KOLMOGOROV OPERATOR WITH THE VECTOR FIELD IN NASH CLASS

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Abstract. We consider divergence-form parabolic equation with measurable uniformly elliptic matrix and the vector field in a large class containing, in particular, the vector fields in L^p, $p>d$, as well as some vector fields that are not even in $L^{2+\varepsilon}_{\rm loc}$, $\varepsilon>0$. We establish Hölder continuity of the bounded soutions, sharp two-sided Gaussian bound on the heat kernel, Harnack inequality.

1. Introduction

A celebrated result of E. De Giorgi [3] and J. Nash [11] states that the bounded solutions of the parabolic equation

$$(\partial_t + A)u = 0, \quad A = -\nabla \cdot a \cdot \nabla \tag{1}$$

on $[0,\infty[\times\mathbb{R}^d$, $d \geq 3$, with measurable matrix

$$\begin{aligned} &a = a^* : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d, \\ &\sigma I \leq a(x) \leq \xi I \quad \text{for a.e. } x \in \mathbb{R}^d \quad \text{for constants } 0<\sigma<\xi<\infty \end{aligned} \tag{$H_{\sigma,\xi}$}$$

are Hölder continuous, and the heat kernel $e^{-tA}(x,y)$ satisfies two-sided Gaussian bound with constants that depend only on d, σ, ξ. The purpose of this paper is to extend their result to the equation

$$(\partial_t + \Lambda)u = 0 \tag{2}$$

where

$$\Lambda = -\nabla \cdot a \cdot \nabla + b \cdot \nabla$$

with $b : \mathbb{R}^d \to \mathbb{R}^d$ in a large class of locally unbounded measurable vector fields.

1. The existence and the precise form of the relationship between the integral characteristics of the coefficients a and b and the regularity properties of solutions to (1) and (2) is one of the classical and central problems in the theory of elliptic and parabolic PDEs.

By a result of D. G. Aronson [1], the heat kernel $e^{-t\Lambda}(x,y)$ of equation (2) satisfies two-sided Gaussian bound. By a result of S. D. Eidelman-F. O. Porper [4], $t|\partial_t e^{-t\Lambda}(x,y)|$ satisfies the Gaussian upper bound. The constants in their bounds depend on d, σ, ξ, and the following integral characteristics of b:

$$\|b_1\|_p + \|b_2\|_\infty, \quad p>d$$

provided that $b_1 + b_2 = b$.

2010 *Mathematics Subject Classification.* 35K08, 47D07 (primary), 60J35 (secondary).

Key words and phrases. Heat kernel bounds, De Giorgi-Nash theory, Harnack inequality, strong solutions, singular drift, Feller semigroups.

The research of D.K. is supported by grants from the Natural Sciences and Engineering Research Council of Canada (RGPIN-2017-05567) and Fonds de recherche du Québec - Nature et technologies (2019-NC-254946).

Our first goal is to demonstrate, based on ideas of E. De Giorgi and J. Nash, that the constants in the two-sided bound on $e^{-t\Lambda}(x,y)$, in the upper bound on $t|\partial_t e^{-t\Lambda}(x,y)|$, as well as Hölder continuity of bounded solutions to (2) (assuming first that the coefficients a, b are smooth) depend in fact on a much finer characteristic of the vector field b, that is, on its *elliptic Nash norm*:

$$n_e(b,h) := \sup_{x\in\mathbb{R}^d} \int_0^h \sqrt{e^{t\Delta}|b|^2(x)}\, \frac{dt}{\sqrt{t}} \quad (h>0),$$

and only on its elliptic Nash norm (Theorem 3.1).

Next, as is well known, the existence of even strong a priori estimates does not always mean that there is a satisfactory a posteriori regularity theory of the corresponding differential operator. Our second goal is to develop an exhaustive a posteriori theory of (2), including two-sided Gaussian bound on the heat kernel of $-\nabla\cdot a\cdot\nabla + b\cdot\nabla$, assuming only that b is measurable, $|b|\in L^2_{\rm loc}$ and

$$n_e(b,h) \text{ is sufficiently small}$$

for some $h>0$ (Theorem 3.2).

Definition 1.1. A measurable vector field $b:\mathbb{R}^d\to\mathbb{R}^d$ such that $|b|\in L^2_{\rm loc}$ is said to be in the Nash class $\mathbf{N}_e$ if

$$n_e(b,h)<\infty$$

for some $h>0$.

The class $\mathbf{N}_e$ contains the vector fields $b=b_1+b_2$ with $\|b_1\|_p+\|b_2\|_\infty<\infty$, $p>d$. For such b one has $\lim_{h\downarrow 0} n_e(b,h)=0$. The class $\mathbf{N}_e$ also contains some vector fields b with $|b|$ not even in $L^{2+\varepsilon}_{\rm loc}$, $\varepsilon>0$. See more detailed discussion in Section 3. The elliptic Nash norm $n_e(b,h)$ was introduced in [14] where the two-sided Gaussian bound on the heat kernel $e^{-t\Lambda}(x,y)$ was obtained under some additional to $b\in\mathbf{N}_e$ assumptions.

If $a=I$ or a is Hölder continuous, then the condition $|b|\in L^1_{\rm loc}$ and

$$\kappa_{d+1}(b,h) \text{ is sufficiently small}$$

for some $h>0$, where

$$\kappa_{d+1}(b,h) := \sup_{x\in\mathbb{R}^d}\int_0^h e^{t\Delta}|b|(x)\frac{dt}{\sqrt{t}} \qquad \text{(Kato norm of } b\text{)},$$

provides the upper Gaussian bound [13], the Harnack inequality and the lower Gaussian bound on the heat kernel $e^{-t\Lambda}(x,y)$ [16], see also [17]. The class of the vector fields b such that $|b|\in L^1_{\rm loc}$ and

$$\kappa_{d+1}(b,h)<\infty$$

for some $h>0$ is the well known Kato class $\mathbf{K}^{d+1}$. (The results in [16, 17] were obtained, in fact, for $b=b(t,x)$ in the non-autonomous Kato class, itself introduced by Q. S. Zhang.)

Thus, the Nash class $\mathbf{N}_e$ is an analogue of the Kato class $\mathbf{K}^{d+1}$ in case $a=a(x)$ is only measurable. Note that $\mathbf{N}_e\subset\mathbf{K}^{d+1}$ as is immediate from elementary inequality $e^{t\Delta}|b|(x)\le\sqrt{e^{t\Delta}|b|^2(x)}$.

The principal difference between the cases covered by the Nash class $\mathbf{N}_e$ (a is measurable) and the Kato class $\mathbf{K}^{d+1}$ (a is Hölder continuous) is as follows. For Hölder continuous a one can appeal, in the proof of the two-sided bound, to the estimate $|\nabla_x e^{-tA}(x,y)|\le Ct^{-\frac{1}{2}}e^{ct\Delta}(x,y)$, which does not hold for merely measurable a; for such a the role of the previous estimate is assumed by far-reaching inequalities

$$\mathcal{N}(t) \leq \frac{c_0}{t}, \quad \hat{\mathcal{N}}(t) \leq \frac{\hat{c}_0}{t},$$

where $\mathcal{N}(t)$, $\hat{\mathcal{N}}(t)$ are the so-called Nash's functions similar to

$$\langle \nabla_x p \cdot \frac{a(x)}{p} \cdot \nabla_x p \rangle, \quad p \equiv p(t,x,y) = e^{-tA}(x,y)$$

employed by J. Nash in [11]. See Sections 4 and 5 for details.

We comment more on the relationship between the Nash class and the Kato class in Section 9 below.

2. In the context of the semigroup theory of (2), the standard assumption on the vector field b used in the literature is the form-boundedness condition: there exist constants $\delta > 0$ and $c(\delta) \geq 0$ such that the quadratic inequality

$$\|\sqrt{b \cdot a^{-1} \cdot b}\, f\|_2^2 \leq \delta \|A^{\frac{1}{2}} f\|_2^2 + c(\delta)\|f\|_2^2,$$

holds for all $f \in W^{1,2}$. Briefly,

$$b \cdot a^{-1} \cdot b \leq \delta A + c(\delta) \qquad \text{(in the sense of quadratic forms)}$$

(written as $b \in \mathbf{F}_\delta(A)$). This is a large class of singular vector fields containing e.g. the vector fields $b = b_1 + b_2$ with $|b_1|$ in L^d or in the weak L^d class, $|b_2| \in L^\infty$, see discussion below (before Theorem 3.3).

If $b \in \mathbf{F}_\delta(A)$ with $\delta < 1$, then the corresponding to $\Lambda = -\nabla \cdot a \cdot \nabla + b \cdot \nabla$ quadratic form on $W^{1,2}$ is quasi m-accretive, and so it determines an operator Λ_2 in L^2 generating a holomorphic semigroup. The equation (2) with $\Lambda = \Lambda_2$ possesses a detailed regularity theory in L^2 and, moreover, in L^p, $p > \frac{2}{2-\sqrt{\delta}}$, but not in L^1. See Section 9 for more details.

If $b \in \mathbf{N}_e$, then the situation is different: the equation (2) does not seem to admit any L^p theory for $p > 1$ beyond the existence of a semigroup. However, it admits a detailed L^1 theory. In Theorem 3.2 we construct an operator realization Λ_1 of the formal operator Λ in L^1 as the *algebraic sum*

$$\Lambda_1 = A_1 + (b \cdot \nabla)_1, \quad D(\Lambda_1) = D(A_1),$$

where A_1 is the operator realization of $-\nabla \cdot a \cdot \nabla$ in L^1 and $(b \cdot \nabla)_1$ is the closure of $b \cdot \nabla$ in the graph norm of A_1, and show that

$$e^{-t\Lambda_1} = s\text{-}L^1\text{-}\lim_{\varepsilon \downarrow 0} e^{-t\Lambda_1^\varepsilon} \quad \text{(loc. uniformly in } t \geq 0)$$

where $\Lambda_1^\varepsilon = -\nabla \cdot a_\varepsilon \cdot \nabla + b_\varepsilon \cdot \nabla$ of domain $D(\Lambda_1^\varepsilon) = (1-\Delta)^{-1}L^1$ with smooth $(a_\varepsilon, b_\varepsilon)$ approximating (a, b) and essentially non-increasing the Nash norm:

$$n_e(b_\varepsilon, h) \leq n_e(b, h) + \tilde{c}\varepsilon.$$

Armed with the last results and a priori two-sided Gaussian bound on $e^{-t\Lambda^\varepsilon}(x,y)$ of Theorem 3.1, we develop an exhaustive regularity theory of (2), including a posteriori two-sided Gaussian bound on the heat kernel $e^{-t\Lambda}(x,y)$, the Harnack inequality, the Hölder continuity of bounded solutions of (2), the strong Feller property, and the Gaussian upper bound on $t|\partial_t e^{-t\Lambda}(x,y)|$ with the optimal (up to a strict inequality) exponent in the Gaussian factor. We also establish the bounds

$$\|\nabla(\mu+\Lambda_1)^{-\alpha}\|_{1\to 1} \leq C\mu^{-\frac{2\alpha-1}{2}}$$

for $\frac{1}{2}<\alpha\leq 1$, $\mu>\mu_0>0$ (μ_0 depends on $d,\sigma,\xi,n_e(b,h)$), and

$$\|\nabla e^{-t\Lambda_1}\|_{1\to 1} \leq ct^{-\frac{1}{2}}e^{\omega t}, \quad t>0,$$

see Theorem 3.3.

We conclude this introduction by mentioning that the condition $b\in\mathbf{F}_\delta(A)$, $\delta<\infty$ provides two-sided Gaussian bounds on the heat kernel of $-\nabla\cdot a\cdot\nabla+b\cdot\nabla$ but only as long as $\operatorname{div} b$ satisfies additional integral constraints (that is, $\operatorname{div} b$ is in the Kato class $\mathbf{K}^d$, cf. Section 9), see [7].

Contents

2. Preliminaries

We will need the following standard notations and results.

1. Let $\mathcal{B}(X,Y)$ denote the space of bounded linear operators between Banach spaces $X\to Y$, endowed with the operator norm $\|\cdot\|_{X\to Y}$. $\mathcal{B}(X):=\mathcal{B}(X,X)$.

We write $T=s\text{-}X\text{-}\lim_n T_n$ for T, $T_n\in\mathcal{B}(X,Y)$ if

$$\lim_n \|Tf-T_nf\|_Y=0 \quad \text{for every } f\in X.$$

Denote by $[L^p]^d$ and $[L^p]^{d\times d}$ the spaces of the d-vectors and the $d\times d$-matrices with entries in $L^p\equiv L^p(\mathbb{R}^d,dx)$.

Put

$$\langle f,g\rangle=\langle f\bar{g}\rangle:=\int_{\mathbb{R}^d} f\bar{g}dx$$

and $\|\cdot\|_{p\to q}=\|\cdot\|_{L^p\to L^q}$.

$C_\infty := \{f \in C(\mathbb{R}^d) \mid \lim_{|x|\to\infty} f(x) = 0\}$ endowed with the sup-norm.

$\mathcal{W}^{\alpha,1}$, $\alpha > 0$, is the Bessel potential space endowed with norm $\|u\|_{1,\alpha} := \|g\|_1$, $u = (1-\Delta)^{-\frac{\alpha}{2}} g$, $g \in L^1$.

Let $E_\varepsilon f := e^{\varepsilon\Delta} f$ ($\varepsilon > 0$), the De Giorgi mollifier of f.

For a vector field b we put $b^2 := |b|^2$ and $b_a^2 := b \cdot a^{-1} \cdot b$.

We write $c \neq c(\varepsilon)$ to emphasize that c is independent of ε.

Put

$$k_\mu(t,x,y) \equiv k(\mu t, x, y) := (4\pi\mu t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4\mu t}}, \quad \mu > 0.$$

2. Let $a \in (H_{\sigma,\xi})$, $0 < \sigma < \xi < \infty$. Let $p(t,x,y)$ be the heat kernel of $-\nabla \cdot a \cdot \nabla$ (that is, $p(t,x,y) = e^{-tA}(x,y)$ in the notation of the next section).

