FRACTIONAL KOLMOGOROV OPERATOR AND DESINGULARIZING WEIGHTS

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ABSTRACT. We establish sharp upper and lower bounds on the heat kernel of the fractional Laplace operator perturbed by Hardy-type drift by transferring it to appropriate weighted space with singular weight.

1. INTRODUCTION

The fractional Kolmogorov operator $(-\Delta)^{\frac{\alpha}{2}} + \mathsf{f} \cdot \nabla$, $1 < \alpha < 2$ with a (locally unbounded) vector field $\mathsf{f} : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 3$, plays important role in probability theory where it arises as the generator of symmetric α-stable process with a drift (in contrast to diffusion processes, α-stable process has long range interactions). It has been the subject of intensive study over the past two decades. There is now a well developed theory of this operator with f belonging to the corresponding Kato class. This class, in particular, contains the vector fields f with $|\mathsf{f}| \in L^p$, $p > \frac{d}{\alpha - 1}$ and is, indeed, responsible for existence of the standard (local in time) two-sided bound on the heat kernel $e^{-t\Lambda}(x,y)$, $\Lambda \supset (-\Delta)^{\frac{\alpha}{2}} + \mathsf{f} \cdot \nabla$, in terms of $e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)$, see [BJ].

The authors in [KSS] studied the fractional Kolmogorov operator

$$\Lambda = (-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla, \quad b(x) = \kappa |x|^{-\alpha} x, \quad 0 < \kappa < \kappa_0,$$

where κ_0 is the borderline constant for existence of $e^{-t\Lambda}(x,y) \geq 0$. The model vector field b lies outside of the scope of the Kato class, and exhibits critical behaviour both at $x = 0$ and at infinity making the standard upper bound on $e^{-t\Lambda}(x,y)$ in terms of $e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)$ invalid. Instead, the two-sided bounds $e^{-t\Lambda}(x,y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\varphi_t(y)$ $(y \neq 0)$ hold for an appropriate weight $\varphi_t \geq \frac{1}{2}$ unbounded at $y = 0$ [KSS, Theorem 3].

The present paper continues [KSS]. We study the heat kernel $e^{-t\Lambda}(x,y)$ of the fractional Kolmogorov operator with the drift of opposite sign ("repulsion case")

$$\begin{aligned} \Lambda &= (-\Delta)^{\frac{\alpha}{2}} - b \cdot \nabla, \\ b(x) &= \kappa |x|^{-\alpha} x, \quad 0 < \kappa < \infty. \end{aligned} \tag{1}$$

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Although the standard (global) upper bound in terms of $e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)$ holds true for $e^{-t\Lambda}(x,y)$ (Theorem 3 below), the singularity of b at $x=0$ makes it off the mark. Namely, in Theorem 4 and Theorem 5 below we establish sharp upper and lower bounds

$$e^{-t\Lambda}(x,y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\psi_t(y), \quad x,y \in \mathbb{R}^d, \quad t>0, \tag{ULB_w}$$

where the continuous weight $0 \leq \psi_t(y) \leq 2$ vanishes at $y=0$ as $|y|^\beta$, $\beta>0$ (Theorem 2). (Here notation $a(z) \approx b(z)$ means that $c^{-1}b(z) \leq a(z) \leq cb(z)$ for some constant $c>1$ and all admissible z.) The order of vanishing β ($<\alpha$) depends explicitly on the value of the multiple $\kappa>0$ and tends to α as $\kappa \uparrow \infty$.

The key step in proving the upper and lower bound (ULB_w) is the weighted Nash initial estimate

$$0 \leq e^{-t\Lambda}(x,y) \leq Ct^{-\frac{d}{\alpha}}\psi_t(y), \quad x,y \in \mathbb{R}^d, \quad t>0. \tag{NIE_w}$$

The proof of (NIE_w) uses the method of desingularizing weights [MS0, MS1, MS2] based on ideas set forth by J. Nash [N]: it depends on the "desingularizing" (L^1, L^1) bound on the weighted semigroup $\psi_t e^{-t\Lambda}\psi_t^{-1}$.

The operator (1) in the local case $\alpha=2$ has been studied in [MeSS, MeSS2] by considering it in the space $L^2(\mathbb{R}^d, |x|^\gamma dx)$ for appropriate γ where the operator becomes symmetric. This approach, however, does not work for $\alpha<2$.

Recently, the authors in [CKSV], [JW] considered the fractional Schrödinger operator $H_+ = (-\Delta)^{\frac{\alpha}{2}} + V$, $V(x) = \kappa|x|^{-\alpha}$, $0<\alpha<2$, $\kappa>0$, and established, using different methods, sharp two-sided bounds

$$e^{-tH_+}(x,y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)\psi_t(x)\psi_t(y)$$

for appropriate weights $\psi_t(x)$ vanishing at $x=0$. We apply some ideas from [JW] (in the proof of Theorem 4).

In contrast to the cited papers, this work deals with purely non-local and non-symmetric situation. This leads to new difficulties, and requires new ideas. Even the proof of the standard upper bound $e^{-t\Lambda}(x,y) \leq Ce^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)$ (Theorem 3), as well as the construction of semigroups $e^{-t\Lambda}$, $e^{-t\Lambda^*}$ (Sections 8 and 9) become non-trivial. The same applies to the Sobolev regularity of $e^{-t\Lambda}f$, $f \in C_c^\infty$ established in Section 8.2. We consider these results, along with Theorem 4 and Theorem 5, as the main results of this article.

Below we apply the scheme of the proof of the upper and lower bounds in [KSS], although with comprehensive modifications in the method, both at the level of the abstract desingularization theorem (Theorem 1) and in the proofs of (NIE_w), (ULB_w) and of the standard upper bound.

We note that the heat kernel of the operator $(-\Delta)^{\frac{\alpha}{2}} + \mathsf{f}\cdot\nabla$ with $\operatorname{div}\mathsf{f}=0$ was studied in [MM, MM2]. For properties of the Feller process determined by (1) see [KM].

Let us mention that the vector field $b(x) = \kappa|x|^{-\alpha}x$ exhibits critical behaviour even if we remove the singularity of b at the origin. Namely, if we consider Λ with b bounded in $B(0,1)$ but having slower decay at infinity, $b(x) = \kappa|x|^{-\alpha+\varepsilon}x$, $\varepsilon>0$ for $|x|\geq 1$, then the global in time upper bound $e^{-t\Lambda}(x,y) \leq Ce^{-t(-\Delta)^{\frac{\alpha}{2}}}(x,y)$ of Theorem 3 would no longer be valid.

Contents

2. Desingularization in abstract setting

We first prove a general desingularization theorem in abstract setting, that we will apply in the next section to the fractional Kolmogorov operator.

