STRONG SOLUTIONS OF SDES WITH SINGULAR (FORM-BOUNDED) DRIFT VIA RÖCKNER-ZHAO APPROACH

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Abstract. We use the approach of Röckner-Zhao to prove strong well-posedness for SDEs with singular drift satisfying some minimal assumptions.

1. Introduction and result

1. Consider stochastic differential equation (SDE)

$$X_t^x = x + \int_0^t b(s, X_s^x)ds + W_t, \quad 0 \leq t \leq T, \tag{1}$$

where $x \in \mathbb{R}^d$, $d \geq 3$, $b : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is a Borel measurable vector field (drift), and $\{W_t\}_{0 \leq t \leq T}$ is a Brownian motion on a complete filtered probability space $(\Omega, \{\mathcal{F}_t\}_{0\leq t \leq T}, \mathcal{F}, \mathbf{P})$.

One of the central problems in the theory of diffusion processes is the problem of strong well-posedness of SDE (1) under minimal assumptions on a locally unbounded drift b, for every starting point $x \in \mathbb{R}^d$. The following are the milestone results. Veretennikov [V] was first who proved strong well-posedness of (1) for discontinuous drifts $b \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$. Krylov-Röckner [KrR] established strong well-posedness assuming that the drift in the sub-critical Ladyzhenskaya-Prodi-Serrin class

$$b \in L^p(\mathbb{R}, L^q(\mathbb{R}^d)), \quad \frac{d}{q} + \frac{2}{p} < 1, \quad p > 2, \quad q > d. \tag{2}$$

Beck-Flandoli-Gubinelli-Maurelli [BFGM] established strong existence and uniqueness for drifts in the critical Ladyzhenskaya-Prodi-Serrin class

$$b \in L^p(\mathbb{R}, L^q(\mathbb{R}^d)), \quad \frac{d}{q} + \frac{2}{p} \leq 1, \quad p \geq 2, \quad q \geq d, \tag{LPS}$$

but only for a.e. starting point $x \in \mathbb{R}^d$. A major step forward was made recently by Röckner-Zhao [RZ] who established strong existence and uniqueness for (1) with drift b in the critical Ladyzhenskaya-Prodi-Serrin class (LPS) ($p > 2$) for every $x \in \mathbb{R}^d$. Another major advancement is the series of papers [Kr1, Kr2, Kr3, Kr4] where Krylov proved strong well-posedness of (1), for every $x \in \mathbb{R}^d$, for $|b| \in L^d$ and beyond, in a large Morrey class of time-inhomogeneous drifts (in terms of the Morrey norm (4), one has to have $\|b\|_{M_s}$, $s > \frac{d}{2} \vee 2$, sufficiently small).

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The method of Röckner-Zhao is different from the methods used in the other cited papers, and is based on a relative compactness criterion for random fields on the Wiener-Sobolev space. Their proof of uniqueness uses Cherny's theorem [C] (strong existence + weak uniqueness $\Rightarrow$ strong uniqueness). The method of [RZ] is a far-reaching strengthening of the methods of Meyer-Brandis and Proske [MP], Mohammed-Nilsen-Proske [MNP] (for $b \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$) and Rezakhanlou [R] (for b in (2)). We refer again to [RZ] for a comprehensive survey of these and other important results on strong well-posedness of SDE (1).

We show in this paper that the method of Röckner-Zhao works, with few modifications, for a larger class of form-bounded drifts. Together with the weak uniqueness result from [KM], their method yields strong well-posedness of SDE (1) with form-bounded drift (Theorem 1).

Definition. A locally square integrable vector field $b : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is said to be form-bounded if there exist a constant $\delta > 0$ such that for a.e. $t \in \mathbb{R}$ the following quadratic form inequality holds:

$$\|b(t,\cdot)\varphi\|_2^2 \leq \delta \|\nabla \varphi\|_2^2 + g_\delta(t)\|\varphi\|_2^2 \tag{3}$$

for all $\varphi \in W^{1,2}$, for some function $0 \leq g_\delta \in L^1_{\rm loc}(\mathbb{R})$.

Throughout the paper, $\|\cdot\|_p$ denotes the norm in the Lebesgue space $L^p := L^p(\mathbb{R}^d, dx)$; $W^{1,p} := W^{1,p}(\mathbb{R}^d, dx)$ is the Sobolev space.

Condition (3) will be written as $b \in \mathbf{F}_\delta$. This is essentially the largest class of vector fields b, defined in terms of $|b|$, that provides an L^2 theory of divergence-form operator $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$. See [K2] for detailed discussion.

Example 1. The critical Ladyzhenskaya-Prodi-Serrin class (LPS) is contained in the class of form-bounded vector fields. For $q = d$ and $p = \infty$ this is an immediate consequence of the Sobolev embedding theorem:

$$\|b(t,\cdot)\varphi\|_2^2 \leq \|b(t,\cdot)\|_d^2 \|\varphi\|_{\frac{2d}{d-2}}^2 \leq C_S \|b(t,\cdot)\|_d^2 \|\nabla \varphi\|_2^2,$$

so $\delta = C_S \sup_{t\in\mathbb{R}} \|b(t,\cdot)\|_d^2$ and $g_\delta = 0$ (for $q > d$ and $p < \infty$ using, additionally, a simple interpolation argument, in which case g is in general non-zero, see e.g. [KM] for the proof). Moreover, if e.g. $b \in C_c(\mathbb{R}, L^d(\mathbb{R}^d))$, then form-bound δ can be chosen arbitrarily small at expense of increasing g_δ.

Example 2. Another subclass of (3), which is considerably larger than $L^\infty(\mathbb{R}, L^d)$, consists of vector fields b such that $b(t,\cdot)$ belongs, uniformly in $t \in \mathbb{R}$, to the scaling-invariant Morrey class $M_{2+\varepsilon}$. That is,

$$\sup_{t\in\mathbb{R}} \|b(t,\cdot)\|_{M_{2+\varepsilon}} = \sup_{t\in\mathbb{R}} \sup_{r>0, x\in\mathbb{R}^d} r\left(\frac{1}{|B_r|}\int_{B_r(x)} |b(t,\cdot)|^{2+\varepsilon} dx\right)^{\frac{1}{2+\varepsilon}} < \infty \tag{4}$$

where $B_r(x)$ is the ball of radius r centered at x, and ε is fixed arbitrarily small. Then, by a result in [F] (see also [CFr]),

$$b \in \mathbf{F}_\delta \quad \text{with } \delta = C \sup_{t\in\mathbb{R}} \|b(t,\cdot)\|_{M_{2+\varepsilon}} \text{ and } g_\delta = 0$$

for appropriate constant C. Note that Morrey M_s becomes larger as s becomes smaller.