THEOREM 2.1. *Fix constants $0 < c_2 < \sigma$ and $c_4 > \xi$. There exist constants c_1, $c_3 > 0$ that depend only on d, c_2, c_4 such that, for all $t > 0$, $x, y \in \mathbb{R}^d$,*

$$p(t,x,y) \leq c_3 k_{c_4}(t, x-y) \qquad (\mathrm{UGB}^p)$$

and

$$c_1 k_{c_2}(t, x-y) \leq p(t,x,y). \qquad (\mathrm{LGB}^p)$$

Also, for a given $c_6 > \xi$ there is a generic constant c_5 depending on c_6 such that

$$t|\partial_t p(t,x,y)| \leq c_5 k_{c_6}(t, x-y) \qquad (\mathrm{UGB}^{\partial_t p})$$

for all $t > 0$, $x, y \in \mathbb{R}^d$.

The proof of (UGB^p) and (LGB^p) with *some* constants c_2 and c_4 is due to [1]. The proof of ($\mathrm{UGB}^{\partial_t p}$) with some constant c_6 is due to [4]. The proof of (UGB^p) and ($\mathrm{UGB}^{\partial_t p}$) in the form as stated is due to [9], and in a strengthened form, i.e. with polynomial factor, can be found in [2]. The proof of (LGB^p) as stated is due to [13].

3. Recall that if S and T are linear operators in a Banach space $(Y, \|\cdot\|)$, then S is said to be T-bounded if $D(S) \supset D(T)$ and there exist constants η and c such that

$$\|Sy\| \leq \eta\|Ty\| + c\|y\| \quad \text{for all } y \in D(T).$$

By $T \upharpoonright X$ we denote the restriction of T to a subset $X \subset D(T)$.

By $\big(T \upharpoonright X\big)^{\mathrm{clos}}_{Y \to Y}$ we denote the closure of $T \upharpoonright X$ (when it exists).

Next, let operator T be closed. A subset $D_T \subset D(T)$ is called a core of T if

$$(T \upharpoonright D_T)^{\mathrm{clos}}_{Y\to Y} = T.$$

Let P, Q be linear operators in a Banach space Y. Assume that Q is closed, $D(P)$ contains a core D_Q of Q and $\|Py\| \leq \eta\|Qy\| + c\|y\|$, $y \in D_Q$ (η, c some constants). This inequality extends by continuity to $D(Q)$. An extension of P obtained in this way, say $\tilde{P}$, is Q-bounded.

3. Main results

1. We first prove *a priori* Gaussian lower and upper bounds on the heat kernel of $-\nabla\cdot a\cdot\nabla+b\cdot\nabla$, $a \in (H_{\sigma,\xi})$. In what follows, $d \geq 3$.

Definition 3.1. We say that a constant is generic if it depends only on the dimension d and the constants σ and ξ.

Theorem 3.1. *Let $a \in (H_{\sigma,\xi})$ be smooth, let $b : \mathbb{R}^d \to \mathbb{R}^d$ be smooth and bounded, $\xi_1 > \xi$. There exists a generic constant $\tilde{n} > 0$ such that if the Nash norm of b*

$$n_e(b,h) \equiv \sup_{x \in \mathbb{R}^d} \int_0^h \sqrt{e^{t\Delta}|b|^2(x)}\,\frac{dt}{\sqrt{t}}$$

satisfies

$$n_e(b,h) \leq \tilde{n}$$

for some $h > 0$, then there exist positive constants $\sigma_1 < \sigma$ and c_{σ_1}, $c_{\xi_1} > 0$, $\omega_i \geq 0$, $i = 1, 2$, such that the heat kernel $u(t,x,y)$ of $-\nabla\cdot a\cdot\nabla + b\cdot\nabla$ satisfies the Gaussian lower and upper bounds

$$c_{\sigma_1} e^{-t\omega_1} k_{\sigma_1}(t, x-y) \leq u(t,x,y) \leq c_{\xi_1} e^{t\omega_2} k_{\xi_1}(t, x-y) \qquad (\mathrm{LUGB}^u)$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$. The constants σ_1, $c_{\sigma_1}, c_{\xi_1}, \omega_i$ depend only on d, ξ_1 and $n_e(b,h)$.

Definition 3.2. We say that a constant is generic* if it depends on d, σ, ξ and on the Nash norm $n_e(b,h)$ of the vector field b.

Thus, the constants in (LUGB^u) are generic*. The fact that they do not depend on the smoothness of a, b, coupled with the next Proposition 3.1 and a careful approximation argument, will allow us to establish the corresponding a posteriori heat kernel bounds (Theorem 3.2).

2. Recall that a vector field $b \in [L^2_{\mathrm{loc}}]^d$ is said to be in the Nash class $\mathbf{N}_e$ if

$$n_e(b,h) < \infty$$

for some $h > 0$.

Example 3.1. (1) We have

$$|b| \in L^p,\ p > d \quad \Rightarrow \quad b \in \mathbf{N}_e,$$

as follows easily using $\|e^{t\Delta}\|_{r\to\infty} \leq Ct^{-\frac{d}{2r}}$ upon taking $r = \frac{p}{2}$:

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}\int_0^h \sqrt{e^{t\Delta}|b|^2(x)}\frac{dt}{\sqrt{t}} &\leq \int_0^h \sqrt{\|e^{t\Delta}|b|^2\|_\infty}\frac{dt}{\sqrt{t}} \\ &\leq C^{\frac{1}{2}}\int_0^h \sqrt{t^{-\frac{d}{p}}\|b\|_p^2}\frac{dt}{\sqrt{t}} \\ &= C^{\frac{1}{2}}\frac{2p}{p-d}h^{\frac{p-d}{2p}}\|b\|_p < \infty.\end{aligned}$$

(2) There exist $b \in \mathbf{N}_e$ such that, for any $\varepsilon > 0$, $|b| \notin L^{2+\varepsilon}_{\mathrm{loc}}$, e.g. consider

$$|b(x)| = \mathbf{1}_{B(0,e^{-1})}(x)|x_1|^{-\frac{1}{2}}|\log|x_1||^{-\alpha}, \quad \alpha > \frac{1}{2},$$

where $x = (x_1, \dots, x_d)$.

3. Let $A \equiv A_2$ be the self-adjoint operator in L^2 associated with the quadratic form $\langle \nabla u, a \cdot \nabla u\rangle$, $u \in W^{1,2}$. A standard application of the Beurling-Deny theory yields that the operator A generates a symmetric Markov semigroup e^{-tA}. Then

$$e^{-tA_1} := \Big[e^{-tA} \upharpoonright L^1 \cap L^2\Big]^{\rm clos}_{L^1 \to L^1} \in \mathcal{B}(L^1), \quad t > 0.$$

is a C_0 semigroup (this is a general fact from the theory of symmetric Markov semigroups). Its generator $-A_1$ is an appropriate operator realization of the formal operator $-\nabla \cdot a \cdot \nabla$ in L^1.

Given a vector field $b \in [L^1_{\rm loc}]^d$, we define in L^1 operator $B_{\max} \supset b \cdot \nabla$ of domain

$$D(B_{\max}) := \{f \in L^1 \mid f \in W^{1,1}_{\rm loc} \text{ and } b \cdot \nabla f \in L^1\}.$$

The following result will allow us to construct an operator realization of the formal Kolmogorov operator $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$, with $a \in (H_{\sigma,\xi})$ measurable and $b \in \mathbf{N}_e$ locally unbounded, in L^1.

Proposition 3.1. *Let $b \in \mathbf{N}_e$. Then $D(B_{\max}) \supset D(A) \cap D(A_1)$ and $B_{\max} \upharpoonright D(A_1) \cap D(A)$ extends by continuity in the graph norm of A_1 to A_1-bounded operator $(b \cdot \nabla)_1$:*

$$\|(b \cdot \nabla)_1 f\|_1 \leq \eta \|A_1 f\|_1 + \eta\mu \|f\|_1, \quad f \in D(A_1),$$

with bound $\eta := \frac{1}{1-e^{-\mu h}} \sqrt{\frac{c_0}{\sigma c_4}}\, n_e(b, hc_4)$, $\mu > 0$. Here and below,

$$c_0 := 2c_3c_5 + \frac{d}{2},$$

where c_i $(i = 3, 4, 5)$ are generic constants in the Gaussian bounds on the heat kernel $e^{-tA}(x, y)$ and its time derivative in Theorem 2.1.

We will also need the following standard result. Since e^{-tA_1} and e^{-tA} have the same integral kernel $e^{-tA}(x, y)$ which satisfies $|\partial_t e^{-tA}(x, y)| \leq c_5 t^{-1} k_{c_6}(t, x - y)$ (Theorem 2.1), there exists a generic constant $C > 0$ such that $(CtD_t e^{-tA_1})^n$ are uniformly (in $0 \leq t \leq 1$ and $n = 1, 2 \dots$) bounded in $\mathcal{B}(L^1)$, and so, by a classical result [15, Ch. IX, sect. 10],

$$\|(\zeta + A_1)^{-1}\|_{1 \to 1} \leq \frac{M}{|\zeta|}, \quad \mathrm{Re}\zeta > 0 \tag{3}$$

with generic constant M.

Theorem 3.2. *Let $a \in (H_{\sigma,\xi})$, $b \in \mathbf{N}_e$ with the Nash norm*

$$n_e(b, hc_4) < \sqrt{\frac{\sigma c_4}{c_0}}$$

for some $h > 0$ (the constants c_0, c_4 were introduced above).

The following is true:

(i) The algebraic sum $\Lambda_1 := A_1 + (b \cdot \nabla)_1$, $D(\Lambda_1) = D(A_1)$ generates a quasi bounded holomorphic semigroup $e^{-t\Lambda_1}$ in L^1 with the sector of holomorphy

$$\{z \in \mathbb{C} \mid |\arg z| < \frac{\pi}{2} - \theta\}, \quad \text{where } \tan\theta = \sqrt{2}\Big(\frac{M}{1 - \sqrt{\frac{c_0}{\sigma c_4}} n_e(b, hc_4)} - 1\Big).$$

The operator Λ_1 is an operator realization of the formal Kolmogorov operator $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ in L^1.

(ii)

$$e^{-t\Lambda_1} = s\text{-}L^1\text{-}\lim_{\varepsilon\downarrow 0} e^{-t\Lambda_1^\varepsilon} \quad (\textit{loc. uniformly in } t \geq 0),$$

where

$$\Lambda_1^\varepsilon := -\nabla \cdot a_\varepsilon \cdot \nabla + b_\varepsilon \cdot \nabla, \quad D(\Lambda_1^\varepsilon) = \mathcal{W}^{2,1}$$

are the approximating operators, with smooth matrices $a_\varepsilon \in (H_{\sigma,\xi})$ and smooth bounded vector fields b_ε constructed in such a way that

$$a_\varepsilon \to a \quad \textit{strongly in } [L^2_{\rm loc}]^{d\times d}, \quad b_\varepsilon \to b \quad \textit{strongly in } [L^2_{\rm loc}]^d \quad \textit{as } \varepsilon \downarrow 0,$$

and the Nash norm of b_ε for all small $\varepsilon > 0$ is controlled by the Nash norm of b:

$$n_e(b_\varepsilon, h) \leq n_e(b,h) + \tilde{c}\varepsilon \quad (\tilde{c} \textit{ generic constant}).$$

The semigroup $e^{-t\Lambda_1}$ conserves positivity and is a L^∞ contraction (and so the convergence in (ii) holds for $e^{-t\Lambda_r}$ in L^r for all $1 < r < \infty$).

Moreover, there exists a generic constant $\tilde{n} > 0$ such that if $n_e(b, hc_4) \leq \tilde{n}$, then we further have:

(iii) For every $t > 0$, $e^{-t\Lambda_1}$ is an integral operator.

(iv) The heat kernel $e^{-t\Lambda}(x,y)$ ($\equiv$ the integral kernel of $e^{-t\Lambda_1}$) satisfies, possibly after redefinition on a measure zero set in $\mathbb{R}^d \times \mathbb{R}^d$, the lower and upper Gaussian bounds:

For every $\xi_1 > \xi$ there exist generic constants $\sigma_1 \in]0,\sigma[$ and $c_i > 0$, $\omega_i \geq 0$, $i = 1,2$ such that*

$$c_1 e^{-t\omega_1} k_{\sigma_1}(t, x-y) \leq e^{-t\Lambda}(x,y) \leq c_2 e^{t\omega_2} k_{\xi_1}(t, x-y)$$

for all $t > 0$, $x, y \in \mathbb{R}^d$.

(v) $e^{-t\Lambda_1}$ conserves probability:

$$\langle e^{-t\Lambda}(x,\cdot)\rangle = 1 \quad \textit{for every } x \in \mathbb{R}^d.$$

(vi) For every $f \in L^1$, $u(t,\cdot) := e^{-t\Lambda_1} f(\cdot)$ is Hölder continuous (possibly after redefinition on a measure zero set in $\mathbb{R}^d \times \mathbb{R}^d$), i.e. for every $0 < \alpha < 1$ there exist generic constants $C < \infty$ and $\beta \in]0,1[$ such that for all $z \in \mathbb{R}^d$, $s > R^2$, $0 < R \leq 1$*

$$|u(t,x) - u(t',x')| \leq C \|u\|_{L^\infty([s-R^2,s]\times \bar{B}(z,R))} \left(\frac{|t-t'|^{\frac{1}{2}} + |x-x'|}{R}\right)^\beta$$

for all (t,x), $(t',x') \in [s-(1-\alpha^2)R^2, s] \times \bar{B}(z,(1-\alpha)R)$.