Let X be a locally compact topological space, and μ a σ-finite Borel measure on X. Set $L^p=L^p(X,\mu)$, $p\in[1,\infty]$, a (complex) Banach space. We use the notation

$$\langle u,v\rangle=\langle u\bar{v}\rangle:=\int_X u\bar{v}d\mu,\quad \|\cdot\|_{p\to q}=\|\cdot\|_{L^p\to L^q}.$$

Let $-\Lambda$ be the generator of a contraction C_0 semigroup $e^{-t\Lambda}$, $t>0$, in L^2.

Assume that, for some constants $M\geq 1$, $c_S>0$, $j>1$, c,

$$\|e^{-t\Lambda}f\|_1\leq M\|f\|_1,\quad t\geq 0,\quad f\in L^1\cap L^2. \tag{B_{11}}$$

$$\text{Sobolev embedding property:}\quad \mathrm{Re}\langle\Lambda u,u\rangle\geq c_S\|u\|_{2j}^2,\quad u\in D(\Lambda). \tag{B_{12}}$$

$$\|e^{-t\Lambda}\|_{2\to\infty}\leq ct^{-\frac{j'}{2}},\quad t>0,\quad j'=\frac{j}{j-1}. \tag{B_{13}}$$

Assume also that there exists a family of real valued weights $\psi=\{\psi_s\}_{s>0}$ on X such that, for all $s>0$,

$$0\leq\psi_s,\psi_s^{-1}\in L^1_{\mathrm{loc}}(X-N,\mu),\quad \text{where } N \text{ is a closed null set}, \tag{B_{21}}$$

and there exist constants $\theta\in]0,1[$, $\theta\neq\theta(s)$, $c_i\neq c_i(s)$ $(i=2,3)$ and a measurable set $\Omega^s\subset X$ such that

$$\psi_s(x)^{-\theta}\leq c_2 \text{ for all } x\in X-\Omega^s, \tag{B_{22}}$$

$$\|\psi_s^{-\theta}\|_{L^{q'}(\Omega^s)}\leq c_3 s^{j'/q'},\ \text{where } q'=\frac{2}{1-\theta}. \tag{B_{23}}$$

Theorem 1. *In addition to* $(B_{11})-(B_{23})$ *assume that there exists a constant* $c_1 \neq c_1(s)$ *such that, for all* $\frac{s}{2} \leq t \leq s$,

$$\|\psi_s e^{-t\Lambda}\psi_s^{-1} f\|_1 \leq c_1\|f\|_1, \quad f \in L^1. \tag{B_3}$$

Then there is a constant C *such that, for all* $t>0$ *and* μ *a.e.* $x,y\in X$,

$$|e^{-t\Lambda}(x,y)| \leq Ct^{-j'}\psi_t(y).$$

Remark 1. In application of Theorem 1 to concrete operators, the main difficulty is in verification of the assumption (B_3).

Proof of Theorem 1. Set $\psi \equiv \psi_s$ and put $L^2_\psi := L^2(X,\psi^2 d\mu)$. Define a unitary map $\Psi : L^2_\psi \to L^2$ by $\Psi f = \psi f$. Set $\Lambda_\psi = \Psi^{-1}\Lambda\Psi$ of domain $D(\Lambda_\psi) = \Psi^{-1}D(\Lambda)$. Then

$$e^{-t\Lambda_\psi} = \Psi^{-1}e^{-t\Lambda}\Psi, \quad \|e^{-t\Lambda_\psi}\|_{2,\psi\to 2,\psi} = \|e^{-t\Lambda}\|_{2\to 2}, \quad t \geq 0.$$

Here and below the subscript ψ indicates that the corresponding quantities are related to the measure $\psi^2 d\mu$.

Set $u_t = e^{-t\Lambda_\psi}f$, $f \in L^2_\psi \cap L^1_\psi$. Applying (B_{12}), and then the Hölder inequality, we have

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\langle u_t,u_t\rangle_\psi &= \mathrm{Re}\langle \Lambda_\psi u_t, u_t\rangle_\psi \\ &= \mathrm{Re}\langle \Lambda\psi u_t, \psi u_t\rangle \\ &\geq c_S\|\psi u_t\|^2_{2j} \\ &\geq c_S\frac{\langle u_t,u_t\rangle_\psi^r}{\|\psi u_t\|_q^{2(r-1)}}, \end{aligned}$$

where $q = \frac{2}{1+\theta}(<2)$ and $r = \frac{(1+\theta)j-1}{j\theta}$.

Noticing that (B_{11}) + (B_{12}) implies the bound $\|e^{-t\Lambda}\|_{1\to 2} \leq \hat{c}t^{-\frac{j'}{2}}$ (for details, if needed, see Remark 2 below), we have by the interpolation inequality

$$\|e^{-t\Lambda}\|_{1\to q} \leq c_4 t^{-\frac{j'}{q'}}, \quad q' = \frac{q}{q-1}, \quad c_4 = M^{\frac{2}{q}-1}\hat{c}^{\frac{2}{q'}};$$

also, by (B_{11}) and interpolation, $\|e^{-t\Lambda}\|_{q\to q} \leq M^{\frac{2}{q}-1}$. Therefore,

$$\begin{aligned} \|\psi u_t\|_q &= \|e^{-t\Lambda}\psi f\|_q = \|e^{-t\Lambda}|\psi|^{-\theta}|\psi|^{\frac{2}{q}}f\|_q \\ &\quad \text{(we are applying } (B_{22}),(B_{23})) \\ &\leq c_2\|e^{-t\Lambda}\|_{q\to q}\|f\|_{q,\psi} + \|e^{-t\Lambda}\|_{1\to q}\||\psi|^{-\theta}\|_{L^{q'}(\Omega^s)}\|f\|_{q,\psi} \\ &\leq \big(c_2M^{\frac{2}{q}-1} + c_3c_4(s/t)^{\frac{j'}{q'}}\big)\|f\|_{q,\psi}. \end{aligned}$$

Thus, setting $w = \langle u_t,u_t\rangle_\psi$, we obtain

$$\frac{d}{dt}w^{1-r} \geq 2(r-1)c_S\big(c_2M^{\frac{2}{q}-1} + c_3c_4(s/t)^{\frac{j'}{q'}}\big)^{-2(r-1)}\|f\|_{q,\psi}^{-2(r-1)}.$$

Integrating this differential inequality yields

$$\|u_t\|_{2,\psi_s} \leq C_1 t^{-j'\left(\frac{1}{q}-\frac{1}{2}\right)}\|f\|_{q,\psi_s}, \quad s/2 \leq t \leq s.$$

The last inequality and (B_3) rewritten in the form $\|u_t\|_{1,\psi} \leq c_1\|f\|_{1,\psi}$ yield according to the Coulhon-Raynaud Extrapolation Theorem (Theorem 13 in Appendix B)

$$\|u_t\|_{2,\psi_s} \leq C_2 t^{-\frac{j'}{2}}\|f\|_{1,\psi_s}, \quad s/2 \leq t \leq s,$$

or

$$\|e^{-t\Lambda}h\|_2 \leq C_2 t^{-\frac{j'}{2}}\|h\|_{1,\sqrt{\psi_s}}, \quad h \in L^2 \cap L^1_{\sqrt{\psi_s}}, \quad s/2 \leq t \leq s, \tag{2}$$

where $L^1_{\sqrt{\psi_s}} := L^1(X, \psi_s d\mu)$.

