Theorem 7 in the case drift b satisfies condition (\mathbb{B}_2). We start with the proof of assertion (ii). Put

$$w := \nabla u, \quad w_i := \nabla_i u.$$

Multiplying equation $(\mu - \Delta + b \cdot \nabla)u = f$ by the test function

$$\phi := -\sum_{i=1}^{d} \nabla_i(w_i | w |^{q-2}) = -\nabla \cdot (w | w |^{q-2})$$

and integrating by parts twice in $\langle -\Delta u, \phi \rangle$, i.e.

$$\langle -\Delta u, -\sum_{i=1}^{d} \nabla_{i}(w_{i}|w|^{q-2}) \rangle = \sum_{i=1}^{d} \langle \nabla_{i}w, \nabla(w_{i}|w|^{q-2}) \rangle \equiv \sum_{i=1}^{d} \langle \nabla w_{i}, \nabla(w_{i}|w|^{q-2}) \rangle$$

$$= \sum_{i=1}^{d} \langle |\nabla w_{i}|^{2}|w|^{q-2} \rangle + (q-2) \sum_{i=1}^{d} \langle \nabla w_{i}, w_{i}|w|^{q-3} \nabla |w| \rangle$$

$$= \sum_{i=1}^{d} \langle |\nabla w_{i}|^{2}|w|^{q-2} \rangle + (q-2) \langle \frac{1}{2} \nabla |w|^{2}, |w|^{q-3} \nabla |w| \rangle,$$

we obtain

$$\mu\langle |w|^q \rangle + I_q + (q-2)J_q + \langle b \cdot w, \phi \rangle = \langle f, \phi \rangle, \tag{15.2}$$

where

$$I_q := \sum_{i=1}^d \langle |\nabla w_i|^2, |w|^{q-2} \rangle, \quad J_q := \langle |\nabla |w||^2, |w|^{q-2} \rangle.$$

Step 1. Regarding term $\langle b \cdot w, \phi \rangle$ in (15.2), we have

$$\langle b \cdot w, \phi \rangle = \langle \tilde{B}w, w | w |^{q-2} \rangle + \langle b \cdot \nabla | w |, | w |^{q-1} \rangle \qquad \tilde{B} := (\nabla_k b_i)_{k,i=1}^d$$
$$= \langle \tilde{B}w, w | w |^{q-2} \rangle - \frac{1}{q} \langle \operatorname{div} b, | w |^q \rangle$$
$$\geq -\langle B_- w, w | w |^{q-2} \rangle.$$

Hence, applying (15.1), we arrive at

$$\langle b \cdot w, \phi \rangle \ge -\nu \langle \left| \nabla |w|^{\frac{q}{2}} \right|^{2} \rangle - c_{\nu} \langle |w|^{q} \rangle$$
$$= -\nu \frac{q^{2}}{4} J_{q} - c_{\nu} \langle |w|^{q} \rangle,$$

so (15.2) yields

$$(\mu - c_{\nu})\langle |w|^{q} \rangle + I_{q} + \left(q - 2 - \frac{q^{2}}{4}\nu\right)J_{q} \le \langle f, \phi \rangle. \tag{15.3}$$

Step 2. Let us estimate $\langle f, \phi \rangle$ in the previous inequality. To this end, we evaluate ϕ :

$$\langle f, \phi \rangle = -\langle f, |w|^{q-2} \Delta u \rangle - (q-2) \langle f, |w|^{q-3} w \cdot \nabla |w| \rangle. \tag{15.4}$$

(a) We estimate

$$|\langle f, |w|^{q-2} \Delta u \rangle| \le \varepsilon_0 \langle |w|^{q-2} |\Delta u|^2 \rangle + \frac{1}{4\varepsilon_0} \langle f^2, |w|^{q-2} \rangle, \tag{15.5}$$

where $\varepsilon_0 > 0$ will be chosen sufficiently small.

Let us deal with the first term in the RHS of (15.5). Representing $|\Delta u|^2 = |\nabla \cdot w|^2$ and integrating by parts twice, we obtain

$$\langle |w|^{q-2}|\Delta u|^2\rangle = -\langle \nabla |w|^{q-2} \cdot w, \Delta u\rangle + \sum_{i=1}^d \langle w_i \nabla |w|^{q-2}, \nabla w_i\rangle + I_q$$

$$\leq (q-2) \left[\frac{1}{4\varkappa} \langle |w|^{q-2} |\Delta u|^2\rangle + \varkappa J_q \right] + (q-2) \left(\frac{1}{2} I_q + \frac{1}{2} J_q \right) + I_q.$$