Furthermore, $u \geq 0$ satisfies the Harnack inequality: Let $0 < \alpha < \beta < 1$ and $\gamma \in]0,1[$, then there exists a constant $K = K(d,\sigma,\xi,\alpha,\beta,\gamma) < \infty$ such that for all $(s,x) \in]R^2,\infty[\times\mathbb{R}^d$, $0 < R \leq 1$ one has

$$u(t,y) \leq K u(s,x)$$

for all $(t,y) \in [s-\beta R^2, s-\alpha^2 R^2] \times \bar{B}(x,\delta R)$.

(vii)

$$e^{-t\Lambda_{C_\infty}} := \left[e^{-t\Lambda_1} \upharpoonright C_\infty \cap L^1\right]^{\rm clos}_{C_\infty \to C_\infty}, \quad t > 0$$

is a Feller semigroup in C_∞ having the property $e^{-t\Lambda_{C_\infty}}[L^\infty \cap L^1] \subset C_\infty$, $t > 0$. Moreover,

$$e^{-t\Lambda_{C_u}} f(x) := \langle e^{-t\Lambda}(x,\cdot) f(\cdot)\rangle, \quad t > 0$$

is a Feller semigroup on C_u, the space of bounded uniformly continuous functions on $\mathbb{R}^d$.

(viii) For every $c_6 > \xi$ there exists a generic constant c_5 such that*

$$|\partial_t e^{-t(\omega_2+\Lambda_1)}(x,y)| \leq c_5 t^{-1} k_{c_6}(t, x-y)$$

for all $t > 0$, x, $y \in \mathbb{R}^d$.

(ix) For every $1 < p < \infty$,

$$e^{-t\Lambda_p} := \left[e^{-t\Lambda_1} \upharpoonright L^1 \cap L^p\right]^{\text{clos}}_{L^p \to L^p}$$

is a quasi bounded holomorphic semigroup with the same sector of holomorphy as in (i).

(x) For every $\frac{1}{2} < \alpha \leq 1$,

$$\|\nabla(\zeta + \Lambda_1)^{-\alpha}\|_{1\to 1} \leq C(\mathrm{Re}\zeta)^{-\alpha+\frac{1}{2}}.$$

4. Recall that a vector field b is said to be form-bounded (with respect to $A \equiv A_2$) if there exist finite constants $\delta > 0$ and $c(\delta) \geq 0$ such that the quadratic inequality

$$\|b_a f\|_2^2 \leq \delta \|A^{\frac{1}{2}} f\|_2^2 + c(\delta)\|f\|_2^2$$

is valid for all $f \in D(A^{\frac{1}{2}}) \equiv W^{1,2}$, where $b_a := \sqrt{b \cdot a^{-1} \cdot b}$. We write $b \in \mathbf{F}_\delta(A)$.

It is easily seen that

$$b \in \mathbf{F}_\delta(-\Delta) \quad \Rightarrow \quad b \in \mathbf{F}_{\delta_a}(A) \text{ with } \delta_a = \sigma^{-2}\delta.$$

The class $\mathbf{F}_\delta(A)$ contains, in particular, the vector fields

$$b = b_1 + b_2, \quad |b_1| \in L^d, \quad |b_2| \in L^\infty,$$

and for every such b the form-bound δ can be chosen arbitrarily small. The class $\mathbf{F}_\delta(A)$ also contains vector fields having critical-order singularities. For instance,

$$b(x) = \pm\sqrt{\delta}\frac{d-2}{2}|x|^{-2}x \in \mathbf{F}_\delta(-\Delta) \quad \text{ with } c(\delta) = 0$$

(by Hardy's inequality). More generally, $\mathbf{F}_\delta(A)$ contains the vector fields $b = b_1 + b_2$ with $|b_1|$ in the weak L^d class or the Campanato-Morrey class, and $|b_2| \in L^\infty$, with δ depending on the norm of $|b_1|$ in the respective classes. Moreover, for every $\varepsilon > 0$ one can find vector fields $b \in \mathbf{F}_\delta(A)$ such that $|b| \notin L^{2+\varepsilon}_{\text{loc}}$. We refer to [8, sect. 4] for details and other examples.

THEOREM 3.3. *Let $d \geq 3$, assume that $b \in \mathbf{N}_e$ with the same norm $n_e(b,h)$ as in Theorem 3.2(iii)-(x) for some $h > 0$. Additionally, assume that $b \in \mathbf{F}_\beta(-\Delta)$ for some $\beta < \infty$. Then*

$$\|\nabla e^{-t\Lambda_1}\|_{1\to 1} \leq C t^{-\frac{1}{2}} e^{\omega_2 t}, \quad t > 0, \tag{4}$$

with constant C depending on d, σ, ξ, $n_e(b,h)$, β and $c(\beta)$.

Remark 3.1. It is not clear how to extend (4) and the bound in Theorem 3.2(x) to

$$\|\nabla e^{-t\Lambda_p}\|_{p\to p} \leq C_p t^{-\frac{1}{2}} e^{\nu_p t}, \quad \|\nabla(\zeta+\Lambda_p)^{-1}\|_{p\to p} \leq c_p(\mathrm{Re}\zeta)^{-\frac{1}{2}} \tag{$*$}$$

for *some* $p>1$. Of course, if also $b \in \mathbf{F}_\beta(A)$ with $\beta<1$, then by standard theory $\|\nabla e^{-t\Lambda_2}\|_{2\to 2} \leq C_2 t^{-\frac{1}{2}} e^{\nu_2 t}$, $t>0$ for constants C_2, ν_2 depending on d, ξ, σ, β and $c(\beta)$, and so $(*)$ follows by interpolation for all $p \in [1,2]$ (similarly for $\nabla(\zeta+\Lambda_p)^{-1}$).

4. Nash's function $\mathcal{N}_\delta$

Put $p(t,x,y) \equiv p_\varepsilon(t,x,y) := e^{-tA^\varepsilon}(x,y)$, where $A^\varepsilon := -\nabla\cdot a_\varepsilon \cdot \nabla$, $a_\varepsilon \equiv E_\varepsilon a$ (the De Giorgi mollifier, see above). Below we write for brevity $a \equiv a_\varepsilon$.

Define Nash's function

$$\mathcal{N}_\delta(t,x) := \big\langle \nabla_. p(t,\cdot,x) \cdot \frac{a(\cdot)}{k_\delta(t,x-\cdot)} \cdot \nabla_. p(t,\cdot,x)\big\rangle, \quad \delta>0.$$

In what follows, we use function $\mathcal{N}_\delta$ (and its counterpart $\hat{\mathcal{N}_\delta}$, see Section 5) with essentially the same purpose as J. Nash did himself in [11].

Proposition 4.1. *If $\delta = c_4$ then there exists a generic constant c_0 such that*

$$\mathcal{N}_\delta(t,x) \leq \frac{c_0}{t}, \quad (t,x) \in]0,\infty[\times\mathbb{R}^d.$$

Proof. Write $\mathcal{N}_\delta = \big\langle \nabla p \cdot \frac{a}{k_\delta} \cdot \nabla p\big\rangle$. Integrating by parts and using the equation $\big(\partial_t + A^\varepsilon\big)p(t,\cdot,x)=0$, we have

$$\mathcal{N}_\delta = \big\langle -\partial_t p, \frac{p}{k_\delta}\big\rangle + \big\langle \nabla p \cdot \frac{ap}{k_\delta^2} \cdot \nabla k_\delta\big\rangle.$$

Let us show that the RHS is finite. By (UGB^p), (UGB$^{\partial_t p}$) and by our choice of δ,

$$\big|\langle -\partial_t p, \frac{p}{k_\delta}\rangle\big| \leq c_3 c_5 t^{-1} \big\langle \frac{k_{c_6} k_{c_4}}{k_\delta}\big\rangle = \frac{c_3 c_5}{t};$$

Due to (UGB^p) and a *qualitative* bound $|\nabla_x p(t,x,y)| \leq C t^{-1/2} k_c(t,x,y)$ (i.e. the constants C, c depend on ε), we have $|\big\langle \nabla p \cdot \frac{ap}{k_\delta^2} \cdot \nabla k_\delta \big\rangle| < \infty$ and hence $\mathcal{N}_\delta < \infty$.

By quadratic inequalities and (UGB^p),

$$\big|\big\langle \nabla p \cdot \frac{ap}{k_\delta^2} \cdot \nabla k_\delta\big\rangle\big| \leq c_3 \mathcal{N}_\delta^{\frac{1}{2}} \big\langle \nabla k_\delta \cdot \frac{a}{k_\delta}\left(\frac{k_{c_4}}{k_\delta}\right)^2 \cdot \nabla k_\delta\big\rangle^{\frac{1}{2}},$$

$$\langle \nabla k_\delta \cdot \frac{a k_{c_4}^2}{k_\delta^3} \cdot \nabla k_\delta\rangle \leq \xi \big\langle \frac{(\nabla k_\delta)^2}{k_\delta}\big\rangle = \frac{\xi d}{2\delta}\frac{1}{t} < \frac{d}{2}\frac{1}{t}.$$

and so

$$\mathcal{N}_\delta \leq 2\big\langle -\partial_t p, \frac{p}{k_\delta}\big\rangle + c_3^2 \big\langle \nabla k_\delta \cdot \frac{a}{k_\delta} \cdot \nabla k_\delta\big\rangle \leq \frac{c_0}{t}, \qquad \text{where } c_0 = 2c_3c_5 + \frac{d}{2}.$$

□

5. Proof of Theorem 3.1

5.1. **Auxiliary estimates.** For a given $\lambda > 0$, denote

$$k_\lambda := k_\lambda(\tau - s, y - \cdot) \quad \text{and} \quad \hat{k}_\lambda := k_\lambda(t-\tau, x - \cdot), \qquad s < \tau < t$$

and

$$\langle \frac{(\nabla k_\lambda)^2}{k_\lambda} \rangle := \langle \frac{(\nabla . k_\lambda(\tau - s, y - \cdot))^2}{k_\lambda(\tau - s, y - \cdot)} \rangle.$$

The next three facts are evident:

($\mathbf{a_1}$)

$$\langle \frac{(\nabla k_\lambda)^2}{k_\lambda} \rangle = \frac{d}{2\lambda}\frac{1}{\tau - s} = \langle \big(\frac{y - \cdot}{2\lambda(\tau - s)}\big)^2 k_\lambda(\tau - s, y - \cdot) \rangle,$$

$$\langle \frac{(\nabla \hat{k}_\lambda)^2}{\hat{k}_\lambda} \rangle = \frac{d}{2\lambda}\frac{1}{t - \tau}.$$

($\mathbf{a_2}$) If $\lambda < \lambda_1$, then $k_\lambda \leq \big(\frac{\lambda_1}{\lambda}\big)^{\frac{d}{2}} k_{\lambda_1}$.

($\mathbf{a_3}$) If $2\delta > c_4$, then

$$\frac{k_{c_4}^2}{k_\delta} = \big(\frac{\delta^2}{(2\delta - c_4)c_4}\big)^{\frac{d}{2}} k_{\frac{\delta c_4}{2\delta - c_4}}.$$

$$(\mathbf{a_4^-}) \begin{cases} 0 < 2\delta < \lambda \\ 0 < \varepsilon < 1 \\ 0 < \tau - s < (t-s)\varepsilon \end{cases} \Rightarrow \begin{cases} \hat{k}_\lambda^2 k_\delta \leq c_-^2 k_{\frac{\lambda\delta}{\lambda - 2\delta}} \cdot k_\lambda^2(t-s, x-y), \\ \text{where } c_- := (1-\varepsilon)^{-d/2}\big(\frac{\lambda}{\lambda - 2\delta}\big)^{d/4}. \end{cases}$$

$$(\mathbf{a_4^+}) \begin{cases} 0 < 2\delta < \lambda \\ \frac{\lambda}{2(\lambda - \delta)} < \varepsilon < 1 \\ (t-s)\varepsilon < \tau - s < t - s \end{cases} \Rightarrow \begin{cases} \hat{k}_\lambda k_{2\delta}^2 \leq c_+^2 \hat{k}_{\frac{\lambda}{r}} \cdot k_\lambda^2(t-s, x-y), \\ \text{where } c_+ := \varepsilon^{-d/2}\big(\frac{\lambda}{2\delta}\big)^{d/2} r^{-d/2}, r = \frac{2(\lambda-\delta)\varepsilon - \lambda}{\lambda - 2\delta\varepsilon}. \end{cases}$$

Proof of ($\mathbf{a_4^-}$). Using $ab \leq a^2 + 4^{-1}b^2$ and $t - \tau \geq (1-\varepsilon)(t-s)$ we have, for any $\alpha \in \mathbb{R}^d$, $\alpha \neq 0$,

$$\begin{aligned} & e^{\alpha\cdot(x-y)}\hat{k}_\lambda^2 k_\delta = e^{\alpha\cdot(x-\cdot)}\hat{k}_\lambda^2 e^{\alpha\cdot(\cdot - y)} k_\delta \\ & \leq (1-\varepsilon)^{-d}\big(4\pi\lambda(t-s)\big)^{-d} e^{\alpha^2\frac{\lambda}{2}(t-\tau)} \cdot \big(4\pi\delta(\tau - s)\big)^{-d/2} e^{\alpha^2\frac{\lambda}{2}(\tau - s)} e^{-\frac{|\cdot - y|^2}{4(\tau - s)}\left(\frac{1}{\delta} - \frac{2}{\lambda}\right)} \\ & = (1-\varepsilon)^{-d}\big(\lambda/(\lambda - 2\delta)\big)^{d/2} k_{\frac{\lambda\delta}{\lambda - 2\delta}} \cdot \big(4\pi\lambda(t-s)\big)^{-d} e^{\alpha^2\frac{\lambda}{2}(t-s)}; \end{aligned}$$

Therefore,

$$\hat{k}_\lambda^2 k_\delta \leq (1-\varepsilon)^{-d}\big(\lambda/(\lambda - 2\delta)\big)^{d/2} k_{\frac{\lambda\delta}{\lambda - 2\delta}} \cdot \big(4\pi\lambda(t-s)\big)^{-d} e^{-\alpha\cdot(x-y) + \alpha^2\frac{\lambda}{2}(t-s)}$$