EXAMPLE 3. Morrey class (4) contains vector fields b with $\|b\|_{L^\infty(\mathbb{R},L^{d,w})} < \infty$.

Recall that the norm in the weak L^d space is defined as

$$\|h\|_{L^{d,w}} := \sup_{s>0} s|\{x \in \mathbb{R}^d : |h(x)| > s\}|^{1/d}.$$

(Clearly, $L^d \subset L^{d,w}$, but not vice versa, e.g. $h(x) = |x|^{-1}$ is in $L^{d,w}$ but not in L^d.)

Let us add that the attracting drift

$$b(x) = -\frac{d-2}{2}\sqrt{\delta}|x|^{-2}x,$$

which is contained[1] in $\mathbf{F}_\delta$ with $g_\delta = 0$ (and is contained in Examples 2 and 3, but not in Example 1) has critical singularity at the origin. That is, if $\delta > 0$ is too large, then SDE (1) with starting point $x = 0$ does not even have a weak solution. But, if δ is sufficiently small, then this SDE is strongly well-posed, see Theorem 1. (In fact, the critical value of δ for weak solvability, at least in high dimensions, is $\delta = 4$, see [KS].)

An equivalent form of the a.e. inequality (3) is: for every $-\infty < t_1 < t_2 < \infty$,

$$\int_{t_1}^{t_2} \|b(t)\psi(t)\|_2^2 dt \leq \delta \int_{t_1}^{t_2} \|\nabla\psi(t)\|_2^2 dt + \int_{t_1}^{t_2} g_\delta(t)\|\psi(t)\|_2^2 dt$$

for all $\psi \in L^\infty(\mathbb{R}, W^{1,2})$.

The class of form-bounded drifts is well known in the literature on parabolic equations, see Semënov [S] and references therein.

2. Our goal here is to prove a principal result: the SDE (1) with drift b having form-bounded singularities is strongly well-posed. So, we will require in this paper, for simplicity,

(A) b has compact support and $g_\delta = 0$ (the last assumption can be removed, see Remark 2).

Fix $T > 0$.

Theorem 1. *Let $d \geq 3$. Assume that $b \in \mathbf{F}_\delta$ and satisfies (A). Then, provided that form-bound δ is sufficiently small, for every $x \in \mathbb{R}^d$, SDE (1) has a strong solution X_t^x. This strong solution satisfies the following Krylov-type bounds:*

1) *For a given $q \in]d, \delta^{-\frac{1}{2}}[$ and any vector field $\mathsf{g} \in \mathbf{F}_{\delta_1}$, $\delta_1 < \infty$,*

$$\mathbf{E}\int_0^T |\mathsf{g}h|(\tau, X_{0,\tau}^x)d\tau \leq c\|\mathsf{g}|h|^{\frac{q}{2}}\|^{\frac{2}{q}}_{L^2([0,T]\times\mathbb{R}^d)} \quad \textit{for all } h \in C_c([0,T]\times\mathbb{R}^d). \tag{5}$$

2) *For a given $\mu > \frac{d+2}{2}$, there exists constant C such that*

$$\mathbf{E}\Big[\int_0^T |h(\tau, X_{0,\tau}^x)|d\tau\Big] \leq C\|h\|_{L^\mu([0,T]\times\mathbb{R}^d)} \quad \textit{for all } h \in C_c([0,T]\times\mathbb{R}^d). \tag{6}$$

[1] and not contained in any $\mathbf{F}_{\delta'}$ with $\delta' < \delta$ regardless of the choice of $g_{\delta'}$

Solution X_t^x is unique among strong solutions to (1) *that satisfy* (5) *for some* $q \in]d, \delta^{-\frac{1}{2}}[$ *with* $\mathsf{g} = 1$ *and with* $\mathsf{g} = b$.

If, in addition to our hypothesis on b, one has $|b| \in L^{\frac{d+2}{2}+\varepsilon}$ *for some* $\varepsilon > 0$, *then* X_t^x *is unique among strong solutions to* (1) *that satisfy* (6).

The proof of Theorem 1 follows closely [RZ], except the proof of Proposition 1 (this is Lemma 4.2(a) in [RZ]). In [RZ], this result is proved using Sobolev regularity estimates for solutions of parabolic equations with distributional right-hand side (these estimates, developing earlier work of Krylov, are quite strong and are interesting on their own). We prove Proposition 1 using a simpler argument which uses weaker estimates on solutions of parabolic equations, and thus allows to treat a larger class of form-bounded drifts. We also use some estimates from paper [KM] that deals with weak well-posedness of SDE (1) with drift $b \in \mathbf{F}_\delta$.

It should be added that for the drifts $b \in C([0,T], L^d)$ or $b \in$ (LPS) $(2 < p < \infty)$ considered in [RZ] the form-bound δ can be chosen arbitrarily small. In other words, replacing drift b by cb, for arbitrarily large constant c, does not affect strong well-posedness of SDE (1). The latter is important in [RZ] since they apply their strong well-posedness result to Navier-Stokes equations.

One can also prove strong well-posedness of SDE (1) with form-bounded drift $b = b(x)$ using the approach of [BFGM], but only for a.e. $x \in \mathbb{R}^d$, see [KSS].

Remark 1 (On weak solutions). Weak existence and uniqueness for (1) is known to hold for larger classes of drifts than the class $\mathbf{F}_\delta$, see [KS2] dealing with weakly form-bounded drifts (time-homogeneous case) and [K] dealing with time-inhomogeneous drifts in essentially the largest possible Morrey class. See also [RZ2]. In a recent paper [Kr5], Krylov proved weak existence and uniqueness for SDEs with VMO diffusion coefficients and time-inhomogeneous drift in a large Morrey class containing (LPS) (in terms of Example 2, this is the Morrey class with exponent $2+\varepsilon$ replaced by $\frac{d}{2}+\varepsilon$; note that in dimension $d=3$ Krylov's Morrey class is larger than $\mathbf{F}_\delta$). We refer to [RZ2] for a survey of the literature on weak solutions of (1).

2. Proof of Theorem 1

2.1. **Notations.** Set $\Delta_n(T_0,T_1) := \{(t_1,\dots,t_n) \mid T_0 \leq t_1 \leq \dots \leq t_n \ \leq T_1\}$ and put $\Delta_n(T) := \Delta_n(0,T)$.

Let $\nabla_i := \partial_{x_i}$, $x = (x_1,\dots,x_d) \in \mathbb{R}^d$.

Let $\mathbf{E}_{\mathcal{F}_t}$ denote conditional expectation with respect to σ-algebra $\mathcal{F}_t$.