Since $\|e^{-2t\Lambda}h\|_\infty \leq \|e^{-t\Lambda}\|_{2\to\infty}\|e^{-t\Lambda}h\|_2$, we have, employing (B_{13}),

$$\|e^{-2t\Lambda}h\|_\infty \leq cC_2 t^{-j'}\|h\|_{1,\sqrt{\psi_s}},$$

and so the assertion of Theorem 1 follows. □

Remark 2. The standard argument yields: $(B_{11}) + (B_{12}) \Rightarrow \|e^{-t\Lambda}\|_{1\to 2} \leq \hat{c}t^{-\frac{j'}{2}}$, $t > 0$. Indeed, setting $u_t := e^{-t\Lambda}f$, $f \in L^2 \cap L^1$, we have applying (B_{12}), Hölder's inequality and (B_{11})

$$\begin{aligned}
-\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 &= \mathrm{Re}\langle \Lambda u_t, u_t\rangle \\
&\geq c_S\|u_t\|_{2j}^2 \\
&\geq c_S\|u_t\|_2^{2+\frac{2}{j'}}\|u_t\|_1^{-\frac{2}{j'}} \\
&\geq c_S M^{-\frac{2}{j'}}\|u_t\|_2^{2+\frac{2}{j'}}\|f\|_1^{-\frac{2}{j'}}.
\end{aligned}$$

Thus, $w := \|u_t\|_2^2$ satisfies $\frac{d}{dt}w^{-\frac{1}{j'}} \geq C\|f\|_1^{-\frac{2}{j'}}$, $C = \frac{2c_S M^{-\frac{2}{j'}}}{j'}$, so integrating this inequality we obtain $\|e^{-t\Lambda}\|_{1\to 2} \leq C^{-\frac{j'}{2}}t^{-\frac{j'}{2}}$.

It is now seen that $(B_1) \equiv (B_{11}) + (B_{12}) + (B_{13})$ implies the bound $e^{-t\Lambda}(x,y) \leq \tilde{c}t^{-j'}$.

3. Heat kernel $e^{-t\Lambda}(x,y)$ for $\Lambda = (-\Delta)^{\frac{\alpha}{2}} - \kappa|x|^{-\alpha}x\cdot\nabla$, $1 < \alpha < 2$, $\kappa > 0$

We now state in detail our main result concerning the fractional Kolmogorov operator $(-\Delta)^{\frac{\alpha}{2}} - \kappa|x|^{-\alpha}x\cdot\nabla$, $1 < \alpha < 2$, $\kappa > 0$.

1. Let us outline the construction of an appropriate operator realization Λ_r of $(-\Delta)^{\frac{\alpha}{2}} - \kappa|x|^{-\alpha}x\cdot\nabla$ in L^r, $1 \leq r < \infty$. Set

$$b_\varepsilon(x) := \kappa|x|_\varepsilon^{-\alpha}x, \quad |x|_\varepsilon := \sqrt{|x|^2 + \varepsilon},\ \varepsilon > 0,$$

define the approximating operators in L^r

$$\Lambda^\varepsilon \equiv \Lambda_r^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon\cdot\nabla, \quad D(\Lambda_r^\varepsilon) = \mathcal{W}^{\alpha,r} := \big(1 + (-\Delta)^{\frac{\alpha}{2}}\big)^{-1}L^r, \quad 1 \leq r < \infty,$$

and in C_u (the space of uniformly continuous bounded functions with standard sup-norm),

$$\Lambda^\varepsilon \equiv \Lambda_{C_u}^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon\cdot\nabla, \quad D(\Lambda_{C_u}^\varepsilon) = D((-\Delta)_{C_u}^{\frac{\alpha}{2}}).$$

The operator $-\Lambda^\varepsilon$ is the generator of a holomorphic semigroup in L^r and in C_u. For details, if needed, see Section 8 below.

It is well known that

$$e^{-t\Lambda^\varepsilon}L_+^r \subset L_+^r \text{ and } e^{-t\Lambda^\varepsilon}C_u^+ \subset C_u^+$$

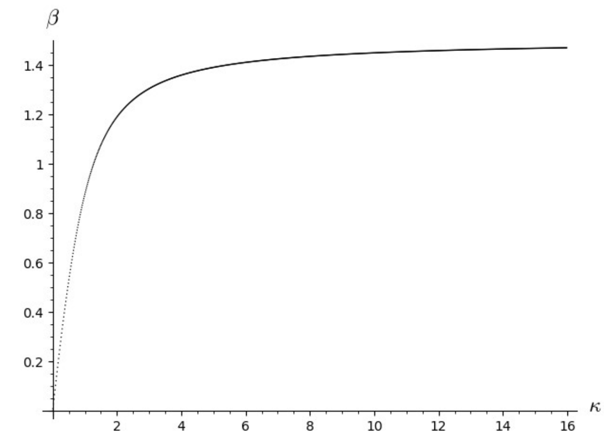

FIGURE 1. The function $\kappa \mapsto \beta$ for $d=3$ and $\alpha=\frac{3}{2}$.

where $L^r_+ := \{f \in L^r \mid f \geq 0\}$, $C_u^+ := \{f \in C_u \mid f \geq 0\}$. Also

$$\|e^{-t\Lambda^\varepsilon} f\|_\infty \leq \|f\|_\infty, \quad f \in L^r \cap L^\infty, \text{ or } f \in C_u.$$

In Proposition 10 below we show that, for every $r \in [1,\infty[$, the limit

$$s\text{-}L^r\text{-}\lim_{\varepsilon \downarrow 0} e^{-t\Lambda_r^\varepsilon} \quad (\text{loc. uniformly in } t \geq 0)$$

exists and determines a positivity preserving, contraction C_0 semigroup in L^r, say $e^{-t\Lambda_r}$; the (minus) generator Λ_r is an appropriate operator realization of the fractional Kolmogorov operator $(-\Delta)^{\frac{\alpha}{2}} - \kappa|x|^{-\alpha}x \cdot \nabla$ in L^r; there exists a constant c such that

$$\|e^{-t\Lambda_r}\|_{r \to q} \leq ct^{-\frac{d}{\alpha}(\frac{1}{r}-\frac{1}{q})}, \quad t>0,$$

for all $1 \leq r < q \leq \infty$; by construction, the semigroups $e^{-t\Lambda_r}$ are consistent:

$$e^{-t\Lambda_r} \upharpoonright L^r \cap L^p = e^{-t\Lambda_p} \upharpoonright L^r \cap L^p.$$

Using Proposition 10, we obtain

$$\langle \Lambda_r u, h\rangle = \langle u, (-\Delta)^{\frac{\alpha}{2}} h\rangle + \langle u, b \cdot \nabla h\rangle + \langle u, (\operatorname{div} b) h\rangle, \quad u \in D(\Lambda_r), \quad h \in C_c^\infty$$

(cf. [KSS, Prop. 9]).