Set $\alpha = \frac{x-y}{\lambda(t-s)}$. □

Proof of ($\mathbf{a_4^+}$). Using $ab \leq a^2 + 4^{-1}b^2$ and $\varepsilon(t-s) \leq \tau - s$ we have, for any $\alpha \in \mathbb{R}^d$, $\alpha \neq 0$ and $r \in]0,1[$,

$$\begin{aligned} & e^{\alpha\cdot(x-y)}\hat{k}_\lambda k_{2\delta}^2 = e^{\alpha\cdot(\cdot - y)} k_{2\delta}^2 e^{\alpha\cdot(x - \cdot)}\hat{k}_\lambda \\ & \leq \varepsilon^{-d}(\lambda/(2\delta))^d\big(4\pi\lambda(t-s)\big)^{-d} e^{\alpha^2\delta(\tau - s)} \cdot \big(4\pi\lambda(t-\tau)\big)^{-d/2} e^{\alpha\cdot(x-\cdot) - \frac{|x - \cdot|^2}{4\lambda(t-\tau)}(1 - r + r)} \\ & \leq \varepsilon^{-d}(\lambda/(2\delta)^d r^{-d/2}\hat{k}_{\frac{\lambda}{r}} \cdot \big(4\pi\lambda(t-s)\big)^{-d} e^{\alpha^2\delta(\tau - s) + \alpha^2\frac{\lambda}{1-r}(t-\tau)}; \end{aligned}$$

Using $t-\tau \leq (1-\varepsilon)(t-s)$ and taking into account our choice of r and ε, we have

$$\begin{aligned}\delta(\tau-s)+\frac{\lambda}{1-r}(t-\tau) &= \delta(t-s)+\big(\frac{\lambda}{1-r}-\delta\big)(t-\tau) \\ &\leq \delta(t-s)+\big(\frac{\lambda}{1-r}-\delta\big)(1-\varepsilon)(t-s)=\frac{\lambda}{2}(t-s).\end{aligned}$$

Therefore

$$\hat{k}_\lambda k_{2\delta}^2 \leq \varepsilon^{-d}(\lambda/(2\delta)^d r^{-d/2}\hat{k}_{\frac{\lambda}{r}} \cdot \big(4\pi\lambda(t-s)\big)^{-d} e^{-\alpha\cdot(x-y)+\alpha^2\frac{\lambda}{2}(t-s)}.$$

Set $\alpha=\frac{x-y}{\lambda(t-s)}$. □

5.2. **Nash's function $\hat{\mathcal{N}}_\delta$.** Let $p(t,x,y)$ denote the heat kernel of $\partial_t + A^\varepsilon$, $A^\varepsilon \equiv -\nabla\cdot a_\varepsilon \cdot \nabla$. Put for brevity $a\equiv a_\varepsilon$. Define

$$\hat{\mathcal{N}}_\delta(t-\tau,\tau-s,x,y) := \Big\langle \nabla_\cdot p(\tau-s,\cdot,y)\cdot \frac{a(\cdot)k_\lambda(t-\tau,x,\cdot)}{k_{2\delta}^2(\tau-s,y,\cdot)}\cdot \nabla_\cdot p(\tau-s,\cdot,y)\Big\rangle,$$

for all $s<\tau<t$, $x,y\in\mathbb{R}^d$.

Proposition 5.1. *Let $c_4,c_6<2\delta<\lambda$, fix $0<\varepsilon<1$. There exists a generic constant $\hat{c}_0$ such that*

$$\hat{\mathcal{N}}_\delta(t-\tau,\tau-s,x,y)\leq \frac{\hat{c}_0}{t-\tau}$$

for all $t>s$, $(t-s)\varepsilon<\tau-s<t-s$, $x,y\in\mathbb{R}^d$.

Proof. Write $\hat{\mathcal{N}}_\delta = \langle \nabla p \cdot \frac{a\hat{k}_\lambda}{k_{2\delta}^2}\cdot\nabla p\rangle$. Integrating by parts and using the equation $\big(\partial_\tau + A^\varepsilon\big)p(\tau-s,\cdot,y)=0$, we obtain

$$\hat{\mathcal{N}}_\delta = \big\langle -\partial_\tau p, \frac{\hat{k}_\lambda p}{k_{2\delta}^2}\big\rangle - \big\langle \nabla p\cdot\frac{ap}{k_{2\delta}^2}\cdot\nabla\hat{k}_\lambda\big\rangle + 2\big\langle\nabla p\cdot\frac{ap\hat{k}_\lambda}{k_{2\delta}^3}\cdot\nabla k_{2\delta}\big\rangle.$$

By quadratic inequalities,

$$\begin{aligned}|\big\langle\nabla p\cdot\frac{ap}{k_{2\delta}^2}\cdot\nabla\hat{k}_\lambda\big\rangle| &\leq \frac{1}{4}\hat{\mathcal{N}}_\delta + \big\langle\nabla\hat{k}_\lambda\cdot\frac{ap^2}{k_{2\delta}^2\hat{k}_\lambda}\cdot\nabla\hat{k}_\lambda\big\rangle \\ &\equiv \frac{1}{4}\hat{\mathcal{N}}_\delta + M_1, \\ 2|\big\langle\nabla p\cdot\frac{ap\hat{k}_\lambda}{k_{2\delta}^3}\cdot\nabla k_{2\delta}\big\rangle| &\leq \frac{1}{4}\hat{\mathcal{N}}_\delta + 4\big\langle\nabla k_{2\delta}\cdot\frac{ap^2\hat{k}_\lambda}{k_{2\delta}^4}\cdot\nabla k_{2\delta}\big\rangle \\ &\equiv \frac{1}{4}\hat{\mathcal{N}}_\delta + 4M_2.\end{aligned}$$

Therefore,

$$\hat{\mathcal{N}}_\delta \leq 2\big\langle -\partial_\tau p,\frac{\hat{k}_\lambda p}{k_{2\delta}^2}\big\rangle + 2M_1 + 8M_2. \tag{$*$}$$

Let us estimate the terms in the RHS of $(*)$.

By (UGB^p), (UGB$^{\partial_t p}$) and by our choice of δ,

$$
\begin{aligned}
\big|\big\langle -\partial_\tau p, \frac{\hat{k}_\lambda p}{k_{2\delta}^2}\big\rangle\big| &\leq c_3 c_5 (\tau - s)^{-1} \big\langle \frac{k_{c_6} k_{c_4} \hat{k}_\lambda}{k_{2\delta}^2} \big\rangle \\
&\leq c_3 c_5 (\tau - s)^{-1} \left(\frac{(2\delta)^2}{c_4 c_6} \right)^{\frac{d}{2}} \langle \hat{k}_\lambda \rangle = c_3 c_5 (\tau - s)^{-1} \left(\frac{(2\delta)^2}{c_4 c_6} \right)^{\frac{d}{2}}.
\end{aligned}
$$

Taking into account that $\tau - s > \varepsilon(t-s) \Rightarrow \frac{1}{\tau - s} < \frac{1-\varepsilon}{\varepsilon} \frac{1}{t - \tau}$, we thus obtain

$$
\big|\big\langle -\partial_\tau p, \frac{\hat{k}_\lambda p}{k_{2\delta}^2}\big\rangle\big| \leq c_3 c_5 \left(\frac{(2\delta)^2}{c_4 c_6} \right)^{\frac{d}{2}} \frac{1-\varepsilon}{\varepsilon} \frac{1}{t-\tau}.
$$

Next, using ($\mathbf{a_1}$)-($\mathbf{a_3}$), we have:

$$
\begin{aligned}
M_1 &\leq \xi c_3^2 \left\langle \left(\frac{k_{c_4}}{k_{2\delta}} \right)^2 \frac{(\nabla \hat{k}_\lambda)^2}{\hat{k}_\lambda} \right\rangle \\
&\leq \xi c_3^2 \left(\frac{2\delta}{c_4} \right)^d \big\langle \frac{(\nabla \hat{k}_\lambda)^2}{\hat{k}_\lambda} \big\rangle \\
&= \xi c_3^2 \left(\frac{2\delta}{c_4} \right)^d \frac{d}{2\lambda} \frac{1}{t-\tau}.
\end{aligned}
$$

$$
M_2 \leq \xi c_3^2 \Big\langle \left(\frac{k_{c_4}}{k_{2\delta}} \right)^2 \hat{k}_\lambda (\nabla \log k_{2\delta})^2 \Big\rangle,
$$

where

$$
\begin{aligned}
\left(\frac{k_{c_4}}{k_{2\delta}} \right)^2 &= \left(\frac{2\delta}{c_4} \right)^d \exp\left[-\frac{|y - \cdot|^2}{4(\tau - s)} \left(\frac{1}{c_4} - \frac{1}{2\delta} \right) 2 \right] \\
&= \left(\frac{2\delta}{c_4} \right)^d \exp\left[-\frac{|y - \cdot|^2}{4\gamma(\tau - s)} \right], \qquad \gamma := \frac{\delta c_4}{2\delta - c_4},
\end{aligned}
$$

$$
(\nabla \log k_{2\delta})^2 = \left(\frac{y - \cdot}{2(2\delta)(\tau - s)} \right)^2 = \frac{|y - \cdot|^2}{4\gamma(\tau - s)} \frac{\gamma}{(2\delta)^2} \frac{1}{\tau - s}.
$$

Since $0 < \eta < e^\eta$, we have therefore

$$
\left\langle \left(\frac{k_{c_4}}{k_{2\delta}} \right)^2 \hat{k}_\lambda (\nabla \log k_{2\delta})^2 \right\rangle \leq \left(\frac{2\delta}{c_4} \right)^d \frac{\gamma}{(2\delta)^2} \frac{1}{\tau - s} \langle \hat{k}_\lambda \rangle,
$$

and so

$$
M_2 \leq \xi c_3^2 \left(\frac{2\delta}{c_4} \right)^d \frac{c_4}{(2\delta - c_4) 4\delta} \frac{1-\varepsilon}{\varepsilon} \frac{1}{t-\tau}.
$$

Substituting the previous estimates into $(*)$, we obtain

$$
\mathcal{N}_\delta \leq 2\, c_3 c_5 \left(\frac{(2\delta)^2}{c_4 c_6} \right)^{\frac{d}{2}} \frac{1-\varepsilon}{\varepsilon} \frac{1}{t-\tau} + c_3^2 \left(\frac{2\delta}{c_4} \right)^d \left(2 \cdot \frac{\xi d}{2\lambda} + 8 \cdot \frac{2\xi}{4\delta} \cdot \frac{c_4}{2\delta - c_4} \cdot \frac{1-\varepsilon}{\varepsilon} \right) \frac{1}{t-\tau},
$$

as claimed. □

5.3. **Proof of the upper bound.** For brevity, $b \equiv b_\varepsilon$. We iterate the Duhamel formula

$$u(t-s,x,y) = p(t-s,x,y) - \int_s^t \langle u(t-\tau,x,\cdot)b(\cdot)\cdot\nabla.p(\tau-s,\cdot,y)\rangle d\tau.$$

We obtain the series

$$l(t-s,x,y) := \sum_{n=0}^{\infty}(-1)^n u_n(t-s,x,y),$$

where $u_0(t-s,x,y) := p(t-s,x,y)$ and, for $n=1,2,\dots,$

$$u_n(t-s,x,y) := \int_s^t \langle u_{n-1}(t-\tau,x,\cdot)b(\cdot)\cdot\nabla.p(\tau-s,\cdot,y)\rangle d\tau.$$

In particular,

$$u_1(t-s,x,y) = \int_s^t \langle p(t-\tau,x,\cdot)b(\cdot)\cdot\nabla.p(\tau-s,\cdot,y)\rangle d\tau,$$

and so

$$|u_1(t-s,x,y)| \leq c_3\int_s^t \langle k_{c_4}(t-\tau,x-\cdot)|b(\cdot)\cdot\nabla.p(\tau-s,\cdot,y)|\rangle d\tau.$$

Suppose that we are able to find generic constants $h>0$ and $C_h<1$ such that the bound:*

$$\int_s^t \langle k_{c_4}(t-\tau,x-\cdot)|b(\cdot)\cdot\nabla.p(\tau-s,\cdot,y)|\rangle d\tau \leq C_h k_{c_4}(t-s,x-y) \qquad (\star^b\star^N)$$

is valid for all $x,y\in\mathbb{R}^d$ and $0<t-s\leq h$.

Then $|u_1(t-s,x,y)| \leq c_3 C_h k_{c_4}(t-s,x-y)$, and by induction,

$$|u_n(t-s,x,y)| \leq c_3\big(C_h\big)^n k_{c_4}(t-s,x-y).$$

Therefore, for all $0<t-s\leq h$ and all $x,y\in\mathbb{R}^d$, the series $l(t-s,x,y)$ is well defined and

$$|l(t-s,x,y)| \leq \frac{c_3}{1-C_h}k_{c_4}(t-s,x-y).$$

Repeating the standard argument we conclude that l satisfies the Duhamel formula provided that $0<t-s\leq h$. Then the uniqueness of $u(t-s,x,y)$ implies

$$u = l \quad (0<t-s\leq h),$$

and the reproduction property of u implies

$$u(t-s,x,y) \leq \frac{c_3}{1-C_h}e^{(t-s)\omega_h}k_{c_4}(t-s,x-y)$$

for all $t-s>h$, where $\omega_h = \frac{1}{h}\log\frac{c_3}{1-C_h}$. Thus, we obtain the upper bound in (LUGB^u) of Theorem 3.1.