So, for any fixed $\varkappa > \frac{q-2}{4}$,

$$\left(1 - \frac{q-2}{4\varkappa}\right) \langle |w|^{q-2} |\Delta u|^2 \rangle \le I_q + (q-2) \left(\varkappa J_q + \frac{1}{2} I_q + \frac{1}{2} J_q\right).$$
(15.6)

Let us handle the second term in the RHS of (15.5):

$$\begin{split} \langle f^2, |w|^{q-2} \rangle &\leq \|f\|_{\frac{qd}{d+q-2}}^2 \|w\|_{\frac{qd}{d-2}}^{q-2} \\ &\leq c_S \|f\|_{\frac{qd}{d+q-2}}^2 \|\nabla |w|^{\frac{q}{2}}\|_2^{2\frac{(q-2)}{q}} = C \|f\|_{\frac{qd}{d+q-2}}^2 J_q^{\frac{q-2}{q}}, \quad C = \frac{c_S q^2}{4} \\ &\leq \frac{q-2}{q} C \varepsilon^{\frac{q}{q-2}} J_q + \frac{2}{q} C \varepsilon^{-\frac{q}{2}} \|f\|_{\frac{qd}{d+q-2}}^q. \end{split}$$

(b) We estimate

$$(q-2)|\langle -f, |w|^{q-3}w \cdot \nabla |w| \rangle| \le (q-2)J_q^{\frac{1}{2}} \langle f^2, |w|^{q-2} \rangle^{\frac{1}{2}}$$

$$\le (q-2) \left(\varepsilon_1 J_q + 4\varepsilon_1^{-1} \langle f^2, |w|^{q-2} \rangle \right),$$

where we estimate the very last term in the same way as above.

Substituting the above estimates in (15.4), we obtain

$$|\langle f, \phi \rangle| \le c\varepsilon_0 (I_q + J_q) + \frac{c_1(\varepsilon, \varepsilon_1)}{\varepsilon_0} J_q + \frac{c_2(\varepsilon, \varepsilon_1)}{\varepsilon_0} ||f||_{\frac{qd}{d+q-2}}^q, \tag{15.7}$$

where $c_1(\varepsilon, \varepsilon_1) > 0$ can be made as small as needed by first selecting ε_1 sufficiently small, and then selecting ε even smaller.

Step 3. Now, we return to (15.3). By (15.7),

$$(\mu - c_{\nu})\langle |w|^{q} \rangle + (1 - c\varepsilon_{0})I_{q} + (q - 2 - \frac{q^{2}}{4}\nu - c\varepsilon_{0} - \frac{c_{1}(\varepsilon, \varepsilon_{1})}{\varepsilon_{0}})J_{q} \leq \frac{c_{2}(\varepsilon, \varepsilon_{1})}{\varepsilon_{0}} ||f||_{\frac{qd}{d+q-2}}^{q}$$

By the pointwise inequality

$$|\nabla |w||^2 = \left|\frac{\sum_{i=1}^d w_i \nabla w_i}{|w|}\right|^2 \le \left(\frac{\sum_{i=1}^d |w_i| |\nabla w_i|}{|w|}\right)^2 \le \sum_{i=1}^d |\nabla w_i|^2,$$

we have

$$J_a \leq I_a$$
.

In particular, provided ε_0 is sufficiently small so that $1 - c\varepsilon_0 \ge 0$, we have $(1 - c\varepsilon_0)I_q \ge (1 - c\varepsilon_0)J_q$. Therefore,

$$\left(\mu - c_{\nu}\right)\langle |w|^{q}\rangle + \left(q - 1 - \frac{q^{2}}{4}\nu - c(\varepsilon_{0}, \varepsilon, \varepsilon_{1})\right)J_{q} \leq C(\varepsilon_{0}, \varepsilon, \varepsilon_{1})\|f\|_{\frac{qd}{d+\sigma-2}}^{q},\tag{15.8}$$

where constant $c(\varepsilon_0, \varepsilon, \varepsilon_1)$ can be made as small as needed by first selecting ε_1 sufficiently small, and then selecting ε even smaller (ε_0 is already fixed). Take $\mu_0 := c_{\nu}$. Recalling that $J_q = \frac{4}{q^2} \|\nabla |\nabla u|^{\frac{q}{2}}\|_2^2$, we obtain the required gradient estimate from (15.8).