Put

$$\langle f, g \rangle = \langle fg \rangle := \int_{\mathbb{R}^d} fg dx.$$

2.2. **Some estimates.** Let $f_i \in L^2_{\rm loc}(\mathbb{R}^{d+1})$ $(i \geq 1)$ be form-bounded:

$$\|f_i(t,\cdot)\varphi\|_2^2 \leq \nu \|\nabla \varphi\|_2^2 \tag{7}$$

for some $\nu > 0$. Also, in this section, f_i are smooth. Additionally, let us assume that:

(A') all f_i have compact supports contained in $\mathbb{R} \times B_R(0)$ for a fixed $R > 0$ (independent of i).

In this subsection, $b \in \mathbf{F}_\delta$ is additionally assumed to be smooth. However, the constants in the estimates below will not depend on smoothness or boundedness of b and f_i.

By the classical theory, there exists a unique strong solution X_t^x to

$$X_t^x = x + \int_0^t b(\tau, X_\tau^x)d\tau + W_t.$$

Let $0 \leq T_0 \leq T_1 \leq T$.

Proposition 1. *There exist positive constants C_0, K such that, for every $n \geq 1$,*

$$\int_{\mathbb{R}^d} \biggl| \mathbf{E} \int_{\Delta_n(T_0,T_1)} \prod_{i=1}^n \nabla_{\alpha_i} f_i(t_i, X_{t_i}^x) dt_1 \dots dt_n \biggr|^2 dx \leq C_0 K^n (T_1 - T_0),$$

where $1 \leq \alpha_i \leq d$ $(i \geq 1)$. Moreover, K can be made as small as needed by assuming that form-bounds δ and ν in (3), (7) *are sufficiently small.*

Proof. Fix n, put $u_{n+1} = 1$ and define consecutively

$$g_k = (\nabla_{\alpha_k} f_k) u_{k+1}, \quad k = 1, \dots, n,$$

where u_k solves the terminal-value problem on $[T_0, T_1]$

$$\partial_t u_k + \frac{1}{2}\Delta u_k + b \cdot \nabla u_k + g_k = 0, \quad u_k(T_1) = 0. \tag{8}$$

Then, repeating the argument in [RZ, Proof of Lemma 4.2],

$$\mathbf{E}_{\mathcal{F}_{T_0}} \int_{\Delta_n(T_0,T_1)} \prod_{i=1}^n \nabla_{\alpha_i} f_i(t_i, X_{t_i}^x) dt_1 \dots dt_n = u_1(T_0, X_{T_0}^x).$$

Again as in [RZ], let U be the solution to the initial-value problem on $[0, T_1]$,

$$\partial_t U - \frac{1}{2}\Delta U - B \cdot \nabla U - G = 0, \quad U(0) = 0, \tag{9}$$

where

$$B(t,\cdot) = b(T_1 - t, \cdot)\mathbf{1}_{[0,T_1-T_0]}(t) + b(t + T_0 - T_1, \cdot)\mathbf{1}_{]T_1-T_0,T_1]}(t),$$
$$G(t,\cdot) := g_1(T_1 - t, \cdot)\mathbf{1}_{[0,T_1-T_0]}(t).$$

One has $U(t,\cdot) = u_1(T_1 - t, \cdot)$, $t \in [0, T_1 - T_0]$. Further, $V(t,x) := U(t + T_1 - T_0)$ solves on $[0, T_0]$

$$\partial_t V - \frac{1}{2}\Delta V - b \cdot \nabla V = 0, \quad V(0,\cdot) = U(T_1 - T_0, \cdot) = u_1(T_0, \cdot).$$

Therefore,

$$\int_{\mathbb{R}^d} \left| \mathbf{E} \int_{\Delta_n(T_0,T_1)} \prod_{i=1}^{n} \nabla_{\alpha_i} f_i(t_i, X_{t_i}^x) dt_1 \dots dt_n \right|^2 dx$$
$$= \int_{\mathbb{R}^d} |\mathbf{E} u_1(T_0, X_{T_0}^x)|^2 dx = \int_{\mathbb{R}^d} |V(T_0,x)|^2 dx = \|U(T_1,\cdot)\|_2^2.$$

We estimate $\|U(T_1,\cdot)\|_2^2$ in three steps:

1. We multiply equation (9) by U and integrate over $[0,T_1]\times\mathbb{R}^d$, arriving at

$$\frac{1}{2}\langle U^2(T_1,\cdot)\rangle - 0 + \frac{1}{2}\int_0^{T_1} \langle |\nabla U|^2\rangle ds = \int_0^{T_1} \langle B\cdot\nabla U, U\rangle ds + \int_0^{T_1} \langle g_1(T_1-s,\cdot)\mathbf{1}_{[0,T_1-T_0]}(s), U(s)\rangle ds. \tag{10}$$

The first term in the RHS of (10) is estimated, using the quadratic inequality $ac \leq \frac{1}{2\sqrt{\delta}}a^2 + \frac{\sqrt{\delta}}{2}c^2$ and the form-boundedness $b \in \mathbf{F}_\delta$, as follows:

$$\int_0^{T_1} \langle B\cdot\nabla U, U\rangle ds \leq \frac{1}{2\sqrt{\delta}} \int_0^{T_1} \langle B^2, U^2\rangle ds + \frac{\sqrt{\delta}}{2}\int_0^{T_1} \langle |\nabla U|^2\rangle ds$$
$$\leq \sqrt{\delta}\int_0^{T_1} \langle |\nabla U|^2\rangle ds. \tag{11}$$

The second term in the RHS of (10):

$$\int_0^{T_1} \langle g_1(T_1-s,\cdot)\mathbf{1}_{[0,T_1-T_0]}(s), U(s)\rangle ds = \int_0^{T_1} \langle \nabla_{\alpha_1} f_1(T_1-s,\cdot), u_2(T_1-s,\cdot)\mathbf{1}_{[0,T_1-T_0]}(s)U(s)\rangle ds$$
$$= -\int_0^{T_1} \langle f_1(T_1-s,\cdot), (\nabla_{\alpha_1} u_2(T_1-s,\cdot))\mathbf{1}_{[0,T_1-T_0]}(s)U(s,\cdot)\rangle ds$$
$$- \int_0^{T_1} \langle f_1(T_1-s,\cdot), u_2(T_1-s,\cdot)\mathbf{1}_{[0,T_1-T_0]}(s)\nabla_{\alpha_1} U(s,\cdot)\rangle ds$$