It remains to prove $(\star^b\star^N)$. Without loss of generality, $s=0$. Set $b_a^2 := b\cdot a^{-1}\cdot b$ and denote

$$\langle k_\mu b_a^2\rangle := \langle k_\mu(\tau,y-\cdot)b_a^2(\cdot)\rangle, \qquad \langle\hat{k}_\mu b_a^2\rangle := \langle k_\mu(t-\tau,x-\cdot)b_a^2(\cdot)\rangle.$$

Set

$$I := \int_0^t \langle k_\lambda(t-\tau,x-\cdot)|b(\cdot)\cdot\nabla.p(\tau,\cdot,y)|\rangle d\tau.$$

Lemma 5.1. *Fix $\lambda > \xi$ and select constants δ, c_4 such that*

$$\lambda > 2\delta > c_4 > \xi.$$

Let $\frac{\lambda}{2(\lambda-\delta)} < \varepsilon < 1$, $r = \frac{2(\lambda-\delta)\varepsilon-\lambda}{\lambda-2\delta\varepsilon}$, and let $c_\pm$ be the constants defined in ($\mathbf{a_4^\pm}$). Then, for all $x, y \in \mathbb{R}^d$ and $t > 0$,

$$I \leq (c_- M^- + c_+ M^+) k_\lambda(t,x,y),$$

where

$$M^- := \int_0^{t\varepsilon} \sqrt{\langle k_{\frac{\lambda\delta}{\lambda-2\delta}} b_a^2\rangle} \sqrt{\frac{c_0}{\tau}}\, d\tau,$$
$$M^+ := \int_{t\varepsilon}^t \sqrt{\langle \hat{k}_{\frac{\lambda}{r}} b_a^2\rangle} \sqrt{\frac{\hat{c}_0}{t-\tau}}\, d\tau.$$

Proof. Using quadratic inequality, we bound $\langle \hat{k}_\lambda |b\cdot\nabla p|\rangle^2$ in two ways:

$$\langle \hat{k}_\lambda |b\cdot\nabla p|\rangle^2 \leq \langle \hat{k}_\lambda^2 k_\delta b_a^2\rangle \langle \nabla p \cdot \frac{a}{k_\delta} \cdot \nabla p\rangle$$

and

$$\langle \hat{k}_\lambda |b\cdot\nabla p|\rangle^2 \leq \langle \hat{k}_\lambda k_{2\delta}^2 b_a^2\rangle \langle \nabla p \cdot \frac{a\hat{k}_\lambda}{k_{2\delta}^2} \cdot \nabla p\rangle,$$

and hence

$$I \equiv \int_0^t \langle \hat{k}_\lambda |b\cdot\nabla p|\rangle\, d\tau \leq I_\varepsilon^- + I_\varepsilon^+,$$

where

$$I_\varepsilon^- := \int_0^{t\varepsilon} \sqrt{\langle \hat{k}_\lambda^2 k_\delta b_a^2\rangle} \sqrt{\langle \nabla p \cdot \frac{a}{k_\delta} \cdot \nabla p\rangle}\, d\tau$$
$$I_\varepsilon^+ := \int_{t\varepsilon}^t \sqrt{\langle \hat{k}_\lambda k_{2\delta}^2 b_a^2\rangle} \sqrt{\langle \nabla p \cdot \frac{a\hat{k}_\lambda}{k_{2\delta}^2} \cdot \nabla p\rangle}\, d\tau$$

Now the assertion of Lemma 5.1 follows directly from ($\mathbf{a_4^\mp}$) and Propositions 4.1 and 5.1. (Here we apply Propositions 4.1 with δ chosen as in Proposition 5.1, but it is not difficult to see, using ($\mathbf{a_3}$), that its proof works for all $\delta > \frac{c_4}{2}$ although with different generic constant c_0.) ☐

It remains to note that both M_+, M_- in Lemma 5.1 are majorated by $c\, n_e(b,h)$ for appropriate multiple $c > 0$. Provided that $n_e(b,h)$ is sufficiently small, i.e. so that $C_h := (c_- + c_+)cn_e(b,h) < 1$, we obtain ($\star^b\star^N$).

5.4. **Proof of the lower bound.** The analysis of the previous section and the Gaussian upper bound (UGB^p) of Theorem 2.1 yield for $|x-y|^2 \leq t \leq h$

$$\begin{aligned} u(t,x,y) &\geq p(t,x,y) - \sum_{n\geq 1} |u_n(t,x,y)| \\ &\geq c_1 k_{c_2}(t,x-y) - \frac{c_3 C_h}{1-C_h} k_{c_4}(t,x-y) \\ &\geq \left(c_1 c_2^{-\frac{d}{2}} e^{-\frac{1}{4c_2}} - \frac{c_3 C_h}{1-C_h} c_4^{-\frac{d}{2}} \right) (4\pi t)^{-\frac{d}{2}} \\ &\equiv r t^{-\frac{d}{2}}, \end{aligned} \tag{$**$}$$

where $r>0$ provided that C_h is small enough, i.e. $\frac{C_h}{1-C_h} < \frac{c_1}{c_3}\left(\frac{c_4}{c_2}\right)^{\frac{d}{2}} e^{-\frac{1}{4c_2}}$.

Now the standard argument ("small gains yield large gain", see e.g. [2, Theorem 3.3.4]) yields for all $x,y\in\mathbb{R}^d$, $t>0$,

$$u(t,x,y) \geq r e^{t\nu_h} t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{4c_2 t}\right), \quad \nu_h = \frac{1}{h}\log r.$$

The proof of Theorem 3.1 is completed.

6. Proof of Proposition 3.1

1. Let $\mathbf{1}_\varepsilon$, $\varepsilon>0$ be the indicator of $\{x\in\mathbb{R}^d \mid |x|\leq \varepsilon^{-1}, |b(x)|\leq\varepsilon^{-1}\}$. Define

$$b_\varepsilon := E_{\nu_\varepsilon}(\mathbf{1}_\varepsilon b),$$

where, recall, $E_\nu \equiv e^{\nu\Delta}$, and ε, $\nu_\varepsilon>0$.

Define also $(b^2)_\varepsilon = E_{\nu_\varepsilon}(\mathbf{1}_\varepsilon b^2)$ and set $g_{1,\varepsilon} := b_\varepsilon - \mathbf{1}_\varepsilon b$ and $g_{2,\varepsilon} := |(b^2)_\varepsilon - \mathbf{1}_\varepsilon b^2|$.

In what follows, we select $\{\nu_\varepsilon\}$ so that $\nu_\varepsilon \downarrow 0$ sufficiently rapidly as $\varepsilon\downarrow 0$ so that $\|g_{1,\varepsilon}\|_2\leq\varepsilon$ and $\|g_{2,\varepsilon}\|_q \leq \varepsilon^2$ for some $q\geq d$. Note that $(b^2)_\varepsilon \leq g_{2,\varepsilon}+b^2$. Since $\|\mathbf{1}_{B(0,R)}(b_\varepsilon - b)\|_2 \leq \|g_{1,\varepsilon}\|_2 + \|\mathbf{1}_{B(0,R)}(\mathbf{1}_\varepsilon b - b)\|_2$, we have

$$b_\varepsilon \to b \quad \text{strongly in } [L^2_{\rm loc}]^d.$$

The Nash norm of b_ε is controlled by the Nash norm of b:

LEMMA 6.1. *$n_e(b_\varepsilon,h)\leq n_e(b,h)+c_d h^{\frac14}\varepsilon$, $\varepsilon>0$.*

Proof. Clearly, $(b_\varepsilon)^2 \leq (b^2)_\varepsilon$, and so

$$\begin{aligned} n_e(b_\varepsilon,h) &\equiv \sup_{x\in\mathbb{R}^d}\int_0^h \sqrt{e^{t\Delta}(b_\varepsilon)^2(x)}\frac{dt}{\sqrt{t}} \\ &\leq n_e(b,h) + \sup_{x\in\mathbb{R}^d}\int_0^h \sqrt{e^{t\Delta}g_{2,\varepsilon}(x)}\frac{dt}{\sqrt{t}}, \end{aligned}$$

where

$$\begin{aligned} \sup_{x\in\mathbb{R}^d}\int_0^h \sqrt{e^{t\Delta}g_{2,\varepsilon}(x)}\frac{dt}{\sqrt{t}} &\leq \int_0^h \sqrt{\|e^{t\Delta}g_{2,\varepsilon}\|_\infty}\frac{dt}{\sqrt{t}} \leq C_d\int_0^h \sqrt{t^{-\frac{d}{2q}}\|g_{2,\varepsilon}\|_q}\frac{dt}{\sqrt{t}} \\ &\leq \sqrt{\|g_{2,\varepsilon}\|_q}\,C_d\frac{2}{1-\frac{d}{2q}}h^{\frac12-\frac{d}{4q}} \leq 4C_d h^{\frac14}\varepsilon. \end{aligned}$$

□

2. Now we can give

Proof of Proposition 3.1. Set $\delta := c_4$. We will construct $(b\cdot\nabla)_1$ and prove

$$\|(b\cdot\nabla)_1 g\|_1 \leq \eta\|(\zeta+A_1)g\|_1, \quad g\in D(A_1), \tag{5}$$

with $\eta := \frac{1}{1-e^{-{\rm Re}\zeta h}}\sqrt{\frac{c_0}{\sigma\delta}}\, n_e(b,h\delta)$, for all ${\rm Re}\zeta>0$, so taking $\zeta:=\mu>0$ we obtain the assertion of the proposition.

Step 1. Put $B_1^\varepsilon := [b_\varepsilon \cdot \nabla \restriction C_c^1]^{\rm clos}_{L^1 \to L^1}$ of domain $\mathcal{W}^{1,1}$, and

$$T_1^\varepsilon := B_1^\varepsilon(\zeta + A_1^\varepsilon)^{-1} \in \mathcal{B}(L^1),$$

where, recall, $A_1^\varepsilon := -\nabla \cdot a_\varepsilon \cdot \nabla$, $a_\varepsilon \equiv E_\varepsilon a$, $D(A_1^\varepsilon) = \mathcal{W}^{2,1}$. Since B_1^ε is closed, we can write

$$T_1^\varepsilon f(x) = \int_0^\infty e^{-\zeta t} B_1^\varepsilon e^{-tA_1^\varepsilon} f(x) dt = \int_0^\infty e^{-\zeta t} \langle b_\varepsilon(x) \cdot \nabla_x p_\varepsilon(t,x,\cdot) f(\cdot)\rangle dt, \quad f \in \mathcal{W}^{1,1}.$$

Denote $\mu := \mathrm{Re}\zeta$. We have

$$\begin{aligned}\|T_1^\varepsilon f\|_1 &\leq \sum_{j=0}^\infty e^{-j\mu h} \int_{jh}^{(j+1)h} \|B_1^\varepsilon e^{-tA_1^\varepsilon} f\|_1 dt \\ &= \sum_{j=0}^\infty e^{-j\mu h} \int_0^h \|B_1^\varepsilon e^{-tA_1^\varepsilon} e^{-jhA_1^\varepsilon} f\|_1 dt.\end{aligned}$$

By the Fubini Theorem and the Cauchy-Bunyakovsky inequality,

$$\begin{aligned}\int_0^h \|B_1^\varepsilon e^{-tA^\varepsilon} e^{-jhA_1^\varepsilon} f\|_1 dt &\leq \Big\langle \int_0^h \langle |b_\varepsilon(x) \cdot \nabla_x p_\varepsilon(t,x,y)|\rangle_x dt |e^{-jhA_1^\varepsilon} f(y)|\Big\rangle_y \\ &\leq \sup_{y \in \mathbb{R}^d} \int_0^h \langle |b_\varepsilon(x) \cdot \nabla_x p_\varepsilon(t,x,y)|\rangle_x dt \|f\|_1 \\ &\leq \sup_{y \in \mathbb{R}^d} \int_0^h \sqrt{\langle k_\delta(t,x-y)(b_\varepsilon \cdot a_\varepsilon^{-1} \cdot b_\varepsilon)(x)\rangle_x} \sqrt{\mathcal{N}_\delta(t,y)} dt \|f\|_1,\end{aligned}$$

where $\mathcal{N}_\delta(t,y) \equiv \big\langle \nabla_x p_\varepsilon(t,x,y) \cdot \frac{a_\varepsilon(x)}{k_\delta(t,x-y)} \cdot \nabla_x p_\varepsilon(t,x,y)\big\rangle_x \leq \frac{c_0}{t}$ by Proposition 4.1. Therefore,

$$\begin{aligned}\int_0^h \|B_1^\varepsilon e^{-tA_1^\varepsilon} e^{-jhA_1^\varepsilon} f\|_1 dt &\leq \sqrt{\frac{c_0}{\sigma\delta}}\, n_e(b_\varepsilon, h\delta) \|f\|_1 \\ &\qquad \text{(we are applying lemma above)} \\ &\leq \sqrt{\frac{c_0}{\sigma\delta}} \big(n_e(b, h\delta) + c_d h^{\frac{1}{4}} \delta^{\frac{1}{4}} \varepsilon\big) \|f\|_1.\end{aligned}$$

Thus,

$$\|T_1^\varepsilon f\|_1 \leq \eta_\varepsilon \|f\|_1, \quad f \in L^1, \quad \eta_\varepsilon := \eta + \tilde{c}\varepsilon, \quad \mathrm{Re}\zeta > 0.$$

Step 2. Set $Tf := b \cdot \nabla(\zeta + A)^{-1} f$, $f \in L^2$ and note that $\nabla(\zeta + A^\varepsilon)^{-1} \to \nabla(\zeta + A)^{-1}$ strongly in $[L^2]^d$. [The proof is standard: For $1 \leq i \leq d$, $f \in W^{-1,2}$, $\|\nabla_i(\zeta + A^\varepsilon)^{-1} f - \nabla_i(\zeta + A)^{-1} f\|_2 =: M_\varepsilon(f)$,

$$\begin{aligned}M_\varepsilon(f) &:= \|\nabla_i(\zeta + A^\varepsilon)^{-1} \nabla \cdot (a - a_\varepsilon) \cdot \nabla(\zeta + A)^{-1} f\|_2 \\ &\leq \|\nabla_i(\zeta + A^\varepsilon)^{-1} \nabla\|_{2 \to 2} \|(a - a_\varepsilon) \cdot \nabla(\zeta + A)^{-1} f\|_2,\end{aligned}$$

where $\|\nabla_i(\zeta + A^\varepsilon)^{-1}\nabla\|_{2\to 2} \leq \|\nabla(\zeta + A^\varepsilon)^{-\frac{1}{2}}\|_{2\to 2}^2 \leq C$, $C \neq C(\varepsilon)$ and $\|(a - a_\varepsilon) \cdot \nabla(\zeta + A)^{-1} f\|_2 \to 0$ (e.g. using the Dominated Convergence Theorem), so $M_\varepsilon(f) \to 0$ as $\varepsilon \downarrow 0$, in particular, for $f \in L^2$.]