Proof of assertion (i). Steps 1 and 3 do not change. Step 2 now consists of estimating $\langle |\mathbf{g}|f,\phi\rangle$, which we represent as

$$\langle |\mathsf{g}|f,\phi\rangle = -\langle |\mathsf{g}|f,|w|^{q-2}\Delta u\rangle - (q-2)\langle |\mathsf{g}|f,|w|^{q-3}w\cdot\nabla |w|\rangle.$$

(a') We have

$$|\langle |\mathsf{g}|f, |w|^{q-2} \Delta u \rangle| \le \varepsilon_0 \langle |w|^{q-2} |\Delta u|^2 \rangle + \frac{1}{4\varepsilon_0} \langle |\mathsf{g}|^2 f^2, |w|^{q-2} \rangle, \quad \varepsilon_0 > 0,$$

where $\langle |w|^{q-2}|\Delta u|^2\rangle$ is estimates in the same way as in (a) above, and

$$\begin{split} \langle |\mathsf{g}|^2 f^2, |w|^{q-2} \rangle &= \langle |\mathsf{g}|^{2-\frac{4}{q}} |w|^{q-2}, |\mathsf{g}|^{\frac{4}{q}} f^2 \rangle \\ &\leq \frac{q-2}{q} \varepsilon^{\frac{q}{q-2}} \langle |\mathsf{g}|^2 |w|^q \rangle + \frac{2}{q} \varepsilon^{-\frac{q}{2}} \langle \rho |\mathsf{g}|^2 f^q \rangle \\ &\quad (\text{we are using } \mathsf{g} \in \mathbf{F}_{\delta_1}) \\ &\leq \frac{q-2}{q} \varepsilon^{\frac{q}{q-2}} \left[\delta_1 \frac{q^2}{4} J_q + c_{\delta_1} \langle |w|^q \rangle \right] + \frac{2}{q} \varepsilon^{-\frac{q}{2}} \langle |\mathsf{g}|^2 f^q \rangle. \end{split}$$

(b') We estimate

$$(q-2)|\langle |\mathsf{g}|f, |w|^{q-3}w \cdot \nabla |w| \rangle| \le (q-2)J_q^{\frac{1}{2}} \langle |\mathsf{g}^2|f^2, |w|^{q-2} \rangle^{\frac{1}{2}}$$

$$\le (q-2)(\varepsilon_1 J_q + 4\varepsilon_1^{-1} \langle |\mathsf{g}|^2 f^2, |w|^{q-2} \rangle),$$

where we bound $\langle |\mathbf{g}|^2 f^2, |w|^{q-2} \rangle$ as in (a').

Now, arguing as above, we arrive at

$$\left(\mu - c_{\nu} - c_{0}(\varepsilon_{0}, \varepsilon_{1}, \varepsilon)\right) \langle |w|^{q} \rangle + \left(q - 1 - \frac{q^{2}}{4}\nu - c(\varepsilon_{0}, \varepsilon, \varepsilon_{1})\right) J_{q} \leq C(\varepsilon_{0}, \varepsilon, \varepsilon_{1}) \langle |\mathsf{g}|^{2} f^{q} \rangle,$$

where constant $c(\varepsilon_0, \varepsilon, \varepsilon_1)$ can be made as small as needed by selecting ε_1 sufficiently small and then selecting ε even smaller. So, taking $\mu_0 := c_\mu + c_0(\varepsilon_0, \varepsilon_1, \varepsilon)$, we obtain the required gradient estimate.

Proof of Theorem 7 in the case drift b satisfies condition (\mathbb{B}_1). One needs to estimate term $\langle b \cdot w, \phi \rangle$ in (15.2) differently. Indeed, b is no longer differentiable and hence one cannot integrate by parts. Instead, arguing as in [41], we evaluate the test function ϕ as

$$\langle b \cdot w, \phi \rangle = -\langle b \cdot w, |w|^{q-2} \Delta u \rangle - (q-2) \langle b \cdot w, |w|^{q-3} w \cdot \nabla |w| \rangle,$$

and then re-uses the elliptic equation to express Δu in terms of μu , $b \cdot w$ and f (or $|\mathsf{g}|f$). Then we repeat the argument from [41] up to the estimates on $|\langle f, \phi \rangle|$ (assertion (i)), which we take from Step 2 above.

16. Proof of Theorem 4

Let b_n be constructed as in Lemma 3, i.e.

$$b_n = E_{\varepsilon_n} b, \quad \varepsilon_n \downarrow 0,$$

so that b_n are bounded, smooth, converge to b locally in L^2 and, crucially, do not increase neither form-bound δ of b nor constant c_{δ} .