2. We now introduce the desingularizing weights for $e^{-t\Lambda}$. Define β by

$$\beta \frac{d+\beta-2}{d+\beta-\alpha} \frac{\gamma(d+\beta-2)}{\gamma(d+\beta-\alpha)} = \kappa,$$

where

$$\gamma(\alpha) := \frac{2^\alpha \pi^{\frac{d}{2}} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{d}{2}-\frac{\alpha}{2})}.$$

Direct calculations show that $\beta \in]0,\alpha[$ exists (see Figure 1), and that $|x|^\beta$ is a Lyapunov's function of the formal adjoint operator $\Lambda^* = (-\Delta)^{\frac{\alpha}{2}} + \nabla \cdot b$, i.e. $\Lambda^* |x|^{-\beta} = 0$.

Set $\psi(x) \equiv \psi_s(x) := \eta(s^{-\frac{1}{\alpha}}|x|)$, where η is given by

$$\eta(t) = \begin{cases} t^\beta, & 0 < t < 1, \\ \beta t(2 - \frac{t}{2}) + 1 - \frac{3}{2}\beta, & 1 \leq t \leq 2, \\ 1 + \frac{\beta}{2}, & t \geq 2. \end{cases}$$

Applying Theorem 1 to the operator Λ_r and the weights ψ_s, we obtain

Theorem 2. *$e^{-t\Lambda_r}$ is an integral operator for each $t > 0$ with integral kernel $e^{-t\Lambda}(x, y) \geq 0$. There exists a constant $c_{N,w}$ such that the weighted Nash initial estimate*

$$e^{-t\Lambda}(x, y) \leq c_{N,w} t^{-\frac{d}{\alpha}} \psi_t(y). \tag{NIE_w}$$

is valid for all $x, y \in \mathbb{R}^d$ and $t > 0$.

The next step is to deduce the following global in time "standard" upper bound on $e^{-t\Lambda}(x, y)$.

Theorem 3. *(i) There is a constant C_1 such that, for all $t > 0$, $x, y \in \mathbb{R}^d$,*

$$e^{-t\Lambda}(x, y) \leq C_1 e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x, y).$$

(ii) Moreover, for a given $\delta \in]0, 1[$, there is a constant $D = D_\delta > 0$ such that

$$e^{-t\Lambda}(x, y) \leq (1 + \delta) e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x, y), \qquad |x| > Dt^{\frac{1}{\alpha}}, \ \ y \in \mathbb{R}^d.$$

Theorem 2 and Theorem 3 are the key tools which allow us to establish the upper bound on $e^{-t\Lambda}(x, y)$:

Theorem 4. *There is a constant C such that, for all $t > 0$, $x, y \in \mathbb{R}^d$,*

$$e^{-t\Lambda}(x, y) \leq C e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x, y)\psi_t(y). \tag{UB_w}$$

Using Theorem 4, we prove the lower bound on $e^{-t\Lambda}(x, y)$:

Theorem 5. *There is a constant $\tilde{C} > 0$ such that, for all $t > 0$, $x, y \in \mathbb{R}^d$,*

$$e^{-t\Lambda}(x, y) \geq \tilde{C} e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x, y)\psi_t(y). \tag{LB_w}$$

4. Proof of Theorem 2: The weighted Nash initial estimate

The proof follows by applying Theorem 1 to $e^{-t\Lambda_r}$.

The conditions (B_{11}) and (B_{13}) (with $j' = \frac{d}{\alpha}$) are satisfied by Proposition 10. Let us prove (B_{12}). By Proposition 8 ($\Lambda^\varepsilon \equiv \Lambda_2^\varepsilon$),

$$\mathrm{Re}\big\langle \Lambda^\varepsilon (1 + \Lambda^\varepsilon)^{-1} g, (1 + \Lambda^\varepsilon)^{-1} g \big\rangle \geq c_S \|(1 + \Lambda^\varepsilon)^{-1} g\|_{2j}^2, \quad g \in L^2, \quad j = \frac{d}{d - \alpha}, \quad c_S \neq c_S(\varepsilon),$$

i.e.

$$\mathrm{Re}\big\langle g - (1 + \Lambda^\varepsilon)^{-1} g, (1 + \Lambda^\varepsilon)^{-1} g \big\rangle \geq c_S \|(1 + \Lambda^\varepsilon)^{-1} g\|_{2j}^2.$$

Using the convergence $(1 + \Lambda^\varepsilon)^{-1} \overset{s}{\to} (1 + \Lambda)^{-1}$ in L^2 as $\varepsilon \downarrow 0$ (Proposition 10), we pass to the limit $\varepsilon \downarrow 0$ in the last inequality to obtain $\mathrm{Re}\big\langle \Lambda(1 + \Lambda)^{-1} g, (1 + \Lambda)^{-1} g \big\rangle \geq c_S \|(1 + \Lambda)^{-1} g\|_{2j}^2$ for all $g \in L^2$, and so (B_{12}) is proven.

The condition (B_{21}) is evident from the definition of the weights ψ_s. It is easily seen that $(B_{22}),(B_{23})$ hold with $\Omega^s = B(0, s^{\frac{1}{\alpha}})$ and $\theta = \frac{(2-\alpha)d}{(2-\alpha)d+8\beta}$. It remains to prove the desingularizing (L^1, L^1) bound (B_3), which presents the main difficulty.

Proof of (B_3). We modify the proof of the analogous (L^1, L^1) bound in [KSS] (see also Remark 6 below). We will appeal to the Lumer-Phillips Theorem applied to specially constructed C_0 semigroups in L^1, corresponding to operators with smooth coefficients and smooth weights, which approximate $\psi_s e^{-t\Lambda}\psi_s^{-1}$.

Recall that $b_\varepsilon(x) := \kappa|x|_\varepsilon^{-\alpha}x$, $|x|_\varepsilon := \sqrt{|x|^2+\varepsilon}$, $\varepsilon > 0$,

$$\Lambda^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon \cdot \nabla, \quad D(\Lambda^\varepsilon) = \mathcal{W}^{\alpha,1} := \big(1+(-\Delta)^{\frac{\alpha}{2}}\big)^{-1}L^1,$$

$$(\Lambda^\varepsilon)^* = (-\Delta)^{\frac{\alpha}{2}} + \nabla \cdot b_\varepsilon, \quad D(\Lambda^\varepsilon) = \mathcal{W}^{\alpha,1}.$$

By the Hille Perturbation Theorem, for each $\varepsilon > 0$, both $e^{-t\Lambda^\varepsilon}$, $e^{-t(\Lambda^\varepsilon)^*}$ can be viewed as C_0 semigroups in L^1 and C_u (see Sections 8 and 9).