Therefore, since $b_\varepsilon \to b$ strongly in $[L^2_{\rm loc}]^d$,

$$T^\varepsilon f \to Tf \quad \text{strongly in } L^1_{\rm loc} \text{ as } \varepsilon \downarrow 0. \tag{6}$$

Passing to a subsequence in ε, if necessary, we have $T^\varepsilon f \to Tf$ $\mathcal{L}^d$ a.e. Applying Fatou's Lemma, we have by Step 1, for all $f \in L^1 \cap L^2$,

$$\|Tf\|_1 \leq \liminf_{\varepsilon} \|T^\varepsilon f\|_1 \leq \eta \|f\|_1. \tag{7}$$

Let T_1 denote the extension of $T \upharpoonright L^1 \cap L^2$ by continuity to L^1.

Step 3. Since, by Step 2, $\|b \cdot \nabla(\zeta + A)^{-1} f\|_1 \leq \eta \|f\|_1$ for all $f \in L^1 \cap L^2$, $\mathrm{Re}\zeta > 0$, the operator $B := b \cdot \nabla \upharpoonright D(A_1) \cap D(A) : L^1 \to L^1$, and

$$\|b \cdot \nabla h\|_1 \leq \eta \|(\zeta + A_1)h\|_1, \quad h \in D(A_1) \cap D(A).$$

Since $D(A_1) \cap D(A)$ $(= (1 + A)^{-1}[L^1 \cap L^2])$ is a core of A_1, B extends by continuity in the graph norm of A_1 to A_1-bounded operator $(b \cdot \nabla)_1$. The proof of Proposition 3.1 is completed. □

Remark 6.1. The proof above can be extended to non-local operators of the type $\Lambda = (\mu - \nabla \cdot a \cdot \nabla)^{\frac{\alpha}{2}} + b \cdot \nabla$, $1 < \alpha < 2$, with b in an appropriate modification of the elliptic Nash class.

That is, assume that $b \in [L^2_{\mathrm{loc}}]^d$ satisfies

$$\tilde{n}^\alpha(b, \mu) = \sup_{y \in \mathbb{R}^d} \int_0^\infty e^{-\mu t} \sqrt{e^{t\Delta}|b|^2(y)} \frac{dt}{t^{\frac{3-\alpha}{2}}} < \infty, \quad \mu > 0.$$

Put $T_1^\varepsilon := b_\varepsilon \cdot \nabla(\mu + A_1^\varepsilon)^{-\frac{\alpha}{2}}$. A key bound $\|T_1^\varepsilon f\|_1 \leq \tilde{\eta} \|f\|_1$, $f \in L^1$ remains valid with $\tilde{\eta} = \delta^{\frac{1-\alpha}{2}} \sqrt{\frac{c_0}{\sigma}} \tilde{n}^\alpha(b, \mu\delta^{-1})$. Namely,

$$\begin{aligned}\|T_1^\varepsilon f\|_1 &\leq \left(\sup_y \int_0^\infty e^{-\mu t} t^{\frac{\alpha}{2}-1} \sqrt{\langle k_\delta(t, y - \cdot) b_a^2(\cdot)\rangle} \sqrt{\mathcal{N}_\delta(t, y)} dt \right) \|f\|_1 \qquad (b_a^2 = b \cdot a^{-1} \cdot b) \\ &\leq \delta^{\frac{1-\alpha}{2}} \sqrt{\frac{c_0}{\sigma}} \tilde{n}^\alpha(b, \mu\delta^{-1}) \|f\|_1.\end{aligned}$$

Above one can replace $\tilde{n}^\alpha(b, \mu)$ by $n^\alpha(b, h) := \sup_{y \in \mathbb{R}^d} \int_0^h \sqrt{e^{t\Delta}|b|^2(y)} \frac{dt}{t^{\frac{3-\alpha}{2}}}$.

7. Proof of Theorem 3.2

In the proof of Proposition 3.1 we established: $T_1^\varepsilon := b_\varepsilon \cdot \nabla(\zeta + A_1^\varepsilon)^{-1}$, $T_1 := (b \cdot \nabla)_1 (\zeta + A_1)^{-1}$, $\mathrm{Re}\zeta > 0$ satisfy $T_1 \in \mathcal{B}(L^1)$ and

$$\|T_1^\varepsilon\|_{1 \to 1} \leq \eta + \tilde{c}\varepsilon, \quad \|T_1\|_{1 \to 1} \leq \eta.$$

Proposition 7.1. $T_1 = s\text{-}L^1\text{-}\lim_{\varepsilon \downarrow 0} T_1^\varepsilon$.

Proof of Proposition 7.1. Under the additional assumption $b^2 \in L^1 + L^\infty$, the assertion of the proposition is evident (use (6) in the proof of Proposition 3.1). In general one has to employ the separation property of e^{-tA}, as is done below.

Since $\sup_{\varepsilon > 0} \|T_1^\varepsilon\|_{1 \to 1}, \|T_1\|_{1 \to 1} < \infty$, it suffices to prove the claimed convergence on C_c^∞. Fix $f \in C_c^\infty$ and then $r > 0$ by $B(0, r) \supset \mathrm{sprt}\, f$. Since by (6) $T_1^\varepsilon f \to T_1 f$ strongly in L^1_{loc}, the required convergence in (ii) would follow from (7) once we show that, for every $\theta > 0$, there exists $R = R(r, \theta) > 0$ such that

$$\|\mathbf{1}_{B^c(0,R)} T_1^\varepsilon f\|_1 \leq \theta \|f\|_1 \quad \text{for all } \varepsilon > 0 \text{ sufficiently small.}$$

Here $B^c(0, R) := \mathbb{R}^d - B(0, R)$.

To prove the latter, we write

$$\mathbf{1}_{B^c(0,R)}T_1^\varepsilon f(x)=\int_0^\infty e^{-\zeta t}\langle \mathbf{1}_{B^c(0,R)}(x)b_\varepsilon(x)\cdot\nabla_x p_\varepsilon(t,x,\cdot)f(\cdot)\rangle dt,$$

where $p_\varepsilon(t,x,y)=e^{-tA_1^\varepsilon}(x,y)$. Put $\mu:=\mathrm{Re}\zeta$. Then

$$\begin{aligned}\|\mathbf{1}_{B^c(0,R)}T_1^\varepsilon f\|_1 &\leq \sum_{j=0}^\infty e^{-j\mu h}\int_{jh}^{(j+1)h}\|\mathbf{1}_{B^c(0,R)}B_1^\varepsilon e^{-tA_1^\varepsilon}f\|_1 dt\\ &=\sum_{j=0}^\infty e^{-j\mu h}\int_0^h \|\mathbf{1}_{B^c(0,R)}B_1^\varepsilon e^{-tA_1^\varepsilon}e^{-jhA_1^\varepsilon}f\|_1dt\\ &=\sum_{j=0}^\infty e^{-j\mu h}\bigg[\int_0^h\|\mathbf{1}_{B^c(0,R)}B_1^\varepsilon e^{-tA_1^\varepsilon}\mathbf{1}_{B(0,mr)}e^{-jhA_1^\varepsilon}f\|_1dt\\ &+\int_0^h\|\mathbf{1}_{B^c(0,R)}B_1^\varepsilon e^{-tA_1^\varepsilon}\mathbf{1}_{B^c(0,mr)}e^{-jhA_1^\varepsilon}f\|_1dt\bigg]=:\sum_{j=0}^\infty e^{-j\mu h}\big[I_j+J_j\big],\end{aligned}$$

where constant $m\geq 1$ is to be chosen. Arguing as in the proof of Step 1 of the proof of Proposition 3.1 and putting $\delta:=c_4$, we obtain, for all $j\geq 0$,

$$\begin{aligned}I_j&\leq\sqrt{\frac{c_0}{\sigma\delta}}\sup_{y\in B(0,mr)}\int_0^h\sqrt{\langle k_\delta(t,y,\cdot)\mathbf{1}_{B^c(0,R)}(\cdot)|b_\varepsilon(\cdot)|^2\rangle}\frac{dt}{\sqrt{t}}\,\|e^{-khA_1^\varepsilon}f\|_1\\ &\leq\left(\sqrt{\frac{c_0}{\sigma\delta}}M_R+4C_d(h\delta)^{\frac14}\varepsilon\right)\|f\|_1,\end{aligned}$$

where $M_R:=\sup_{y\in B(0,mr)}\int_0^h\sqrt{\langle k_\delta(t,y,\cdot)\mathbf{1}_{B^c(0,R)}(\cdot)|b(\cdot)|^2\rangle}\frac{dt}{\sqrt{t}}$, $R>mr$.

Clearly, $J_0=0$. For all $j\geq 1$ and $\eta_0=\sqrt{\frac{c_0}{\sigma\delta}}n_e(b,h\delta)$,

$$\begin{aligned}J_j&\leq\eta_0\|\mathbf{1}_{B^c(0,mr)}e^{-jhA_1^\varepsilon}f\|_1\\ &\quad(\text{we are applying (UGB}^p\text{) to } e^{-jhA_1^\varepsilon}(x,y))\\ &\leq\eta_0c_3(4\pi c_4jh)^{-\frac d2}e^{-\frac{(m-1)^2r^2}{4c_4jh}}\|f\|_1.\end{aligned}$$

Thus, we have

$$\|\mathbf{1}_{B^c(0,R)}T_1^\varepsilon f\|_1\leq\theta\|f\|_1,$$

where

$$\theta:=\left(\sqrt{\frac{c_0}{\sigma\delta}}M_R+4C_d(h\delta)^{\frac14}\varepsilon\right)\frac{1}{1-e^{-\mu h}}+C_g\sum_{j=1}^\infty e^{-\mu jh}(jh)^{-\frac d2}e^{-\frac{(m-1)^2r^2}{4c_4jh}}.$$

It is clear that selecting m sufficiently large, we can make the second term in the RHS as small as needed.

We are left to prove the convergence $M_R\to 0$ as $R\to\infty$.

(a_1) Fix $n>0$ by $k_\delta(t,z,y)\leq C_nk_\delta(t,z,0)$ for all $t>0$, $z\in B^c(0,(m+n)r)$, $y\in B(0,mr)$. Then

$$M_R\leq C_n\int_0^h\sqrt{\langle k_\delta(t,0,\cdot)\mathbf{1}_{B^c(0,R)}(\cdot)|b(\cdot)|^2\rangle}\frac{dt}{\sqrt{t}}\quad\forall R>(m+n)r.$$

(a_2) Due to $b \in \mathbf{N}_e$ the function

$$w_R(t) := \sqrt{\langle k_\delta(t,\cdot,0)\mathbf{1}_{B^c(0,R)}(\cdot)|b(\cdot)|^2\rangle}\frac{1}{\sqrt{t}}$$

is in $L^1([0,h])$ for every $R \geq 1$. Moreover, it is seen from the definition of w_R that for every $0 < t_1 < t_2 \leq h$, $w_R(t_1) \leq C_{t_1,t_2-t_1} w_R(t_2)$, $C_{t_1,t_2-t_1} < \infty$. Thus, $w_R(t)$ is finite for all $0 < t \leq h$.

(a_3) $w_R(t) \to 0$ as $R \to \infty$ for every $0 < t \leq h$.

Indeed, fix $t \in]0,h]$. Set $v_R(x) := k_\delta(t,x,0)\mathbf{1}_{B^c(0,R)}(x)|b(x)|^2$. For a.e. $x \in \mathbb{R}^d$, $v_R(x) \downarrow 0$ as $R \uparrow \infty$, and $v_R \leq v_1$ a.e. on $\mathbb{R}^d$ for all $R \geq 1$, where v_1 is summable. Hence by the Dominated Convergence Theorem, $\langle v_R \rangle \to 0$ as $R \to \infty$, and so $w_R(t) \to 0$ as $R \to \infty$.

(a_4) Due to (a_3) and $w_R \leq w_1$ for $R \geq 1$, the Dominated Convergence Theorem yields

$$\int_0^h w_R(t)dt \to 0 \quad \text{ as } R \to \infty.$$

Thus, $M_R \to 0$ as $R \to \infty$. The proof of Proposition 7.1 is completed. □

We are in position to complete the proof of Theorem 3.2. Recall $\delta := c_4$.

(i) By our assumption on $n_e(b,h\delta)$, there exists $\lambda_0 > 0$ such that

$$\eta := \frac{1}{1-e^{-\lambda_0 h}}\sqrt{\frac{c_0}{\sigma\delta}}n_e(b,h\delta) < 1.$$

By Proposition 3.1, Λ_1 is a closed densely defined operator. Using (5), we obtain that

$$(\zeta + \Lambda_1)^{-1} = (\zeta + A_1)^{-1}(1+T_1)^{-1} \in \mathcal{B}(L^1), \quad \mathrm{Re}\zeta > \lambda_0.$$

Using (3), we obtain

$$\|(\zeta + \Lambda_1)^{-1}\|_{1\to 1} \leq \frac{M}{|\zeta|(1-\eta)}, \quad \mathrm{Re}\zeta > \lambda_0, \tag{8}$$

completing the proof of the first part of assertion (i).