A comment regarding the case when b satisfies condition (\mathbb{B}_2). Below we use gradient bounds from Theorem 7 for vector fields b_n . The proof of these gradient bounds depends on a somewhat less restrictive condition than (\mathbb{B}_2), i.e. $b \in \mathbf{F}_{\delta}$, $\delta < \infty$, and

$$\langle (E_{\varepsilon_n} B_-) h, h \rangle \le \nu \langle |\nabla |h||^2 \rangle + c_\nu \langle |h|^2 \rangle,$$
 (16.1)

where B_{-} is the negative part of matrix $(\nabla_k b^i)_{k,i=1}^d - \frac{\operatorname{div} b}{q}I$. (Indeed, if B_{+} denotes the positive part of the last matrix, we have

$$(\nabla_k b_n^i)_{k,i=1}^d - \frac{\operatorname{div} b_n}{q} I = E_{\varepsilon_n} B_+ - E_{\varepsilon_n} B_-, \quad E_{\varepsilon_n} B_{\pm} \ge 0,$$

and can repeat the proof of Theorem 7 for b_n and $E_{\varepsilon_n}B_{-}$.) By Lemma 12, inequality (16.1) does hold with constants $\nu = \sum_{j=1}^{d} \nu_j$ and $c_{\nu} = \sum_{j=1}^{d} c_{\nu_j}$ that are, obviously, independent of $\{\varepsilon_n\}$, and so the constants in the gradient bounds in Theorem 7 for b_n do not depend on n.

Proof of assertion (i). Let $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ be the strong Markov family of martingale solutions to (4.1) constructed in Theorem 3. Fix some y. Our goal is prove the following estimate: there exists generic $q > (d-2) \vee 2$ and C such that, for all $g \in \mathbf{F}_{\delta_1}$, $\delta_1 < \infty$, and all λ greater than some generic λ_0 ,

$$\mathbb{E}_{\mathbb{P}_y} \int_0^\infty e^{-\lambda s} |\mathsf{g} f|(\omega_s) ds \le C \|\mathsf{g} |f|^{\frac{q}{2}} \|_2^{\frac{2}{q}} \tag{16.2}$$

for all $f \in C_c$. Let g_m the bounded smooth regularization of g constructed according to Lemma 3. Using the gradient estimate of Theorem 7(i), after applying the Sobolev embedding theorem twice, we obtain

$$\mathbb{E}_{\mathbb{P}_y^n} \int_0^\infty e^{-\lambda s} |\mathsf{g}_m f|(\omega_s) ds \leq C \|\mathsf{g}_m |f|^{\frac{q}{2}} \|_2^{\frac{2}{q}}, \quad n, m = 1, 2, \dots,$$

where \mathbb{P}_x^n is the martingale solution of the regularized SDE

$$Y(t) = y - \int_0^t b_n(Y(s))ds + \sqrt{2}B(t), \quad t \ge 0$$

and, by the construction of \mathbb{P}_x in the proof of Theorem 3, $\mathbb{P}_x^n \to \mathbb{P}_x$ weakly (we pass to a subsequence of $\{b_n\}$ if necessary). Thus, we have

$$\mathbb{E}_{\mathbb{P}_y} \int_0^\infty e^{-\lambda s} |\mathsf{g}_m f|(\omega_s) ds \leq C \|\mathsf{g}_m |f|^{\frac{q}{2}} \|_2^{\frac{2}{q}}, \quad m = 1, 2, \dots$$

Fatou's lemma applied in m now yields (16.2) and thus ends the proof of (i).

Proof of assertion (i'). Let $\{\mathbb{P}_y^1\}_{y\in\mathbb{R}^d}$, $\{\mathbb{P}_y^2\}_{y\in\mathbb{R}^d}$ be two Markov families of martingale solutions to SDE

$$Y(t) = y - \int_0^t b(Y(s))ds + \sqrt{2}B(t), \quad t \ge 0.$$

Fix some y. By our assumption, there exists $q > (d-2) \vee 2$ such that, for all $g \in \mathbf{F}_{\delta_1}$, $\delta_1 < \infty$, and all λ greater than some generic λ_0 ,

$$\mathbb{E}_{\mathbb{P}_{y}^{i}} \int_{0}^{\infty} e^{-\lambda s} |\mathsf{g}f|(\omega_{s}) ds \leq C \|\mathsf{g}|f|^{\frac{q}{2}} \|_{2}^{\frac{2}{q}} \tag{16.3}$$