Define approximating weights

$$\phi_{n,\varepsilon} := n^{-1} + e^{-\frac{(\Lambda^\varepsilon)^*}{n}}\psi, \quad \psi = \psi_s.$$

Remark 3. This choice of the regularization of ψ is dictated by the method: $e^{-\frac{(\Lambda^\varepsilon)^*}{n}}$ will be needed below to control the auxiliary potential U_ε. See also Remark 5 below.

In L^1 define operators

$$Q = \phi_{n,\varepsilon}\Lambda^\varepsilon\phi_{n,\varepsilon}^{-1}, \quad D(Q) = \phi_{n,\varepsilon}D(\Lambda^\varepsilon),$$

where $\phi_{n,\varepsilon}D(\Lambda^\varepsilon) := \{\phi_{n,\varepsilon}u \mid u \in D(\Lambda^\varepsilon)\}$,

$$F_{\varepsilon,n}^t = \phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}\phi_{n,\varepsilon}^{-1}.$$

Since $\phi_{n,\varepsilon}, \phi_{n,\varepsilon}^{-1} \in L^\infty$, these operators are well defined. In particular, $F_{\varepsilon,n}^t$ are bounded C_0 semigroups in L^1, say $F_{\varepsilon,n}^t = e^{-tG}$.

Set

$$\begin{aligned} M &:= \phi_{n,\varepsilon}(1+(-\Delta)^{\frac{\alpha}{2}})^{-1}[L^1 \cap C_u] \\ &= \phi_{n,\varepsilon}(\lambda_\varepsilon + \Lambda^\varepsilon)^{-1}[L^1 \cap C_u], \quad 0 < \lambda_\varepsilon \in \rho(-\Lambda^\varepsilon). \end{aligned}$$

Clearly, M is a dense subspace of L^1, $M \subset D(Q)$ and $M \subset D(G)$. Moreover, $Q \upharpoonright M \subset G$. Indeed, for $f = \phi_{n,\varepsilon}u \in M$,

$$Gf = s\text{-}L^1\text{-}\lim_{t\downarrow 0} t^{-1}(1-e^{-tG})f = \phi_{n,\varepsilon}s\text{-}L^1\text{-}\lim_{t\downarrow 0} t^{-1}(1-e^{-t\Lambda^\varepsilon})u = \phi_{n,\varepsilon}\Lambda^\varepsilon u = Qf.$$

Thus $Q \upharpoonright M$ is closable and $\tilde{Q} := (Q \upharpoonright M)^{\rm clos} \subset G$.

Proposition 1. *The range* $R(\lambda_\varepsilon + \tilde{Q})$ *is dense in* L^1.

Proof of Proposition 1. If $\langle(\lambda_\varepsilon + \tilde{Q})h, v\rangle = 0$ for all $h \in D(\tilde{Q})$ and some $v \in L^\infty$, $\|v\|_\infty = 1$, then taking $h \in M$ we would have $\langle(\lambda_\varepsilon + Q)\phi_{n,\varepsilon}(\lambda_\varepsilon + \Lambda^\varepsilon)^{-1}g, v\rangle = 0$, $g \in L^1 \cap C_u$, or $\langle\phi_{n,\varepsilon}g, v\rangle = 0$. Choosing $g = e^{\frac{\Delta}{k}}(\chi_m v)$, where $\chi_m \in C_c^\infty$ with $\chi_m(x) = 1$ when $x \in B(0,m)$, we would have $\lim_{k\uparrow\infty}\langle\phi_{n,\varepsilon}g, v\rangle = \langle\phi_n\chi_m, |v|^2\rangle = 0$, and so $v = 0$. Thus, $R(\lambda_\varepsilon + \tilde{Q})$ is dense in L^1. ☐

Proposition 2. *There are constants $\hat{c} > 0$ and $\varepsilon_n > 0$ such that, for every n and all $0 < \varepsilon \leq \varepsilon_n$,*

$$\lambda + \tilde{Q} \text{ is accretive whenever } \lambda \geq \hat{c}s^{-1} + n^{-1}.$$

Proof of Proposition 2. Recall that both $e^{-t\Lambda^\varepsilon}$, $e^{-t(\Lambda^\varepsilon)^*}$ are holomorphic in L^1 and C_u due to Hille's Perturbation Theorem. We have

$$\psi = \psi_{(1)} + \psi_{(u)}, \qquad 0 \leq \psi_{(1)} \in D((-\Delta)_1^{\frac{\alpha}{2}}), \qquad 0 \leq \psi_{(u)} \in D((-\Delta)_{C_u}^{\frac{\alpha}{2}}).$$

For instance,

$$\psi_{(u)} := 1 + \frac{\beta}{2}, \quad \psi_{(1)} := \psi - 1 - \frac{\beta}{2} \quad (\text{so, sprt}\, \psi_{(1)} \subset B(0, 2s^{\frac{1}{\alpha}})).$$

In $B(0, s^{\frac{1}{\alpha}})$, the weight ψ coincides with $\tilde{\psi}(x) \equiv \tilde{\psi}_s(x) := s^{-\frac{\beta}{\alpha}}|x|^\beta$, so $\psi_{(1)} \in D((-\Delta)_1)$. Thus, $\psi_{(1)} \in D((-\Delta)_1^{\frac{\alpha}{2}})$ (see, e.g. [Ka, Ch.V, sect.3.11]). Therefore,

$$(\Lambda^\varepsilon)^*\psi \ \big(= (\Lambda^\varepsilon)^*_{L^1}\psi_{(1)} + (\Lambda^\varepsilon)^*_{C_u}\psi_{(u)}\big)$$

is well defined and belongs to $L^1 + C_u = \{w + v \mid w \in L^1, v \in C_u\}$.