(we are applying quadratic inequality twice; fix some ε, $\beta > 0$)

$$\leq \varepsilon \int_0^{T_1} \langle f_1^2(T_1-s,\cdot)U^2(s,\cdot)\rangle ds + \frac{1}{4\varepsilon}\int_0^{T_1} \langle |\nabla_{\alpha_1} u_2(T_1-s,\cdot)|^2 \mathbf{1}_{[0,T_1-T_0]}(s)\rangle ds$$
$$+ \beta \int_0^{T_1} \langle f_1^2(T_1-s,\cdot), u_2^2(T_1-s,\cdot)\mathbf{1}_{[0,T_1-T_0]}(s)\rangle ds + \frac{1}{4\beta}\int_0^{T_1} \langle |\nabla_{\alpha_1} U(s,\cdot)|^2\rangle ds.$$

Therefore, taking into account the indicator function of $[0, T_1 - T_0]$, and using the form-boundedness assumption (7) on f_i, we obtain

$$\begin{aligned}\int_0^{T_1} \langle g_1(T_1 - s, \cdot)\mathbf{1}_{[0,T_1-T_0]}(s), U(s, \cdot)\rangle ds &\leq \varepsilon \int_0^{T_1} \langle f_1^2(T_1 - s, \cdot)U^2(s)\rangle ds + \frac{1}{4\varepsilon} \int_{T_0}^{T_1} \langle |\nabla_{\alpha_1} u_2(s, \cdot)|^2\rangle ds \\ &+ \beta \int_{T_0}^{T_1} \langle f_1^2(s, \cdot), u_2^2(s, \cdot)\rangle ds + \frac{1}{4\beta} \int_0^{T_1} \langle |\nabla_{\alpha_1} U(s, \cdot)|^2\rangle ds \qquad (12) \\ &\leq \left(\varepsilon\nu + \frac{1}{4\beta}\right) \int_0^{T_1} \langle |\nabla U(s, \cdot)|^2\rangle ds + \left(\beta\nu + \frac{1}{4\varepsilon}\right) \int_{T_0}^{T_1} \langle |\nabla u_2(s, \cdot)|^2\rangle ds.\end{aligned}$$

Thus, we obtain from (10):

$$\frac{1}{2}\langle U^2(T_1)\rangle + \left(\frac{1}{2} - \sqrt{\delta} - \varepsilon\nu - \frac{1}{4\beta}\right) \int_0^{T_1} \langle |\nabla U(s)|^2\rangle ds \leq \left(\beta\nu + \frac{1}{4\varepsilon}\right) \int_{T_0}^{T_1} \langle |\nabla u_2(s)|^2\rangle ds.$$

Now, selecting ε and β large, and requiring the form-bounds δ and ν to be sufficiently small, we arrive at

$$\langle U^2(T_1)\rangle + C_1 \int_0^{T_1} \langle |\nabla U(s)|^2\rangle ds \leq C_2 \int_{T_0}^{T_1} \langle |\nabla u_2(s)|^2\rangle ds \qquad (13)$$

for constants $0 < C_2 < C_1$ independent of smoothness or boundedness of b and f_i. Moreover, it is clear that we can make $\frac{C_2}{C_1}$ arbitrarily small by selecting δ and ν even smaller.

2. Now, we repeat this procedure for u_2 in place of U. That is, we multiply equation (8) (for $k = 2$) by u_2 and integrate over $[T_0, T_1] \times \mathbb{R}^d$ to obtain

$$\frac{1}{2}\langle u_2^2(T_0)\rangle + \frac{1}{2} \int_{T_0}^{T_1} \langle |\nabla u_2|^2\rangle ds = \int_{T_0}^{T_1} \langle b \cdot \nabla u_2, u_2\rangle ds + \int_{T_0}^{T_1} \langle g_2, u_2\rangle ds.$$

We estimate the first term in the RHS as in (11), using quadratic inequality and the assumption $b \in \mathbf{F}_\delta$. The second term in the RHS:

$$\begin{aligned}\int_{T_0}^{T_1} \langle g_2, u_2\rangle ds &= \int_{T_0}^{T_1} \langle (\nabla_{\alpha_2} f_2) u_3, u_2\rangle ds \\ &\leq -\int_{T_0}^{T_1} \langle f_2, (\nabla_{\alpha_2} u_3) u_2\rangle ds - \int_{T_0}^{T_1} \langle f_2, u_3 \nabla_{\alpha_2} u_2\rangle ds \\ &\leq \varepsilon \int_{T_0}^{T_1} \langle f_2^2, u_2^2\rangle ds + \frac{1}{4\varepsilon} \int_{T_0}^{T_1} \langle |\nabla_{\alpha_2} u_3(s, \cdot)|^2\rangle ds \\ &+ \beta \int_{T_0}^{T_1} \langle f_2^2, u_3^2\rangle ds + \frac{1}{4\beta} \int_{T_0}^{T_1} \langle |\nabla_{\alpha_2} u_2|^2\rangle ds \\ &\text{(we are using } f_2 \in \mathbf{F}_\nu) \\ &\leq \left(\varepsilon\nu + \frac{1}{4\beta}\right) \int_{T_0}^{T_1} \langle |\nabla u_3(s)|^2\rangle ds + \left(\beta\nu + \frac{1}{4\varepsilon}\right) \int_{T_0}^{T_1} \langle |\nabla u_2(s)|^2\rangle ds,\end{aligned}$$

as in the previous step. Thus, we arrive at

$$\int_{T_0}^{T_1} \langle |\nabla u_2|^2 \rangle ds \leq \frac{C_2}{C_1} \int_{T_0}^{T_1} \langle |\nabla u_3|^2 \rangle ds.$$

If $n > 3$, we repeat this $n-3$ more times:

$$\int_{T_0}^{T_1} \langle |\nabla u_2|^2 \rangle ds \leq \left(\frac{C_2}{C_1}\right)^{n-2} \int_{T_0}^{T_1} \langle |\nabla u_n|^2 \rangle ds$$

and so, in view of (13),

$$\langle U^2(T_1) \rangle \leq C_2 \left(\frac{C_2}{C_1}\right)^{n-2} \int_{T_0}^{T_1} \langle |\nabla u_n|^2 \rangle ds.$$

3. Finally, we estimate $\int_{T_0}^{T_1} \langle |\nabla u_n(s)|^2 \rangle ds$. Arguing as above, we have (recall that $u_{n+1} = 1$)

$$\int_{T_0}^{T_1} \langle |\nabla u_n(s)|^2 \rangle ds \leq C_3 \int_{T_0}^{T_1} \langle \nabla_{\alpha_n} f_n(s,\cdot), u_n(s,\cdot) \rangle ds$$

$$= -C_3 \int_{T_0}^{T_1} \langle f_n(s,\cdot), \nabla_{\alpha_n} u_n(s,\cdot) \rangle ds$$