for all $f \in C_c$. Let v_n be the classical solution to equation

$$(\lambda - \Delta + b_n \cdot \nabla)v_n = -F,$$

where $F \in C_c(\mathbb{R}^d)$. We will need the weight $\rho(x) = (1 + k|x|^2)^{-\beta}$, k > 0, where constant β is fixed greater than $\frac{d}{2}$ so that $\rho \in L^1(\mathbb{R}^d)$. By Itô's formula applied to $e^{-\lambda t}\rho v_n$, we have

$$\mathbb{E}_{\mathbb{P}_{y}^{i}}[e^{-\lambda t}(\rho v_{n})(\omega_{t})] = \rho(y)v_{n}(y) + \mathbb{E}_{\mathbb{P}_{y}^{i}}\int_{0}^{t} \rho e^{-\lambda s}(-\lambda + \Delta - b \cdot \nabla)v_{n}(\omega_{s})ds - S_{n},$$

where S_n is the remainder term given by

$$S_n := \mathbb{E}_{\mathbb{P}_y^i} \int_0^t e^{-\lambda s} [-(\Delta \rho) v_n - 2\nabla \rho \cdot \nabla v_n + b \cdot (\nabla \rho) v_n](\omega_s) ds.$$

So,

$$\mathbb{E}_{\mathbb{P}_{y}^{i}}[e^{-\lambda t}(\rho v_{n})(\omega_{t})] = \rho(y)v_{n}(y) + \mathbb{E}_{\mathbb{P}_{y}^{i}} \int_{0}^{t} \rho e^{-\lambda s} F(\omega_{s}) ds$$
$$- \mathbb{E}_{\mathbb{P}_{y}^{i}} \int_{0}^{t} [e^{-\lambda s} \rho(b - b_{n}) \cdot \nabla v_{n}](\omega_{s}) ds - S_{n}. \tag{16.4}$$

Proposition 7. For every k > 0,

$$\mathbb{E}_{\mathbb{P}_{y}^{i}} \int_{0}^{t} e^{-\lambda s} [\rho(b-b_{n}) \cdot \nabla v_{n}](\omega_{s}) ds \to 0$$

as $n \uparrow \infty$ uniformly in t > 0.

Proof. We have

$$|\mathbb{E}_{\mathbb{P}_{y}^{i}} \int_{0}^{t} [e^{-\lambda s} \rho(b - b_{n}) \cdot \nabla v_{n}](\omega_{s}) ds| \leq |\mathbb{E}_{\mathbb{P}_{y}^{i}} \int_{0}^{\infty} [e^{-\lambda s} \rho(b - b_{n}) \cdot \nabla v_{n}](\omega_{s}) ds|$$
(we apply (16.3) with $g := \rho(b - b_{n}) \in \mathbf{F}_{2\delta}$)
$$\leq K \|\rho(b - b_{n})| \nabla v_{n}|^{\frac{q}{2}} \|_{2}^{\frac{q}{q}}.$$

In turn, for a $0 < \theta < 1$, we have

$$\|\rho(b-b_n)|\nabla v_n|^{\frac{q}{2}}\|_2 \le \|\rho(b-b_n)\|_2^{\theta}\|\rho(b-b_n)|\nabla v_n|^{\frac{q}{2(1-\theta)}}\|_2^{1-\theta}.$$
(16.5)

Regarding the second multiple in the RHS of (16.5): we assume that θ is chosen to be sufficiently close to 0 so that $\frac{q}{1-\theta} > (d-2) \vee 2$. Then, by $b-b_n \in \mathbf{F}_{2\delta}$,

$$\|\rho(b-b_n)|\nabla v_n|^{\frac{q}{2(1-\theta)}}\|_2^2 \le \|(b-b_n)|\nabla v_n|^{\frac{q}{2(1-\theta)}}\|_2^2$$

$$\le 2\delta \|\nabla|\nabla v_n|^{\frac{q}{2(1-\theta)}}\|_2^2 + 2c_\delta \||\nabla v_n|^{\frac{q}{2(1-\theta)}}\|_2^2.$$

Hence, by the gradient estimate of Theorem 7(i), $\sup_n \|\rho(b-b_n)|\nabla v_n|^{\frac{q}{2(1-\theta)}}\|_2^2 < \infty$. The first multiple in the RHS of (16.5):

$$\|\rho(b-b_n)\|_2^2 \le \langle \mathbf{1}_{B_R(0)}|b-b_n|^2 \rangle + \langle (1-\mathbf{1}_{B_R(0)})\rho, \rho|b-b_n|^2 \rangle$$

$$\le \langle \mathbf{1}_{B_R(0)}|b-b_n|^2 \rangle + (1+kR^2)^{-\beta} \langle \rho|b-b_n|^2 \rangle.$$