We verify that $\mathrm{Re}\langle(\lambda + \tilde{Q})f, \frac{f}{|f|}\rangle \geq 0$ for all $f \in D(\tilde{Q})$. For $f = \phi_{n,\varepsilon}u \in M$, we have

$$\begin{aligned}
\langle Qf, \frac{f}{|f|}\rangle &= \langle \phi_{n,\varepsilon}\Lambda^\varepsilon u, \frac{f}{|f|}\rangle = \lim_{t\downarrow 0} t^{-1}\langle \phi_{n,\varepsilon}(1 - e^{-t\Lambda^\varepsilon})u, \frac{f}{|f|}\rangle, \\
\mathrm{Re}\langle Qf, \frac{f}{|f|}\rangle &\geq \lim_{t\downarrow 0} t^{-1}\langle (1 - e^{-t\Lambda^\varepsilon})|u|, \phi_{n,\varepsilon}\rangle \\
&= \lim_{t\downarrow 0} t^{-1}\langle (1 - e^{-t\Lambda^\varepsilon})|u|, n^{-1}\rangle + \lim_{t\downarrow 0} t^{-1}\langle (1 - e^{-t\Lambda^\varepsilon})e^{-\frac{\Lambda^\varepsilon}{n}}|u|, \psi\rangle \\
&= \lim_{t\downarrow 0} t^{-1}\langle |u|, (1 - e^{-t(\Lambda^\varepsilon)^*})n^{-1}\rangle + \lim_{t\downarrow 0} t^{-1}\langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|, (1 - e^{-t(\Lambda^\varepsilon)^*})\psi\rangle \\
&= \langle |u|, (\Lambda^\varepsilon)^* n^{-1}\rangle + \langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|, (\Lambda^\varepsilon)^*\psi\rangle,
\end{aligned}$$

where the first term is positive since $(\Lambda^\varepsilon)^* n^{-1} = n^{-1}\mathrm{div}\, b_\varepsilon = n^{-1}\big(d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2\big) \geq n^{-1}(d - \alpha)|x|_\varepsilon^{-\alpha} \geq 0$. Thus,

$$\mathrm{Re}\langle Qf, \frac{f}{|f|}\rangle \geq \langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|, (\Lambda^\varepsilon)^*\psi\rangle, \tag{3}$$

so it remains to bound $J := \langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|, (\Lambda^\varepsilon)^*\psi\rangle$ from below. For that, we estimate from below

$$(\Lambda^\varepsilon)^*\psi = (-\Delta)^{\frac{\alpha}{2}}\psi + \mathrm{div}\,(b_\varepsilon\psi).$$

Claim 1. $(-\Delta)^{\frac{\alpha}{2}}\psi \geq -\beta(d + \beta - 2)\frac{\gamma(d+\beta-2)}{\gamma(d+\beta-\alpha)}|x|^{-\alpha}\tilde{\psi}$.

Proof of Claim 1. All identities are in the sense of distributions:

$$\begin{aligned}
(-\Delta)^{\frac{\alpha}{2}}\psi &= -I_{2-\alpha}\Delta\psi \\
&= -I_{2-\alpha}\Delta\tilde{\psi} - I_{2-\alpha}\Delta(\psi - \tilde{\psi}),
\end{aligned}$$

where $I_\nu = (-\Delta)^{-\frac{\nu}{2}}$ is the Riesz potential, and we evaluate the first term

$$\begin{aligned}
-I_{2-\alpha}\Delta\tilde{\psi} &= -s^{-\frac{\beta}{\alpha}}\beta(d + \beta - 2)I_{2-\alpha}|x|^{\beta-2} \\
&= -s^{-\frac{\beta}{\alpha}}\beta(d + \beta - 2)\frac{\gamma(d + \beta - 2)}{\gamma(d + \beta - \alpha)}|x|^{\beta-\alpha},
\end{aligned}$$

while the second term is positive and can be omitted: $-I_{2-\alpha}\Delta(\psi-\tilde{\psi})\geq 0$ (see Remark 4 below for detailed calculation). The proof of Claim 1 is completed. □

Claim 2. $\operatorname{div}(b_\varepsilon\psi)\geq \operatorname{div}(b\tilde{\psi})-U_\varepsilon\tilde{\psi}-\hat{c}s^{-1}\psi$ *for a constant* $\hat{c}\neq\hat{c}(\varepsilon,n)$*, where* $U_\varepsilon(x):=\kappa(d+\beta-\alpha)(|x|^{-\alpha}-|x|_\varepsilon^{-\alpha})>0$.

Proof. We represent

$$\operatorname{div}(b_\varepsilon\psi)=\operatorname{div}(b\tilde{\psi})+\operatorname{div}(b_\varepsilon\psi)-\operatorname{div}(b\tilde{\psi})$$

and estimate the difference $\operatorname{div}(b_\varepsilon\psi)-\operatorname{div}(b\tilde{\psi})$:

$$\begin{aligned}\operatorname{div}(b_\varepsilon\psi)-\operatorname{div}(b\tilde{\psi})&=\operatorname{div}\big[b(\psi-\tilde{\psi})\big]+\operatorname{div}\big[(b_\varepsilon-b)\psi\big]\\&=h_1+\operatorname{div}\big[(b_\varepsilon-b)\psi\big],\end{aligned}$$

where $h_1\in C_\infty$ (continuous functions vanishing at infinity), $h_1=0$ in $B(0,s^{\frac{1}{\alpha}})$. In turn,

$$\begin{aligned}\operatorname{div}\big[(b_\varepsilon-b)\psi\big]&=(b_\varepsilon-b)\cdot\nabla\psi+(\operatorname{div}b_\varepsilon-\operatorname{div}b)\psi\\&=\kappa(|x|_\varepsilon^{-\alpha}-|x|^{-\alpha})x\cdot\nabla\tilde{\psi}+h_2+\kappa\big[d|x|_\varepsilon^{-\alpha}-\alpha|x|_\varepsilon^{-\alpha-2}|x|^2-(d-\alpha)|x|^{-\alpha}\big]\psi\\&\quad(\text{where } h_2:=\kappa(|x|_\varepsilon^{-\alpha}-|x|^{-\alpha})x\cdot\nabla(\psi-\tilde{\psi})\in C_\infty,\ h_2=0\text{ in }B(0,s^{\frac{1}{\alpha}}))\\&=\kappa(|x|_\varepsilon^{-\alpha}-|x|^{-\alpha})\beta\tilde{\psi}+h_2+\kappa\big[d|x|_\varepsilon^{-\alpha}-\alpha|x|_\varepsilon^{-\alpha-2}|x|^2-(d-\alpha)|x|^{-\alpha}\big]\psi\\&\geq\kappa(|x|_\varepsilon^{-\alpha}-|x|^{-\alpha})\beta\tilde{\psi}+h_2+\kappa(d-\alpha)(|x|_\varepsilon^{-\alpha}-|x|^{-\alpha})\psi.\end{aligned}$$

Thus,

$$\operatorname{div}(b_\varepsilon\psi)\geq\operatorname{div}(b\tilde{\psi})+\kappa(d+\beta-\alpha)(|x|_\varepsilon^{-\alpha}-|x|^{-\alpha})\tilde{\psi}+h_1+h_2+h_3,$$

where $h_3:=\kappa(d-\alpha)(|x|_\varepsilon^{-\alpha}-|x|^{-\alpha})(\psi-\tilde{\psi})\in C_\infty$, $h_3=0$ in $B(0,s^{\frac{1}{\alpha}})$.