To prove the second part of (i), note that, in view of (8), the resolvent $\zeta \mapsto (\zeta + \lambda_0 + \Lambda_1)^{-1} = \Theta(\zeta + \lambda_0)$ is holomorphic in the right-half plane $\mathrm{Re}\zeta > 0$ and in $|\zeta - \zeta_0| < \sqrt{2}\big(\frac{M}{1-\eta} - 1\big)|\zeta_0|$ for every ζ_0 with $\mathrm{Re}\zeta_0 = 0$ (see, if needed, the argument in [15, Ch. IX, sect. 10]). Thus, $e^{-z(\lambda_0+\Lambda_1)}$ is holomorphic in the sector

$$\{z \in \mathbb{C} \mid |\arg z| < \frac{\pi}{2} - \theta_{\lambda_0}\}, \quad \text{ where } \tan\theta_{\lambda_0} = \sqrt{2}\bigg(\frac{M}{1-\eta} - 1\bigg).$$

This completes the proof of assertion (i).

(ii) The claimed approximation $\{b_\varepsilon\}$ was constructed in the proof of Proposition 3.1. Let us show that

$$(\lambda + \Lambda_1^\varepsilon)^{-1} \to (\lambda + \Lambda_1)^{-1} \quad \text{strongly in } L^1 \text{ as } \varepsilon \downarrow 0,$$

which, by a standard result, implies the convergence of the semigroups.

Since $(\lambda + \Lambda_1^\varepsilon)^{-1} = (\lambda + A_1^\varepsilon)^{-1}(1+T_1^\varepsilon)^{-1}$, $(\lambda + \Lambda_1)^{-1} = (\lambda + A_1)^{-1}(1+T_1)^{-1}$, it suffices to show that 1) $T_1^\varepsilon \to T_1$ and 2) $(\lambda + A_1^\varepsilon)^{-1} \to (\lambda + A_1)^{-1}$ strongly in L^1 as $\varepsilon \downarrow 0$. 1) is Proposition 7.1. 2) follows immediately from

$$(\lambda + A^\varepsilon)^{-1} \to (\lambda + A)^{-1} \quad \text{strongly in } L^2$$

and $(\lambda + A^\varepsilon)^{-1}(x,y) \leq C(\lambda - c\Delta)^{-1}(x,y)$ for generic constants $0 < c, C < \infty$, an immediate consequence of (UGB^p).

(iii) The upper bound in (LUGB^u) of Theorem 3.1 yields

$$\|e^{-t\Lambda_1^\varepsilon}\|_{1\to\infty} \leq c_2 e^{t\omega_2} t^{-\frac{d}{2}}, \quad t>0, \quad \varepsilon>0$$

with generic* constants c_2, $\omega_2 < \infty$. Using Theorem 3.2(ii) and applying Fatou's lemma, we obtain $\|e^{-t\Lambda_1}\|_{1\to\infty} \leq c_2 e^{t\omega_2} t^{-\frac{d}{2}}$, $t>0$. Hence $e^{-t\Lambda_1}$ is an integral operator for every $t>0$.

(iv) The a priori bounds (LUGB^u) of of Theorem 3.1, and Theorem 3.2(ii), yield for every pair of bounded measurable subsets S_1, $S_2 \subset \mathbb{R}^d$:

$$c_1 e^{t\omega_1}\langle \mathbf{1}_{S_1}, e^{t\sigma_1\Delta}\mathbf{1}_{S_2}\rangle \leq \langle \mathbf{1}_{S_1}, e^{-t\Lambda_1}\mathbf{1}_{S_2}\rangle \leq c_2 e^{t\omega_2}\langle \mathbf{1}_{S_1}, e^{t\xi_1\Delta}\mathbf{1}_{S_2}\rangle.$$

Since $e^{-t\Lambda_1}$ is an integral operator for every $t>0$, assertion (iv) follows by applying the Lebesgue Differentiation Theorem.

(v) For every $\varepsilon>0$, $\langle e^{-t\Lambda^\varepsilon}(x,\cdot)\rangle = 1$, $x \in \mathbb{R}^d$. Fix $t>0$ and $\Omega \subset \mathbb{R}^d$, a bounded open set. By the upper bound (LUGB^u) of Theorem 3.1, for every $\gamma>0$ there exists $R = R(\gamma,t,\Omega)>0$ such that, for every $x \in \Omega$, $\langle e^{-t\Lambda^\varepsilon}(x,\cdot)\mathbf{1}_{B^c(0,R)}(\cdot)\rangle < \gamma$, so $\langle e^{-t\Lambda^\varepsilon}(x,\cdot)\mathbf{1}_{B(0,R)}(\cdot)\rangle \geq 1-\gamma$. Hence

$$\langle \mathbf{1}_\Omega e^{-t\Lambda^\varepsilon}\mathbf{1}_{B(0,R)}\rangle \geq (1-\gamma)|\Omega|.$$

Applying Theorem 3.2(ii), we obtain

$$\frac{1}{|\Omega|}\langle \mathbf{1}_\Omega e^{-t\Lambda}1\rangle \geq \frac{1}{|\Omega|}\langle \mathbf{1}_\Omega e^{-t\Lambda}\mathbf{1}_{B(0,R)}\rangle \geq 1-\gamma.$$

Applying the Lebesgue Differentiation Theorem, we obtain $\langle e^{-t\Lambda}(x,\cdot)\rangle \geq 1-\gamma$ for a.e. $x \in \mathbb{R}^d$. In turn, the opposite inequality $\langle e^{-t\Lambda}(x,\cdot)\rangle \leq 1$ for a.e. $x \in \mathbb{R}^d$ follows easily using Theorem 3.2(ii), and hence $1 \geq \langle e^{-t\Lambda}(x,\cdot)\rangle \geq 1-\gamma$. The proof of (v) is completed.

(vi) Put $u_\varepsilon(t,x) := e^{-t\Lambda^\varepsilon}f(x)$. Repeating the argument in [5, sect. 3] which appeals to the ideas of E. De Giorgi, we obtain assertion (vi) for u_ε. The result now follows upon applying Theorem 3.2(ii) and the Arzelà-Ascoli Theorem.

(vii) follows from (iv), (v) and (vi) using a standard argument for mollifiers.

$(viii)$ is proved repeating the argument in [2, sect. 2].

(ix) follows repeating the argument in [12].

(x) In the proof of (i) we obtain the resolvent representation as the K. Neumann series

$$(\zeta+\Lambda_1)^{-1} = (\zeta+A_1)^{-1}(1+T_1)^{-1} \in \mathcal{B}(L^1), \quad \mathrm{Re}\zeta \geq \lambda_0,$$

where $\lambda_0 = \lambda_0\big(n_e(b,h)\big) > 0$, $T_1 := (b\cdot\nabla)_1(\zeta+A_1)^{-1} \in \mathcal{B}(L^1)$. The latter yields $\|\nabla(\zeta+\Lambda_1)^{-1}\|_{1\to1} \leq c(\mathrm{Re}\zeta)^{-\frac{1}{2}}$. Indeed, $\|\nabla(\zeta+A_1)^{-1}\|_{1\to1} \leq c(\mathrm{Re}\zeta)^{-\frac{1}{2}}$ (integrating ($\star$) in $t \in [0,\infty[$ in the proof of Theorem 3.3), so the resolvent representation yields the required bound. The latter now easily yields the case $1/2 < \alpha < 1$.

8. Proof of Theorem 3.3

It suffices to carry out the proof on C_c^∞ for smooth bounded $a \in (H_{\sigma,\xi})$, b, and then apply Theorem 3.2(ii) using the closedness of the gradient.

First, let $0 < t \leq h$.

The Duhamel formula for $\nabla e^{-t\Lambda_1}$ yields:

$$\|\nabla e^{-t\Lambda_1} f\|_1 \leq \|\nabla e^{-tA_1} f\|_1 + \int_0^t \|\nabla e^{-(t-\tau)A_1}\|_{1\to 1} \|b \cdot \nabla e^{-\tau\Lambda_1} f\|_1 d\tau, \quad f \in C_c^\infty. \tag{9}$$

We will need (proved below):

$$\|\nabla e^{-tA_1}\|_{1\to1} \leq C/\sqrt{t}, \tag{$\star$}$$

$$\int_0^t \frac{C}{\sqrt{t-\tau}} \|b \cdot \nabla e^{-\tau\Lambda_1} f\|_1 d\tau \leq C \sup_{x\in\mathbb{R}^d} \int_0^t \frac{1}{\sqrt{t-\tau}} \sqrt{e^{\delta\tau\Delta} b_a^2(x)} \sqrt{\mathcal{N}_\delta^u(\tau,x)} d\tau \, \|f\|_1, \tag{$\star\star$}$$

$$\mathcal{N}_\delta^u(\tau,x) \leq \frac{C_2}{\tau}, \tag{$\star\star\star$}$$

where $\mathcal{N}_\delta^u(\tau,x) := \langle \nabla u(\tau,x,\cdot) \cdot \frac{a(\cdot)}{k_\delta(\tau,x,\cdot)} \cdot \nabla u(\tau,x,\cdot)\rangle$, $u(\tau,x,y) = e^{-\tau\Lambda}(x,y)$, $\delta > \xi$, the constants C_1, C_2, ω are generic. We estimate the RHS of ($\star\star$): write $\int_0^t = \int_0^{t/2} + \int_{t/2}^t$ and use ($\star\star\star$) to obtain

$$\begin{aligned}\sup_{x\in\mathbb{R}^d} \int_0^{t/2} \frac{1}{\sqrt{t-\tau}} \sqrt{e^{\delta\tau\Delta} b_a^2(x)} \sqrt{\mathcal{N}_\delta^u(\tau,x)} d\tau &\leq \frac{\sqrt{2C_2}}{\sqrt{t}} \sup_{x\in\mathbb{R}^d} \int_0^{t/2} \sqrt{e^{\delta\tau\Delta} b_a^2(x)} \frac{d\tau}{\sqrt{\tau}} \\ &\leq \frac{\sqrt{2C_2}}{\sqrt{\delta t}} n_e(b, \frac{\delta h}{2}),\end{aligned}$$

$$\begin{aligned}\sup_{x\in\mathbb{R}^d} \int_{t/2}^t \frac{1}{\sqrt{t-\tau}} \sqrt{e^{\delta\tau\Delta} b_a^2(x)} \sqrt{\mathcal{N}_\delta^u(\tau,x)} d\tau &\leq \sqrt{C_2} \sup_{x\in\mathbb{R}^d} \int_{t/2}^t \frac{1}{\sqrt{t-\tau}} \sqrt{e^{\delta\tau\Delta} b_a^2(x)} \frac{d\tau}{\sqrt{\tau}} \\ &\text{(we are using } e^{\delta\tau\Delta} b_a^2(x) \leq \frac{\xi d\beta}{8\delta}\frac{1}{\tau} + c(\beta) \text{ since } b \in \mathbf{F}) \\ &\leq \tilde{C} \int_{t/2}^t \frac{1}{\sqrt{t-\tau}} \frac{d\tau}{\tau} \leq \tilde{C} \frac{1}{\sqrt{t}}.\end{aligned}$$

Substituting ($\star$), ($\star\star$) and the last two estimates into (9), we have $\|\nabla e^{-t\Lambda_1}\|_{1\to1} \leq \frac{c}{\sqrt{t}}$ for $0 < t \leq h$.

Also, for all $t > h$, $\|\nabla e^{-t\Lambda_1}\|_{1\to1} \leq \|\nabla e^{-h\Lambda_1}\|_{1\to1} \|e^{-(t-h)\Lambda_1}\|_{1\to1} \leq \frac{\tilde{c}}{\sqrt{h}} e^{(t-h)\omega_2}$ (cf. Theorem 3.2). The latter yields the assertion of Theorem 3.3 for all $t > 0$.

It remains to prove ($\star$)-($\star\star\star$).

Proof of ($\star$): We have for $\mathsf{h} \in \mathbb{R}^d$, $\mathsf{h} = (0,\dots,1,\dots,0)$ (1 is in the i-th coordinate, $1 \leq i \leq d$)

$$\begin{aligned}\|\mathsf{h} \cdot \nabla e^{-tA_1} f\|_1 &\leq \sup_{x\in\mathbb{R}^d} \sqrt{\langle k_\delta(t,x,\cdot)(\mathsf{h}\cdot a^{-1}(\cdot)\cdot \mathsf{h})\rangle} \sqrt{\mathcal{N}_\delta(t,x)} \|f\|_1 \\ &\leq \sigma^{-\frac{1}{2}} \sup_{x\in\mathbb{R}^d} \sqrt{\mathcal{N}_\delta(t,x)} \|f\|_1 = \sigma^{-\frac{1}{2}} \sqrt{\sup_{x\in\mathbb{R}^d} \mathcal{N}_\delta(t,x)} \|f\|_1,\end{aligned}$$

and so by Proposition 4.1

$$\|\nabla e^{-tA_1} f\|_1 \leq \frac{d\sqrt{\sigma^{-1}c_0}}{\sqrt{t}} \|f\|_1.$$

The estimate ($\star\star$) follows using quadratic inequality.