(we are applying quadratic inequality)

$$\leq C_4 \int_{T_0}^{T_1} \langle f_n^2 \rangle ds + \frac{1}{2} \int_{T_0}^{T_1} \langle |\nabla u_n(s)|^2 \rangle ds$$

(we are using assumption (A') that all f_i have support in $B_R(0)$, and apply (7) to $\int_{T_0}^{T_1} \langle f_n^2 \varphi^2 \rangle ds \geq \int_{T_0}^{T_1} \langle f_n^2 \rangle ds$ for a smooth $\varphi \geq \mathbf{1}_{B_R(0)}$)

$$\leq C_5(T_1 - T_0) + \frac{1}{2} \int_{T_0}^{T_1} \langle |\nabla u_n(s)|^2 \rangle ds.$$

Thus, $\frac{1}{2} \int_{T_0}^{T_1} \langle |\nabla u_n(s)|^2 \rangle ds \leq C_5(T_1 - T_0)$. Combining this with the previous estimate, we obtain $\langle U^2(T_1) \rangle \leq C_2 (\frac{C_2}{C_1})^{n-2} 2C_5(T_1 - T_0)$, which gives the required estimate with $K := \frac{C_2}{C_1}$. □

Remark 2. Let us comment on what happens if in Theorem 1 we assume that g_δ is non-zero. We have to assume that

$$0 \leq g_\delta \in L^{1+\varepsilon}_{\rm loc}(\mathbb{R}), \quad \text{for a fixed } \varepsilon > 0.$$

(It should be added that this $\varepsilon > 0$ does not allow to include completely the critical Ladyzhenskaya-Prodi-Serrin class (LPS) even with $p > 2$ there, as is assumed in [RZ]. It does include, however, the case that interests us the most: $p = \infty$, $q = d$. It also includes with case $p > 2$, $q = \infty$).

Only the proof of Proposition 1 has to be changed, where we assume in (7) $0 \leq g_\nu \in L^{1+\varepsilon}_{\rm loc}(\mathbb{R})$. Then the estimate of Proposition 1 changes to

$$\int_{\mathbb{R}^d} \left| \mathbf{E} \int_{\Delta_n(T_0,T_1)} \prod_{i=1}^n \nabla_{\alpha_i} f_i(t_i, X^x_{t_i}) dt_1 \dots dt_n \right|^2 dx \leq C_0' K^n (T_1 - T_0)^{\frac{\varepsilon}{1+\varepsilon}}, \tag{14}$$

which does not affect the validity of the result of the proof. The proof of (14) goes as follows. Put $F(t) := \lambda \int_0^t [g_\delta(s) + g_\nu(s)]ds$, where λ is to be fixed sufficiently large (depending on the values of δ and ν). We multiply equation (9) for U by e^{-F}, obtaining

$$\partial_t(e^{-F}U) + F'e^{-F}U - \frac{1}{2}\Delta e^{-F}U - B\cdot\nabla e^{-F}U - e^{-F}G = 0, \quad U(0) = 0,$$

where $e^{-F}G = (\partial_{\alpha_1} f_1(T_1 - t, \cdot))\mathbf{1}_{[0,T_1-T_0]}(t)e^{-F(t)}u_2(T_1 - t, \cdot)$. After multiplying the previous equation by U, integrating and fixing $\lambda > \frac{1}{2\sqrt{\delta}} + \varepsilon + \beta$, one sees that the term

$$\int_0^{T_1} \langle F'e^{-F}U^2\rangle ds = \lambda \int_0^{T_1} \langle (g_\delta + g_\nu)e^{-F}U^2\rangle ds$$

will absorb the "new" terms $\frac{1}{2\sqrt{\delta}} \int_0^{T_1} \langle g_\delta e^{-F}U^2\rangle ds$ and $(\varepsilon + \beta) \int_0^{T_1} \langle g_\nu e^{-F}U^2\rangle ds$ that will now appear in (11) and (12). This will give us, instead of (13), the estimate:

$$\langle e^{-F(T_1)}U^2(T_1)\rangle + C_1 \int_0^{T_1} \langle e^{-F}|\nabla U|^2\rangle ds \le C_2 \int_{T_0}^{T_1} \langle e^{-\tilde{F}}|\nabla u_2(s)|^2\rangle ds,$$

where $\tilde{F}(t) := F(T_1 - s)$.

In turn, the multiple $e^{-\tilde{F}(t)}$ factors through all equations (8) with the same effect of absorbing the "new" terms containing g_δ and g_ν, that is, we get

$$\int_{T_0}^{T_1} \langle e^{-\tilde{F}}|\nabla u_2|^2\rangle ds \le \frac{C_2}{C_1} \int_{T_0}^{T_1} \langle e^{-\tilde{F}}|\nabla u_3|^2\rangle ds,$$

and so on:

$$\int_{T_0}^{T_1} \langle |e^{-\tilde{F}}\nabla u_2|^2\rangle ds \le \left(\frac{C_2}{C_1}\right)^{n-2} \int_{T_0}^{T_1} \langle e^{-\tilde{F}}|\nabla u_n|^2\rangle ds.$$

Finally, $e^{-\tilde{F}}$ does not affect the estimate on u_n, only the constant C_5. Thus, we arrive at (14) with the same constant K that does not depend on g_δ or g_ν.

For a given vector field $Y = (Y_i)_{i=1}^d : \mathbb{R}^k \to \mathbb{R}^m$, denote

$$\nabla Y = \nabla_x Y(x) := \begin{pmatrix} \nabla_1 Y_1 & \nabla_2 Y_1 & \dots & \nabla_k Y_1 \\ & & \dots & \\ \nabla_1 Y_m & \nabla_2 Y_m & \dots & \nabla_k Y_m \end{pmatrix}. \tag{15}$$

Proposition 2. *For every $r \ge 1$, there exist constants K_1, K_2 (independent of smoothness or boundedness of b) such that*

(i) $\|\nabla X_t^x - I\|_{L^{2r}(\mathbb{R}^d, L^r(\Omega))} \le K_1 t^{\frac{1}{2r}}$ *for all* $0 \le t \le T$*;*

(ii) $\|D_s X_t^x - I\|_{L^{2r}(\mathbb{R}^d, L^r(\Omega))} \le K_1 (t-s)^{\frac{1}{4r}}$ *for a.e.* $s \in [0,T]$ *and* $0 \le s \le t \le T$*;*