Since $b_n \to b$ in L^2_{loc} , the first integral can be made as small as needed (uniformly in R) by selecting n sufficiently large. In the second integral $\sup_n \langle \rho | b - b_n |^2 \rangle < \infty$, since, by $b - b_n \in \mathbf{F}_{2\delta}$,

$$\langle \rho | b - b_n |^2 \rangle \le 2\delta \langle (\nabla \sqrt{\rho})^2 \rangle + 2c_\delta \langle \rho \rangle,$$

so it remains to apply $|\nabla \rho| \leq \beta \sqrt{k} \rho$. At the same time, $(1 + kr^2)^{-\beta}$ can be made as small as needed by selecting r sufficiently large. This completes the proof.

Proposition 8. $S_n \to 0$ as $k \downarrow 0$ uniformly in n and t.

Proof. Using $|\nabla \rho| \leq \beta \sqrt{k\rho}$, $|\Delta \rho| \leq \beta^2 k$, we have

$$|S_n| \le \sqrt{k} C \mathbb{E}_{\mathbb{P}^i_y} \int_0^t [\rho |v_n| + 2\rho |\nabla v_n| + \rho |b| |v_n|](\omega_s) ds.$$

Now we can argue as in the proof of the previous proposition, using additionally $||v_n||_{\infty} \leq \lambda^{-1}||F||_{\infty}$, to show that $\sup_n \mathbb{E}_{\mathbb{P}^i_y} \int_0^t [\rho|v_n|+2\rho|\nabla v_n|+\rho|b||v_n|](\omega_s)ds < \infty$. In fact, in this case the proof is easier since none of the terms contains simultaneously b and ∇v_n . Selecting k sufficiently small, we can make S_n as small as needed.

We now complete the proof of assertion (i'). Let us note that, for every k > 0,

$$\mathbb{E}_{\mathbb{P}_n^i}[e^{-\lambda t}\rho v_n(\omega_t)] \to 0$$
 as $t \to \infty$ uniformly in n .

Indeed, $||v_n||_{\infty} \leq \lambda^{-1} ||F||_{\infty}$, so $|\mathbb{E}_{\mathbb{P}^i_y}[e^{-\lambda t}\rho v_n(\omega_t)]| \leq \lambda^{-1}e^{-\lambda t}$, which yields the required. Combining this result with Propositions 7 and 8, and taking into account that, by Theorem 5(iv), $\{v_n\}$ converge uniformly as $n \to \infty$ to a continuous function v, we obtain from (16.4) upon taking $n \to \infty$ and then taking $k \downarrow 0$:

$$0 = v(y) + \mathbb{E}_{\mathbb{P}_y^i} \int_0^\infty e^{-\lambda s} F(\omega_s) ds, \quad i = 1, 2.$$

Taking into account the continuity of F and ω , and invoking the uniqueness of Laplace transform, we obtain that $\mathbb{E}_{\mathbb{P}^1_y}F(\omega_t)=\mathbb{E}_{\mathbb{P}^2_y}F(\omega_t)$ for all $F\in C_c$, t>0. We deduce from here that the one-dimensional distributions of \mathbb{P}^1_y and \mathbb{P}^2_y coincide. Since we are dealing with Markov families of probability measures, we conclude that $\mathbb{P}^1_y=\mathbb{P}^2_y$ for every $y\in\mathbb{R}^d$.

Proof of assertion (ii). The proof follows closely the proof of (i), but uses the gradient estimate of Theorem 7(i) for $q > (d-2) \vee 2$ chosen closely to $(d-2) \vee 2$. In fact, this proof is easier since we no longer need to take care of extra form-bounded vector fields \mathbf{g} as in (i).

Proof of assertion (ii''). We modify the previous proof of (i'). By our assumption,

$$\mathbb{E}_{\mathbb{P}_{y}^{i}} \int_{0}^{\infty} e^{-\lambda s} |f|(\omega_{s}) ds \leq C \|f\|_{\frac{d}{2-\varepsilon} \wedge \frac{2}{1-\varepsilon}}, \quad \forall f \in C_{c}, \quad \lambda > \lambda_{0}.$$
 (16.6)