A straightforward calculation shows that $h_i\geq -c_i\psi s^{-1}$ with $c_i\neq c_i(\varepsilon,n)$, $i=1,2,3$ (we have used that $h_i=0$ in $B(0,s^{\frac{1}{\alpha}})$). The assertion of Claim 2 follows. □

Now, we combine Claim 1 and Claim 2: In view of the choice of β, $-\beta(d+\beta-2)\frac{\gamma(d+\beta-2)}{\gamma(d+\beta-\alpha)}|x|^{-\alpha}\tilde{\psi}+\operatorname{div}(b\tilde{\psi})=0$ (that is, formally, $\Lambda^*\tilde{\psi}=0$), and so

$$(\Lambda^\varepsilon)^*\psi\geq -U_\varepsilon\tilde{\psi}-\hat{c}s^{-1}\psi.$$

It follows that

$$\begin{aligned}J\equiv\langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|,(\Lambda^\varepsilon)^*\psi\rangle&\geq-\hat{c}s^{-1}\langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|,\psi\rangle-\langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|,U_\varepsilon\tilde{\psi}\rangle\\&\geq-\hat{c}s^{-1}\langle|u|,e^{-\frac{(\Lambda^\varepsilon)^*}{n}}\psi\rangle-\langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|,U_\varepsilon\tilde{\psi}\rangle\\&\geq-\hat{c}s^{-1}\langle|u|,n^{-1}+e^{-\frac{(\Lambda^\varepsilon)^*}{n}}\psi\rangle-\langle e^{-\frac{\Lambda^\varepsilon}{n}}|u|,U_\varepsilon\tilde{\psi}\rangle\\&\quad(\text{recall that }|u|=\phi_{n,\varepsilon}^{-1}|f|\text{ and }\phi_{n,\varepsilon}=n^{-1}+e^{-\frac{(\Lambda^\varepsilon)^*}{n}}\psi)\\&=-\hat{c}s^{-1}\|f\|_1-\langle|u|,e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(U_\varepsilon\tilde{\psi})\rangle.\end{aligned}$$

Now, for every $n \geq 1$, we have

$$\|e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(U_\varepsilon\tilde{\psi})\|_\infty \leq \|e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(\mathbf{1}_{B^c(0,R)}U_\varepsilon\tilde{\psi})\|_\infty + \|e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(\mathbf{1}_{B(0,R)}U_\varepsilon\tilde{\psi})\|_\infty$$

(we are using that $e^{-t(\Lambda^\varepsilon)^*}$ is a L^∞ contraction and ultra-contraction, see Proposition 11)

$$\leq \|\mathbf{1}_{B^c(0,R)}U_\varepsilon\tilde{\psi}\|_\infty + c_N n^{\frac{d}{\alpha}}\|\mathbf{1}_{B(0,R)}U_\varepsilon\tilde{\psi}\|_1$$

(we fix $R = R_n$ such that $\|\mathbf{1}_{B^c(0,R)}U_\varepsilon\tilde{\psi}\|_\infty \leq 2^{-1}n^{-2}$ and choose $\varepsilon_n > 0$ such that for all $\varepsilon \leq \varepsilon_n$ $\|\mathbf{1}_{B(0,R)}U_\varepsilon\tilde{\psi}\|_1 \leq 2^{-1}n^{-2}(c_N n^{\frac{d}{\alpha}})^{-1}$)

$$\leq n^{-2}.$$

Therefore, since $\phi_{n,\varepsilon} \geq n^{-1}$, we have for every n and all $\varepsilon \leq \varepsilon_n$ $\|\phi_{n,\varepsilon}^{-1}e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(U_\varepsilon\tilde{\psi})\|_\infty \leq n^{-1}$ and so $\langle |u|, e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(U_\varepsilon\tilde{\psi})\rangle \leq n^{-1}\|f\|_1$. Thus,

$$J \geq -\big(\hat{c}s^{-1} + n^{-1}\big)\|f\|_1.$$

Returning to (3), one can easily see that the latter yields the assertion of Proposition 2. □

Remark 4. Let us show that $-\Delta(\psi - \tilde{\psi}) \geq 0$. Without loss of generality, $s = 1$. The inequality is evidently true on $\{0 < |x| \leq 1\} \cup \{|x| \geq 2\}$. Now, let $1 < |x| < 2$. Then

$$\begin{aligned}
\Delta(\tilde{\psi} - \psi) &= \beta(d+\beta-2)|x|^{\beta-2} - \eta''(|x|)|x|^{-2} - \eta'(|x|)(d-1)|x|^{-1} \\
&= \beta(d+\beta-2)|x|^{\beta-2} + \beta|x|^{-2} - \beta(2-|x|)(d-1)|x|^{-1} \\
&= \beta|x|^{-2}\big((d+\beta-2)|x|^\beta + 1 - (d-1)(2-|x|)|x|\big) \\
&\geq \beta|x|^{-2}\big((d+\beta-2) + 1 - (d-1)\big) \geq 0.
\end{aligned}$$

□

The fact that $\tilde{Q}$ is closed together with Proposition 1 and Proposition 2 imply $R(\lambda_\varepsilon + \tilde{Q}) = L^1$ (Appendix C). Then, by the Lumer-Phillips Theorem, $\lambda + \tilde{Q}$ is the (minus) generator of a contraction semigroup, and $\tilde{Q} = G$ due to $\tilde{Q} \subset G$. Thus, it follows that, for all n and all $\varepsilon \leq \varepsilon_n$

$$\|e^{-tG}\|_{1\to1} \equiv \|\phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}\phi_{n,\varepsilon}^{-1}\|_{1\to1} \leq e^{\omega t}, \quad \omega = \hat{c}s^{-1} + n^{-1}. \tag{$\star$}$$

To obtain (B_3), it remains to pass to the limit in $(\star)$: first in $\varepsilon \downarrow 0$ and then in $n \to \infty$. It suffices to prove (B_3) on positive functions. By $(\star)$,

$$\|\phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}\phi_{n,\varepsilon}^{-1}f\|_1 \leq e^{\omega t}\|f\|_1, \quad 0 \leq f \in L^1,$$

or taking $f = \phi_{n,\varepsilon}h$, $0 \leq h \in L^1$,

$$\|\phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}h\|_1 \leq e^{\omega t}\|\phi_{n,\varepsilon}h\|_1.$$

Using Proposition 10, we have

$$\|\phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}h\|_1 = \langle n^{-1}e^{-t\Lambda^\varepsilon}h\rangle + \langle\psi, e^{-(t+\frac{1}{n})\Lambda^\varepsilon}h\rangle \to \langle n^{-1}e^{-t\Lambda}h\rangle + \langle\psi, e^{-(t+\frac{1}{n})\Lambda}h\rangle \quad \text{as } \varepsilon \downarrow 0,$$

and

$$\|\phi_{n,\varepsilon}h\|_1 = n^{-1}\langle h\rangle + \langle\psi, e^{-\frac{\Lambda^\varepsilon}{n}}h\rangle \to n^{-1}\langle h\rangle + \langle\psi, e^{-\frac{\Lambda}{n}}h\rangle \quad \text{as } \varepsilon \downarrow 0.$$

Thus,

$$\langle n^{-1}e^{-t\Lambda}h\rangle + \langle \psi, e^{-(t+\frac{1}{n})\Lambda}h\rangle \leq e^{\omega t}\big(n^{-1}\langle h\rangle + \langle \psi, e^{-\frac{\Lambda}{n}}h\rangle\big).$$

Taking $n \to \infty$, we obtain $\langle \psi e^{-t\Lambda}h\rangle \leq e^{\hat{c}s^{-1}t}\langle \psi h\rangle$. (B_3) now follows.