(iii) $\|D_s X_t^x - D_{s'} X_t^x\|_{L^{2r}(\mathbb{R}^d, L^r(\Omega))} \le K_2 |s - s'|^{\frac{1}{4r}}$ *for a.e.* $s, s' \in [0,T]$ *and* $0 \le s, s' \le t \le T$*.*

Proof. The proof repeats [RZ, Proof of Prop. 4.1] essentially word in word. We give an outline of the proof of (i). Since b is bounded and smooth, one has

$$\nabla X_t^x - I = \int_0^t \nabla b(s, X_s^x) \nabla X_s^x ds.$$

The goal is to iterate this identity, obtaining an expression for the left-hand side that one can control:

$$\nabla X_t^x - I = \sum_{n=1}^{\infty} \int_{\Delta_n(t)} \prod_{i=1}^n \nabla b(t_i, X_{t_i}^x) dt_1 \dots dt_n,$$

so

$$\|\nabla X_t^x - I\|_{L^{2r}(\mathbb{R}^d, L^r(\Omega))} \leq \sum_{n=1}^{\infty} \bigg\| \int_{\Delta_n(t)} \prod_{i=1}^n \nabla b(t_i, X_{t_i}^x) dt_1 \dots dt_n \bigg\|_{L^{2r}(\mathbb{R}^d, L^r(\Omega))}. \tag{16}$$

Let us estimate

$$\| \int_{\Delta_n(t)} \prod_{i=1}^n \nabla b(t_i, X_{t_i}^x) dt_1 \dots dt_n \|_{L^{2r}(\mathbb{R}^d, L^r(\Omega))} = \bigg[\int_{\mathbb{R}^d} \bigg[\mathbf{E} \Big(\int_{\Delta_n(t)} \prod_{i=1}^n \nabla b(t_i, X_{t_i}^x) dt_1 \dots dt_n \Big)^r \bigg]^2 dx \bigg]^{\frac{1}{2r}}.$$

First, note that by subdividing $\Delta_n(t) \times \dots \times \Delta_n(t)$ (r times) into sub-simplexes, and recalling definition (15), one can represent

$$(\int_{\Delta_n(t)} \prod_{i=1}^n \nabla b(t_i, X_{t_i}^x) dt_1 \dots dt_n)^r \tag{17}$$

as a sum of at most rn terms of the form

$$\int_{\Delta_{rn}(t)} \prod_{i=1}^n \nabla_{\gamma_1} b_{\beta_1}(t_1, X_{t_1}^x) \dots \nabla_{\gamma_{rn}} b_{\beta_{rn}}(t_{rn}, X_{t_{rn}}^x) dt_1 \dots dt_{rn}, \tag{18}$$

so

$$\begin{aligned} &\| \int_{\Delta_n(t)} \prod_{i=1}^n \nabla b(t_i, X_{t_i}^x) dt_1 \dots dt_n \|_{L^{2r}(\mathbb{R}^d, L^r(\Omega))} \\ &\leq \sum \bigg[\int_{\mathbb{R}^d} \bigg[\sum_{\beta,\gamma} \mathbf{E} \int_{\Delta_{rn}(t)} \prod_{i=1}^n \nabla_{\gamma_1} b_{\beta_1}(t_1, X_{t_1}^x) \dots \nabla_{\gamma_{rn}} b_{\beta_{rn}}(t_{rn}, X_{t_{rn}}^x) dt_1 \dots dt_{rn} \bigg]^2 dx \bigg]^{\frac{1}{2r}} \\ &\leq \sum \bigg[\sum_{\beta,\gamma} \bigg\| \mathbf{E} \int_{\Delta_{rn}(t)} \prod_{i=1}^n \nabla_{\gamma_1} b_{\beta_1}(t_1, X_{t_1}^x) \dots \nabla_{\gamma_{rn}} b_{\beta_{rn}}(t_{rn}, X_{t_{rn}}^x) dt_1 \dots dt_{rn} \bigg\|_{L^2(\mathbb{R}^d)} \bigg]^{\frac{1}{r}}, \end{aligned}$$

where both sums are finite (the first sum comes from the coordinate representation of the product of n matrices $\nabla b(t_i, X_{t_i}^x)$, the second sum contains r^{rn} terms). Finally, applying Proposition 1, one obtains

$$\| \int_{\Delta_n(t)} \prod_{i=1}^n \nabla b(t_i, X_{t_i}^x) dt_1 \dots dt_n \|_{L^{2r}(\mathbb{R}^d, L^r(\Omega))} \leq (C_6 r)^n C_7^{\frac{1}{r}} K^n t^{\frac{1}{r}}.$$

Now, recalling that K can be made as small as needed by assuming that δ is sufficiently small, one has $\| \int_{\Delta_n(t)} \prod_{i=1}^n \nabla b(t_i, X_{t_i}^x) dt_1 \dots dt_n \|_{L^{2r}(\mathbb{R}^d, L^r(\Omega))} \leq C_8^n t^{\frac{1}{r}}$ for a positive constant $C_8 < 1$

Returning to (16), one obtains

$$\|\nabla X_t^x - I\|_{L^{2r}(\mathbb{R}^d, L^r(\Omega))} \leq \sum_{n=1}^{\infty} C_8^n t^{\frac{1}{r}},$$

as needed.

The Malliavin derivative $D_s X_t^x$ satisfies (see e.g. [B])

$$D_s X_t^x - I = \int_s^t \nabla b(\tau, X_\tau^x) D_s X_\tau^x d\tau,$$

so one can iterate this identity and estimate $D_s X_t^x - I$ in the same way as above, which yields (*ii*). The latter yields (*iii*), see [RZ, Proof of Prop. 4.1] for details. □

2.3. **Proof of Theorem 1.** The proof repeats the argument in [RZ]. However, since we will have to use some estimates and some convergence results established in [KM], we included the details for the ease of the reader.

We consider a general $b \in \mathbf{F}_\delta$ as in the assumptions of the theorem. Let us fix an approximation $\{b_m\} \subset C^\infty(\mathbb{R}^{d+1}, \mathbb{R}^d) \cap L^\infty(\mathbb{R}^{d+1}, \mathbb{R}^d)$ of b:

$$b_m \to b \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^{d+1}, \mathbb{R}^d) \text{ as } m \to \infty \tag{19}$$

and for all $t \in \mathbb{R}$

$$\|b_m(t)\varphi\|_2^2 \leq \delta \|\nabla \varphi\|_2^2 \tag{20}$$

Example. It is easy to show that the following b_m, with $\varepsilon_m \downarrow 0$ sufficiently rapidly and $c_m \uparrow 1$ sufficiently slow, satisfy (19), (20).

$$b_m := c_m E^{1+d}_{\varepsilon_m}(\mathbf{1}_m b),$$

where $\mathbf{1}_m$ is the indicator of $\{(t,x) \in [0,T] \times \mathbb{R}^d \mid |b(t,x)| \leq m\}$, E^{1+d}_ε is the Friedrichs mollifier on $\mathbb{R} \times \mathbb{R}^d$, see details e.g. in [KM]. Note that, by selecting $\varepsilon_n \downarrow 0$ rapidly, one can treat b_m as essentially a cutoff of b.