The analogue of Proposition 7 is proved as follows. Clearly, hypothesis

$$(1+|x|^{-2})^{-\beta}|b|^{\frac{d}{2-\varepsilon_1}\vee\frac{2}{1-\varepsilon_1}}\in L^1, \quad \varepsilon_1\in]\varepsilon,1[$$

implies that, for any $k>0,\, \rho|b|^{\frac{d}{2-\varepsilon_1}\vee\frac{2}{1-\varepsilon_1}}\in L^1.$ We have

$$|\mathbb{E}_{\mathbb{P}_{y}^{i}} \int_{0}^{t} [e^{-\lambda s} \rho(b - b_{n}) \cdot \nabla v_{n}](\omega_{s}) ds| \leq |\mathbb{E}_{\mathbb{P}_{y}^{i}} \int_{0}^{\infty} [e^{-\lambda s} \rho(b - b_{n}) \cdot \nabla v_{n}](\omega_{s}) ds|$$
(we apply (16.6) using Fatou's lemma)
$$\leq K \|\rho(b - b_{n}) \cdot \nabla v_{n}\|_{r} \quad r := \frac{d}{2 - \varepsilon} \wedge \frac{2}{1 - \varepsilon}$$

$$\leq K \|\rho(b - b_{n})\|_{s'} \|\nabla v_{n}\|_{s}, \quad \frac{1}{s'} + \frac{1}{s} = \frac{1}{r}, \quad (16.7)$$

where $s' = \frac{d}{2-\varepsilon_1} \vee \frac{2}{1-\varepsilon_1}$ and $s = \frac{q_*d}{d-2}$, where q_* was defined in assertion (ii'') of Theorem 4 that we are proving. Theorem 7(ii), which applies by our assumptions on δ , ν and q_* in the end of assertion (ii''), and the Sobolev embedding theorem, yield

$$\sup_{n} \|\nabla v_n\|_{\frac{q_*d}{d-2}} < \infty.$$

Therefore, the second multiple in the RHS of (16.7) is uniformly (in n) bounded.

In turn, for every fixed k, the first multiple in the RHS of (16.7) tends to zero as $n \to \infty$. Indeed, since $0 < \rho \le 1$, we have

$$\sup_{n} \|\rho^{s'} b_n^{s'}\|_1 \le \sup_{n} \|\rho b_n^{s'}\|_1 < \infty,$$

where the finiteness is seen, after integrating by parts, from $E_{\varepsilon_n} \rho \leq C \rho$ with constant C independent of n (here we simply use the fact that the Friedrichs mollifier is a convolution with a function having compact support) and our hypothesis $\|\rho|b|^{s'}\|_1 < \infty$. Now, we represent

$$\|\rho(b-b_n)\|_{s'} = \|\mathbf{1}_{B_R(0)}(b-b_n)\|_{s'} + \|(1-\mathbf{1}_{B_R(0)})\rho(b-b_n)\|_{s'}$$

$$\leq \|\mathbf{1}_{B_R(0)}(b-b_n)\|_{s'} + (1+kR^2)^{-\beta(s'-1)}(\langle \rho b^{s'} \rangle + \langle \rho b_n^{s'} \rangle).$$

The second term can be made as small as needed by selecting R sufficiently large (uniformly in n). Then, for R thus fixed, the first term can be made as small as needed by selecting n sufficiently large, since $b_n \to b$ in $L_{\text{loc}}^{s'}$ by the properties of Friedrichs mollifier.

Arguing as above, we prove $\sup_n \mathbb{E}_{\mathbb{P}^i_y} \int_0^t [\rho|v_n| + 2\rho|\nabla v_n| + \rho|b_n||v_n|](\omega_s) ds < \infty$, and hence have the analogue of Proposition 8.

The rest of the proof of (ii') repeats the proof of (i').

APPENDIX A. A DESINGULARIZATION THEOREM FROM [38]

Let X be a locally compact topological space, and μ a σ -finite Borel measure on X. In what follows, $L^r = L^r(X, \mu)$ $(1 \le r \le \infty)$. Let j > 1, put $j' := \frac{j}{j-1}$.

Let Λ be the generator of a strongly continuous semigroup $e^{-t\Lambda}$ on L^r for some r > 1, such that for some constants c, j > 1, for all t > 0,

$$||e^{-t\Lambda}||_{r\to\infty} \le ct^{-\frac{j'}{r}}. (S_1)$$

We consider a family of weights $\varphi = \{\varphi_s\}_{s>0}$ in X such that

$$0 \le \varphi_s, \frac{1}{\varphi_s} \in L^1_{loc}(X, \mu) \quad \text{for all } s > 0,$$
 (S₂)

$$\inf_{s>0, x\in X} \varphi_s(x) \ge c_0 > 0. \tag{S_3}$$

Theorem 8. Assume that conditions (S_1) - (S_3) hold and there exists constant c_1 , independent of s, such that, for all $0 < t \le s$,

$$\|\varphi_s e^{-t\Lambda} \varphi_s^{-1} f\|_1 \le c_1 \|f\|_1, \quad f \in L^1 \cap L^\infty.$$
 (S₄)

Then, for each t > 0, $e^{-t\Lambda}$ is an integral operator, and there is a constant $C = C(j, c_1, c_0)$ such that, up to change of $e^{-t\Lambda}(x, y)$ on a measure zero set, the weighted Nash initial estimate

$$|e^{-t\Lambda}(x,y)| \le Ct^{-j'}\varphi_t(y) \tag{A.1}$$

is valid for μ a.e. $x, y \in X$.