Thus, we are left to prove $(\star\star\star)$. Integrating by parts, using the equation for $u(t,x,y)$ and $(\mathrm{UGB}^{\mathrm{u}}), (\mathrm{UGB}^{\partial_t \mathrm{u}})$ (see Theorem 3.2*(iv),(viii)*), we obtain for $0 < t \leq h$ (below c is a generic constant)

$$\mathcal{N}^u_\delta(t,x) = \langle \nabla u \cdot \frac{a}{k_\delta} \cdot \nabla u \rangle = -\langle k_\delta^{-1} u \partial_t u \rangle - \langle k_\delta^{-1} u b \cdot \nabla u \rangle + \langle u k_\delta^{-2} \nabla k_\delta \cdot a \cdot \nabla u \rangle,$$

$$|\langle k_\delta^{-1} u \partial_t u \rangle| \leq \frac{c}{t}, \quad |\langle u k_\delta^{-2} \nabla k_\delta \cdot a \cdot \nabla u \rangle| \leq c |\langle \nabla k_\delta \cdot \frac{a}{k_\delta} \cdot \nabla u \rangle|.$$

Clearly,

$$|\langle \nabla k_\delta \cdot \frac{a}{k_\delta} \cdot \nabla u \rangle| \leq \frac{c}{\sqrt{t}} \sqrt{\mathcal{N}^u_\delta(t,x)}.$$

$$|\langle k_\delta^{-1} u b \cdot \nabla u \rangle| \leq c \sqrt{e^{\delta t \Delta} b_a^2(x)} \sqrt{\mathcal{N}^u_\delta(t,x)} \leq \hat{c} \frac{1}{\sqrt{t}} \sqrt{\mathcal{N}^u_\delta(t,x)}$$

(due to $e^{\delta t \Delta} b_a^2(x) \leq \frac{\xi d \beta}{8\delta} \frac{1}{t} + c(\beta)$, see above). Now $(\star\star\star)$ is evident.

The proof of Theorem 3.3 is completed.

9. Comments

1. The following result was proved in [8] (the reader can compare it with Theorem 3.2). It establishes quantitative dependence of the regularity properties of solutions to $(\partial_t + \Lambda)u = 0$ with $b \in \mathbf{F}_\delta(A)$ on the value of δ.

Theorem 9.1. *Let $d \geq 3$. Assume that $b \in \mathbf{F}_\delta(A)$ for some $0 < \delta < 4$. Set $r_c := \frac{2}{2-\sqrt{\delta}}$ and $b_a^2 := b \cdot a^{-1} \cdot b \in L^2_{\mathrm{loc}}$. The following is true:*

(i) Let $\mathbf{1}_n$ denote the indicator of $\{x \in \mathbb{R}^d \mid b_a(x) \leq n\}$ and set $b_n := \mathbf{1}_n b$. Then the limit

$$s\text{-}L^r\text{-}\lim_{n\to\infty} e^{-t\Lambda_r(a,b_n)}, \quad r \in I_c^o :=]r_c, \infty[,$$

where $\Lambda_r(a,b_n) := A_r + b_n \cdot \nabla$, exists locally uniformly in $t \geq 0$ and determines a positivity preserving, L^∞ contraction, quasi contraction C_0 semigroup on L^r, say, $e^{-t\Lambda_r(a,b)}$.

(ii) Define

$$e^{-t\Lambda_{r_c}(a,b)} := \left[e^{-t\Lambda_r(a,b)} \upharpoonright L^1 \cap L^r\right]^{\mathrm{clos}}_{L^{r_c} \to L^{r_c}}, \qquad r \in I_c^o.$$

Then $e^{-t\Lambda_{r_c}(a,b)}$ is a C_0 semigroup and

$$\|e^{-t\Lambda_r(a,b)}\|_{r\to r} \leq e^{t\omega_r}, \quad \omega_r = \frac{\lambda\delta}{2(r-1)}, \quad r \in I_c := [r_c, \infty[.$$

(iii) The interval I_c is the maximal interval of quasi contractive solvability.

(iv) For each $r \in I_c^o$, $e^{-t\Lambda_r(a,b)}$ is a holomorphic semigroup of quasi contractions in the sector

$$|\arg t| \leq \frac{\pi}{2} - \theta_r, \quad 0 < \theta_r < \frac{\pi}{2}, \ \tan\theta_r \leq \mathcal{K}(2 - r'\sqrt{\delta})^{-1},$$

where $\mathcal{K} = \frac{|r-2|}{\sqrt{r-1}} + r'\sqrt{\delta}$ if $r \leq 2r_c$ and $\mathcal{K} = \frac{r-2+r\sqrt{\delta}}{\sqrt{r-1}}$ if $r > 2r_c$.

(v) $e^{-t\Lambda_r(a,b)}$, $r \in I_c$, extends to a positivity preserving, L^∞ contraction, quasi bounded holomorphic semigroup on L^r for every $r \in I_m :=]\frac{2}{2-\frac{d-2}{d}\sqrt{\delta}}, \infty[$.

(vi) The interval I_m is the maximal interval of quasi bounded solvability.

(vii) For every $r \in I_m$ and $q > r$ there exist constants $c_i = c_i(\delta, r, q)$, $i = 1, 2$ such that the (L^r, L^q) estimate

$$\|e^{-t\Lambda_r(a,b)}\|_{r\to q} \le c_1 e^{c_2 t}\, t^{-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})}$$

is valid for all $t > 0$.

(viii) Let $\delta < 1$, and let $a_n \in (H_{\sigma,\xi})$, $b_n : \mathbb{R}^d \to \mathbb{R}^d$, $n = 1, 2, \dots$ be smooth and such that

$$a_n \to a \text{ strongly in } [L^2_{\rm loc}]^{d\times d}, \quad b_n \to b \text{ strongly in } [L^2_{\rm loc}]^d$$

and $b_n \in \mathbf{F}_\delta(A^n)$ with $c(\delta)$ independent of n, where $A^n \equiv -\nabla \cdot a_n \cdot \nabla$. Then

$$e^{-t\Lambda_r(a,b)} = s\text{-}L^r\text{-}\lim_{n\uparrow\infty} e^{-t\Lambda_r(a_n,b_n)}$$

whenever $r \in I_c^o$, where $\Lambda_r(a_n, b_n) = -\nabla \cdot a_n \cdot \nabla + b_n \cdot \nabla$ of domain $W^{2,r}$.

Remarks. (a) For $\delta < 1$, the corresponding to Λ quadratic form $t[u] = \langle a \cdot \nabla u, \nabla u\rangle + \langle b \cdot \nabla u, u\rangle$, $D(t) = W^{1,2}$ possesses the Sobolev embedding property

$$\mathrm{Re} t[u] \ge c_S \|u\|_{2j}^2, \quad j = \frac{d}{d-2}.$$

This ceases to be true already for $\delta = 1$. The same occurs for $1 < \delta < 4$ and $r = r_c$.

(b) The intervals I_c, I_m are maximal already for $a = I$ and $b(x) = \sqrt{\delta}\frac{d-2}{2}|x|^{-2}x$.

(c) Assertions (i)-(iv) are in fact valid for symmetric $a \in [L^1_{\rm loc}]^{d\times d}$ such that $a \ge \sigma I$, $\sigma > 0$, and $b_a^2 \in L^1 + L^\infty$, see [8, Theorem 4.2].

(d) While for $b \in \mathbf{F}_\delta(A)$, $\delta < 1$ one first constructs the semigroup in L^2 (using the method of quadratic forms) and then proves the corresponding convergence results, in the case $b \in \mathbf{F}_\delta(A)$, $1 \le \delta < 4$ the convergence result of Theorem 9.1(i) becomes the means of construction of the semigroup.

2. Note that $\mathbf{N}_e \cap \mathbf{F} \subset \mathbf{K}^d \subset \mathbf{F}$, where $\mathbf{F} := \cup_{\beta>0}\mathbf{F}_\beta(-\Delta)$, and

$$\mathbf{K}^d := \{|b| \in L^2_{\rm loc} \mid \kappa_d(b,h) := \sup_{x\in\mathbb{R}^d}\int_0^h e^{t\Delta}|b|^2(x)dt < \infty \text{ for some } h > 0\}.$$

Indeed, using $b \in \mathbf{F}$, we have $e^{t\Delta}b^2(x) \equiv \langle k(t,x,\cdot)b^2(\cdot)\rangle \le \beta\|\nabla\sqrt{k(t,x,\cdot)}\|_2^2 + c(\beta) = \frac{\beta d}{8}\frac{1}{t} + c(\beta)$ for some $\beta > 0$ and $c(\beta)$. Therefore, for $0 < t \le h$,

$$e^{t\Delta}b^2(x) \le \sqrt{\frac{\beta d}{8} + c(\beta)h}\,\sqrt{e^{t\Delta}b^2(x)}\frac{1}{\sqrt{t}},$$

and so the condition $b \in \mathbf{N}_e$ now yields the required. In turn, the inclusion $\mathbf{K}^d \subset \mathbf{F}$ is well known (use the fact that $b \in \mathbf{K}^d$ is equivalent to $\||b|^2(\lambda-\Delta)^{-1}\|_{1\to 1} < \infty$, $\lambda > 0$).

3. Let us fix a continuous function $\phi : [0,\infty[\to [0,\infty[$ satisfying the following properties:
1) $\phi(0) = 0$,
2) $\phi(t)/t \in L^1[0,1]$.
Put

$$n_\phi(b,h) = \sup_{x\in\mathbb{R}^d}\int_0^h e^{t\Delta}b^2(x)\frac{dt}{\phi(t)}.$$

If $n_\phi(b,h) < \infty$ for some $h > 0$, then we write $b \in \mathbf{N}_\phi$.

The class $\mathbf{N}_\phi$ arises as the class providing the two-sided Gaussian on the heat kernel of $-\nabla \cdot a(t,x) \cdot \nabla + b(t,x) \cdot \nabla$, where $a(t,x)$ is a measurable uniformly elliptic matrix, see [14], [10]. Since (for $b = b(x)$)

$$\int_0^h \sqrt{e^{t\Delta} b^2(x)} \frac{dt}{\sqrt{t}} \leq \left[\int_0^h e^{t\Delta} b^2(x) \frac{dt}{\phi(t)}\right]^{\frac{1}{2}} \left[\int_0^h \frac{\phi(t)}{t} dt\right]^{\frac{1}{2}},$$

we have $\mathbf{N}_\phi \subset \mathbf{N}_e$ for every admissible ϕ. Moreover, since ϕ is continuous and $\phi(0) = 0$, it is seen that $n_\phi(b,h) > k_d(b,h)$, and so $\mathbf{N}_\phi \subset \mathbf{K}^d$. Thus,

$$\mathbf{N}_\phi \subset \mathbf{N}_e \cap \mathbf{K}^d \subset \mathbf{K}^{d+1} \cap \mathbf{K}^d.$$

The need for more restrictive assumption "$b \in \mathbf{N}_\phi$" when $a = a(t,x)$ is dictated by the subject matter: in the time-dependent case there are no estimates $\mathcal{N}(t)$, $\hat{\mathcal{N}}(t) \leq c(t)$ for any $c(t)$, cf. the previous comment.

4. Let us comment more on classes $\mathbf{K}^{d+1}$ and $\mathbf{F}$.

Note that $\mathbf{K}^{d+1} \not\subset \mathbf{F}$: There are $b \in \mathbf{K}^{d+1}$ such that, for a given $p > 1$, $|b| \notin L^p_{\rm loc}$, e.g. consider

$$|b(x)| = \mathbf{1}_{B(0,1)}(x)|x_1|^{-\alpha_p}, \quad 0 < \alpha_p < 1.$$

On the other hand, already $[L^d]^d \not\subset \mathbf{K}^{d+1}$, and so $\mathbf{F} \not\subset \mathbf{K}^{d+1}$. [Indeed, let

$$|b(x)| = \mathbf{1}_{B(0,e^{-1})}(x)|x|^{-1}|\log|x||^{-\alpha},\ \alpha > d^{-1},\ d \geq 3.$$

Then $\|b\|_d < \infty$ and $k_{d+1}(b,h) = \infty$.]

This dichotomy between the classes $\mathbf{K}^{d+1}$ and $\mathbf{F}$ was resolved in [6, 8] with development of the Sobolev regularity theory of $-\Delta + b \cdot \nabla$ for b in the class

$$\mathbf{F}^{1/2} = \left\{b \in L^1_{\rm loc} \mid \lim_{\lambda \to \infty} \||b|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}}\|_{2 \to 2} < \infty\right\}$$

(introduced in [13] as the class responsible for the (L^p, L^q) estimate on the semigroup) that contains $\mathbf{K}^{d+1} + \mathbf{F} := \{b_1 + b_2 \mid b_1 \in \mathbf{K}^{d+1}, b_2 \in \mathbf{F}\}$.

By analogy, one can ask if it is possible to extend the convergence results in Theorem 3.2 and Theorem 9.1, or (L^p, L^q) estimates, to $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ with a measurable $a \in (H_{\sigma,\xi})$ and $b = b_1 + b_2$ with $b_1 \in \mathbf{N}_e$, $b_2 \in \mathbf{F}_\delta(A)$.

5. Theorem 3.2(*iv*), (*viii*) (the two-sided Gaussian bounds on the heat kernel and its time derivative) can be extended to more general operator

$$\Lambda(a,b,\hat{b}) = -\nabla \cdot a \cdot \nabla + b \cdot \nabla + \nabla \cdot \hat{b}$$

with $a \in (H_{\sigma,\xi})$, and $(b, \hat{b} \in \mathbf{N}_e, \hat{b} \in \mathbf{F})$ or $(b, \hat{b} \in \mathbf{N}_e, b \in \mathbf{F})$, provided that $n(b,h)$, $n(\hat{b},h)$ are sufficiently small. Note that the above assumptions on b and $\hat{b}$ are non-symmetric, i.e. the presence of $b \in \mathbf{N}_e$ forces $\hat{b}$ to be more regular: $\hat{b} \in \mathbf{N}_e \cap \mathbf{F}$, and vice versa. We also note that here the form-boundedness assumption seems to be justified. The proof follows the argument in the present paper but with the Nash's functions $\mathcal{N}$, $\hat{\mathcal{N}}$ defined with respect to $u(t,x,y) := e^{-t\Lambda(a,b)}(x,y)$. We will address this matter in detail elsewhere.

6. The authors do not know if there is a proof of the Harnack inequality for $\Lambda = -\nabla \cdot a \cdot \nabla + b \cdot \nabla$, $a \in (H_{\sigma,\xi})$, $b \in \mathbf{N}_e$ that does not use the lower bound on $e^{-t\Lambda}(x,y)$.

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