(Of course, since by our assumption b in this paper has compact support, in (19) one has convergence in $L^2(\mathbb{R}^{d+1}, \mathbb{R}^d)$.)

Let $\mathsf{f} \in \mathbf{F}_\nu$ be bounded and smooth with function $g_\nu = 0$. (Below we will need $\mathsf{f} = b_m$, in which case $\nu = \delta$, or $\mathsf{f} = b_m - b_k$, in which case $\nu = 2\delta$.) Let us emphasize that the constants in the estimates below do not depend on n or boundedness or smoothness of f. They will depend on the dimension d, T and form-bounds δ and ν.

Since b_n are bounded and smooth, by the classical theory there exists a unique continuous random field $X^m : \Delta_2(T) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ such that

$$X^{x,m}_{s,t} = x + \int_s^t b_m(s, X^{x,m}_{s,r}) dr + W_t - W_s, \quad 0 \leq s \leq t \leq T, \quad x \in \mathbb{R}^d. \tag{21}$$

By Itô's formula, for all $m,k=1,2,\dots,\ s\leq t_1\leq t_2\leq T$,

$$-u_m(t_1,X_{s,t_1}^{x,m})=-\int_{t_1}^{t_2}|\mathsf{f}(t,X_{s,t}^{x,m})|dt+\int_{t_1}^{t_2}\nabla u_m(t,X_{s,t}^{x,m})dW_t,$$

where $u_m(t)$, $s<t\leq t_2$ is the classical solution to

$$\partial_t u_m+\frac{1}{2}\Delta u_m+b_m\cdot\nabla u_m=-|\mathsf{f}|,\quad u_m(t_2)=0.$$

The following estimates on $X_{s,t}^{x,m}$ and u_m are valid:

1)

$$\sup_m\sup_{x\in\mathbb{R}^d}\mathbf{E}\big[\int_{t_1}^{t_2}|\mathsf{f}(t,X_{s,t}^{x,m})|dt\,|\,\mathcal{F}_{t_1}\big]\leq C(t_2-t_1).$$

Indeed,

$$\begin{aligned}\mathbf{E}\Big[\int_{t_1}^{t_2}|\mathsf{f}(t,X_{s,t}^{x,m})|dt\,|\,\mathcal{F}_{t_1}\Big]&=\mathbf{E}\Big[u_m(t_1,X_{s,t_1}^{x,m})\,|\,\mathcal{F}_{t_1}\Big]\\&\leq\|u_m(t_1)\|_\infty\\&\text{(we are applying [KM, Cor. 6.4])}\\&\leq C'\sup_{z\in\mathbb{Z}^d}\|\mathsf{f}\sqrt{\rho_z}\|_{L^2([t_1,t_2],L^2)},\end{aligned}$$

where, recall, $\rho(x)=(1+\kappa|x|^2)^{-\theta}$ with $\theta>\frac{d}{2}$ and $\kappa>0$ fixed sufficiently small, $\rho_z(x)=\rho(x-z)$. In turn, since $\mathsf{f}\in\mathbf{F}_\nu$,

$$\begin{aligned}\|\mathsf{f}\sqrt{\rho_z}\|^2_{L^2([t_1,t_2],L^2)}&\leq\frac{\nu}{4}\int_{t_1}^{t_2}\langle\frac{|\nabla\rho_z|^2}{\rho_z}\rangle dt\\&\leq\frac{\nu}{4}(t_2-t_1)\|\nabla\rho/\sqrt{\rho}\|_2^2\\&\text{(we are using }|\nabla\rho|\leq\theta\sqrt{\kappa}\rho,\ \|\sqrt{\rho}\|_2<\infty)\\&\leq C''(t_2-t_1).\end{aligned}$$

which gives us 1).

2) As a consequence of estimate 1), one has, e.g. for every integer $r\geq 1$,

$$\sup_m\sup_{x\in\mathbb{R}^d}\mathbb{E}\left|\int_{t_1}^{t_2}|\mathsf{f}(t,X_{s,t}^{x,m})|dt\right|^r\leq C_r(t_2-t_1)^r,$$

see proof in [ZZ, Cor. 3.5] (first, one represents $\mathbb{E}\left|\int_{t_1}^{t_2}|\mathsf{f}(t,X_{s,t}^{x,m})|dt\right|^r$ as the expectation of a repeated integral over $\Delta(t_1,t_2)$, cf. transition from (17) to (18), and then uses 1) r times).

3) It follows from 2) (upon selecting $\mathsf{f}=b_m$) that

$$\begin{aligned}\mathbf{E}|X_{s,t_2}^{x,m}-X_{s,t_1}^{x,m}|^r&\leq C\mathbb{E}\left|\int_{t_1}^{t_2}|b_m(t,X_{s,t}^{x,m})|\right|^r+C|W_{t_2}-W_{t_1}|^r\\&\leq C(t_2-t_1)^r.\end{aligned}$$

4) In particular,

$$\sup_m \sup_{0\le s\le t\le T} \sup_{x\in\mathbb{R}^d} \mathbf{E}|X^{x,m}_{s,t}|^r < \infty.$$

5) One has (the Sobolev norm is in the x variable)

$$\sup_m \sup_{0\le s\le t\le T} \sup_{y\in\mathbb{R}^d} \mathbf{E}\int_{B_1(y)} |\nabla_x X^{x,m}_{s,t}|^r dx < \infty.$$

Indeed,

$$\begin{aligned}\mathbf{E}\int_{B_1(y)} |\nabla_x X^{x,m}_{s,t}|^r dx &\le \biggl(\int_{B_1(y)} \bigl(\mathbf{E}|\nabla_x X^{x,m}_{s,t}|^r\bigr)^2 dx\biggr)^{\frac{1}{2}} |B_1(y)|^{\frac{1}{2}} \\ &= c_d \|\nabla X^m_{s,t}\|^r_{L^{2r}(B_1(y),L^r(\Omega))},\end{aligned}$$

so it remains to apply Proposition 2(i).