For the sake of keeping the paper self-contained, we reproduce here the proof of Theorem 8 from [38].

Proof of Theorem 8. 1. We will use a weighted variant of the Coulhon-Raynaud extrapolation theorem. Put

$$0 \leq \psi \in L^1 + L^{\infty}, \quad \|f\|_{p,\sqrt{\psi}} := \langle |f|^p \psi \rangle^{1/p}.$$

Let $U^{t,\theta}$ be a two-parameter family of operators

$$U^{t,\theta}f = U^{t,\tau}U^{\tau,\theta}f, \qquad f \in L^1 \cap L^\infty, \quad 0 \le \theta < \tau < t \le \infty.$$

If for some $1 \le p < q < r \le \infty$, $\nu > 0$

$$||U^{t,\theta}f||_p \le M_1 ||f||_{p,\sqrt{\psi}},$$

$$||U^{t,\theta}f||_r \le M_2 (t-\theta)^{-\nu} ||f||_q$$

for all (t, θ) and $f \in L^1 \cap L^{\infty}$, then

$$||U^{t,\theta}f||_r \le M(t-\theta)^{-\nu/(1-\beta)}||f||_{p,\sqrt{\psi}},$$
 (A.2)

where $\beta = \frac{r}{q} \frac{q-p}{r-p}$ and $M = 2^{\nu/(1-\beta)^2} M_1 M_2^{1/(1-\beta)}$. Here is the proof of (A.2) for reader's convenience. Put $t_{\theta} := \frac{t+\theta}{2}$. We have

$$||U^{t,\theta}f||_r \leq M_2(t-t_{\theta})^{-\nu} ||U^{t_{\theta},\theta}f||_q$$

$$\leq M_2(t-t_{\theta})^{-\nu} ||U^{t_{\theta},\theta}f||_r^{\beta} ||U^{t_{\theta},\theta}f||_p^{1-\beta}$$

$$\leq M_2 M_1^{1-\beta} (t-t_{\theta})^{-\nu} ||U^{t_{\theta},\theta}f||_r^{\beta} ||f||_{p,\sqrt{\psi}}^{1-\beta},$$

and hence

$$(t-\theta)^{\nu/(1-\beta)} \|U^{t,\theta}f\|_r / \|f\|_{p,\sqrt{\psi}} \le M_2 M_1^{1-\beta} 2^{\nu/(1-\beta)} \left[(t-\theta)^{\nu/(1-\beta)} \|U^{t_\theta,\theta}f\|_r / \|f\|_{p,\sqrt{\psi}} \right]^{\beta}.$$

Setting $R_{2T} := \sup_{t-\theta \in]0,T]} \left[(t-\theta)^{\nu/(1-\beta)} \|U^{t,\theta}f\|_r / \|f\|_{p,\sqrt{\psi}} \right]$, we obtain from the last inequality that $R_{2T} \leq M^{1-\beta} (R_T)^{\beta}$. But $R_T \leq R_{2T}$, and so $R_{2T} \leq M$. This gives us (A.2).

2. We are in position to complete the proof of Theorem 8. By (S_4) and (S_3) ,

$$||e^{-t\Lambda}h||_{1} \leq c_{0}^{-1}||\varphi_{s}e^{-t\Lambda}\varphi_{s}^{-1}\varphi_{s}h||_{1}$$

$$\leq c_{0}^{-1}c_{1}||h||_{1,\sqrt{\varphi_{s}}}, \qquad h \in L_{\text{com}}^{\infty}.$$

The latter, (S_1) and the Coulhon-Raynaud extrapolation theorem with $\psi := \varphi_s$ yield

$$||e^{-t\Lambda}f||_{\infty} \le Mt^{-j'}||\varphi_s f||_1, \quad 0 < t \le s, \quad f \in L_c^{\infty}.$$

Note that (S_1) verifies the assumptions of the Dunford-Pettis theorem, which yields that $e^{-t\Lambda}$ is an integral operator. Therefore, taking s = t in the previous estimate, we obtain (A.1).

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