Let now $r>d$. Combining 4) and 5), and using the Sobolev embedding theorem, one obtains (the Hölder norm is in the x variable)

$$\sup_m \sup_{0\le s\le t\le T} \sup_{y\in\mathbb{R}^d} \mathbf{E}\|X^{x,m}_{s,t}\|_{C^{1-\frac{d}{r}}(B_1(y))} < \infty,$$

and so one arrives at:

6) For all $0\le s\le t\le T$, $x,y\in\mathbb{R}^d$ with $|x-y|\le 1$,

$$\mathbf{E}|X^{x,m}_{s,t_2} - X^{y,m}_{s,t_1}|^r \le C|x-y|^{r-d}, \quad r>d.$$

7) Repeating the proof from [RZ] (which is a combination of 3) and 6), by means of the Markov property and the independence of $X^{x,m}_{s_1,s_2}$ and $X^{y,m}_{s_2,t}$), one arrives at

$$\mathbf{E}|X^{x,m}_{s_1,t} - X^{x,m}_{s_2,t}|^r \le C(s_2-s_1)^{r-d}, \quad 0\le s_1\le s_2\le t.$$

Estimates 3), 6), 7) combined yield, for $r>d$,

$$\mathbf{E}|X^{x,m}_{s_1,t_1} - X^{y,m}_{s_2,t_2}|^r \le C(|t_2-t_1|^r + |x-y|^{r-d} + |s_2-s_1|^{r-d}) \tag{22}$$

for all $(s_i,t_i)\in\Delta_2(T)$, $i=1,2$.

Now comes the final stage in the approach of Röckner-Zhao. Proposition 2(i)-(iii) verifies conditions of [RZ, Lemma 3.1], i.e. of the relative compactness criterion for random fields on the Wiener-Sobolev space (see the discussion of history of this type of results in [RZ]). This, and a standard diagonal argument, allow to conclude that there is a subsequence of $\{X^{x,m}_{s,t}\}$ (without loss of generality, still denoted by $\{X^{x,m}_{s,t}\}$) and a countable subset D of $\mathbb{R}^d$ such that

$$X^{x,m}_{s,t} \to X^x_{s,t} \quad \text{ in } L^2(\Omega) \quad \text{ as } m\to\infty$$

for all $(s,t)\in\mathbb{Q}^2\times\Delta_2(T)$, $x\in D$. Moreover, in view of 4), one has $X^{x,m}_{s,t}\to X^x_{s,t}$ in $L^r(\Omega)$, $r\ge 1$. Now (22) yields for $r>d$, upon applying Fatou's lemma, that

$$\mathbf{E}|X^x_{s_1,t_1} - X^y_{s_2,t_2}|^r \le C(|t_2-t_1|^r + |x-y|^{r-d} + |s_2-s_1|^{r-d}) \tag{23}$$

for all $(s_i,t_i) \in \mathbb{Q}^2 \times \Delta_2(T)$, $i=1,2$, $x,y \in D$. Kolmogorov-Chentsov theorem (after selecting $r>d$ even larger) allows to extend $X^x_{s,t}$ to a continuous random field, and yields, together with the equicontinuity estimate (22),

$$X^{x,m}_{s,t} \to X^x_{s,t} \quad \mathbf{P}\text{-a.s. as } m \to \infty$$

for all $(s,t) \in \Delta_2(T)$, $x \in \mathbb{R}^d$.

By [KM, Cor. 6.4],

$$\mathbf{E}\bigg[\int_s^t |\mathsf{f}(\tau, X^{x,m}_{s,\tau})| d\tau\bigg] \leq C_1 \sup_{z \in \mathbb{Z}^d} \|\mathsf{f}\sqrt{\rho_z}\|_{L^2([s,t],L^2)} \tag{24}$$

(cf. proof of 1) above), and so

$$\mathbf{E}\bigg[\int_s^t |\mathsf{f}(\tau, X^{x}_{s,\tau})| d\tau\bigg] \leq C_1 \sup_{z \in \mathbb{Z}^d} \|\mathsf{f}\sqrt{\rho_z}\|_{L^2([s,t],L^2)} \tag{25}$$

where, recall, $\mathsf{f} \in \mathbf{F}_\nu$ is bounded and smooth, but the constant C_r does not depend on smoothness of bundedness of f. Using Fatou's lemma, one can extend (24) to all $\mathsf{f} \in \mathbf{F}_\nu$, i.e. not necessarily smooth. (We will be selecting e.g. $\mathsf{f} = b - b_m$, in which case $\nu = 2\delta$.)

Now, to show that $X^x_{s,t}$ is a strong solution to (21), it remains to show that $\int_s^t b_m(\tau, X^{x,m}_{s,\tau})d\tau \to \int_s^t b(\tau, X^x_{s,\tau})d\tau$ in $L^1(\Omega)$. Indeed,

$$\begin{aligned}\mathbf{E}\bigg|\int_s^t (b_m(\tau, X^{x,m}_{s,\tau})d\tau - \int_s^t b(\tau, X^x_{s,\tau})d\tau\bigg| &\leq \mathbf{E}\bigg|\int_s^t (b_m - b_k)(\tau, X^{x,m}_{s,\tau})d\tau\bigg| \\ &+ \mathbf{E}\bigg|\int_s^t b_k(\tau, X^{x,m}_{s,\tau})d\tau - \int_s^t b_k(\tau, X^x_{s,\tau})d\tau\bigg| \\ &+ \mathbf{E}\bigg|\int_s^t (b_k - b)(\tau, X^x_{s,\tau})d\tau\bigg| =: I_1 + I_2 + I_3.\end{aligned}$$

By (24), $I_1 \leq \sup_{z\in\mathbb{Z}^d} \|(b_m - b_k)\sqrt{\rho_z}\|_{L^2([s,t],L^2)} \to 0$ as $m,k \to \infty$, where the L^2 norm tends to zero since by our assumption b has compact support. Let us fix k sufficiently large. By (25), $I_3 \to 0$ as $m \to \infty$. Finally, $I_2 \to 0$ as $m \to \infty$ (for k fixed above) by the Dominated convergence theorem. This yields that $X^x_{s,t}$ is a strong solution to (1). This strong solution, clearly, satisfies (6).

Finally, regarding uniqueness of $X^x_{s,t}$ in the class of strong solutions satisfying Krylov estimate (6). The proof in [RZ] is based on Cherny's theorem [C] (strong existence + weak uniqueness $\Rightarrow$ strong uniqueness) and the result from [RZ2] on the uniqueness of weak solution to SDE (1) in the class of solutions satisfying a Krylov-type bound. In our setting, it suffices to use instead the weak uniqueness result from [KM], valid for form-bounded drifts. This yields the uniqueness result in Theorem 1 within the class (5); regarding the uniqueness within the class (6), one needs to apply the weak uniqueness result from [K].

□

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