The proof of Theorem 2 is completed. □

Remark 5 (On the choice of the regularization $\phi_{n,\varepsilon}$ of the weight ψ)**.** In [KSS], we construct the regularization of the weight in the same way as above, although there the factor $e^{-\frac{1}{n}(\Lambda^\varepsilon)^*}$ serves a different purpose (in [KSS] the drift term $b \cdot \nabla$ has the opposite sign, and so the corresponding weight is unbounded). (As a by-product, this allows us to consider $(-\Delta)^{\frac{\alpha}{2}}$ perturbed by two drift terms, as in the present paper and as in [KSS], possibly having singularities at different points.)

Remark 6. In the proof of the analogous (L^1, L^1) bound in [KSS, proof of Theorem 2], where we consider the vector field b of the opposite sign, we first pass to the limit in $n \to \infty$, and then in $\varepsilon \downarrow 0$. In the proof of Theorem 2 above this order is naturally reversed.

As a consequence of the (L^1, L^1) bound (B_3), we obtain

Corollary 1. *$\langle e^{-t\Lambda}(\cdot, x)\psi_t(\cdot)\rangle \leq c_1\psi_t(x)$ for all $x \in \mathbb{R}^d$, $x \neq 0$, $t > 0$.*

As a consequence of Corollary 1 and (NIE_w), we obtain

Corollary 2. *$\langle e^{-t\Lambda}(\cdot, x)\rangle = \langle e^{-t\Lambda^*}(x, \cdot)\rangle \leq C_2\psi_t(x)$ for all $x \in \mathbb{R}^d$, $x \neq 0$, $t > 0$.*

Proof. We have

$$\begin{aligned}\langle e^{-t\Lambda^*}(x, \cdot)\rangle &\leq \big\langle \mathbf{1}_{B(0,t^{\frac{1}{\alpha}})}(\cdot)e^{-t\Lambda^*}(x, \cdot)\big\rangle + \big\langle \mathbf{1}_{B^c(0,t^{\frac{1}{\alpha}})}(\cdot)e^{-\Lambda^*}(x, \cdot)\psi_t(\cdot)\big\rangle \\ &=: I_1 + I_2.\end{aligned}$$

By (NIE_w), $I_1 \leq c'\psi_t(x)$, and by Corollary 1, $I_2 \leq c''\psi_t(x)$, for appropriate constants c', $c'' < \infty$. Set $C_2 := c' + c''$. □

5. Proof of Theorem 3: The standard upper bounds

(i) For brevity, put $A := (-\Delta)^{\frac{\alpha}{2}}$. Recall that

$$k_0^{-1}t\big(|x-y|^{-d-\alpha} \wedge t^{-\frac{d+\alpha}{\alpha}}\big) \leq e^{-tA}(x, y) \leq k_0 t\big(|x-y|^{-d-\alpha} \wedge t^{-\frac{d+\alpha}{\alpha}}\big)$$

for all $x, y \in \mathbb{R}^d$, $x \neq y$, $t > 0$, for a constant $k_0 = k_0(d, \alpha) > 1$.

In view of Proposition 10, it suffices to prove the a priori bound

$$e^{-t\Lambda^\varepsilon}(x, y) \leq C_1 e^{-tA}(x, y), \quad x, y \in \mathbb{R}^d, \quad t > 0, \quad C_1 \neq C_1(\varepsilon).$$

By duality, it suffices to prove

$$e^{-t(\Lambda^\varepsilon)^*}(x, y) \leq C_1 e^{-tA}(x, y), \quad x, y \in \mathbb{R}^d, \quad t > 0, \quad C_1 \neq C_1(\varepsilon).$$

Step 1: *For every $D > 1$ and all $t > 0$, $|x| \leq Dt^{\frac{1}{\alpha}}$, $|y| \leq Dt^{\frac{1}{\alpha}}$ the following bound*

$$e^{-t(\Lambda^\varepsilon)^*}(x, y) \leq k_0 c_N (2D)^{d+\alpha} e^{-tA}(x, y)$$

is valid.

In fact, we will prove

Lemma 6. *Let $t>0$ and $D>1$. Then*

$(i)\qquad e^{-t(\Lambda^\varepsilon)^*}(x,y)\leq k_0c_N(2D)^{d+\alpha}e^{-tA}(x,y),\qquad |x|\leq Dt^{\frac{1}{\alpha}},\ |y|\leq Dt^{\frac{1}{\alpha}}.$

$(ii)\qquad e^{-t\Lambda^*}(x,y)\leq k_0c_{N,w}(1+D)^{d+\alpha}e^{-tA}(x,y)\psi_t(x),\qquad |x|\leq t^{\frac{1}{\alpha}},\ |y|\leq Dt^{\frac{1}{\alpha}}.$

Proof. (i) Note that $(|x|\leq Dt^{\frac{1}{\alpha}},\ |y|\leq Dt^{\frac{1}{\alpha}})\Rightarrow t^{-\frac{d}{\alpha}}\leq(2D)^{d+\alpha}t|x-y|^{-d-\alpha}$. The latter means that $t^{-\frac{d}{\alpha}}\leq k_0(2D)^{d+\alpha}e^{-tA}(x,y)$. In Proposition 12, the Nash initial estimate

$$e^{-t(\Lambda^\varepsilon)^*}(x,y)\leq c_Nt^{-\frac{d}{\alpha}},\quad x,y\in\mathbb{R}^d,\quad t>0 \tag{NIE}$$

is proved. Therefore,

$$e^{-t(\Lambda^\varepsilon)^*}(x,y)\leq c_Nt^{-\frac{d}{\alpha}}\leq k_0c_N(2D)^{d+\alpha}e^{-tA}(x,y).$$

(ii) Clearly, $(|x|\leq Dt^{\frac{1}{\alpha}},\ |y|\leq t^{\frac{1}{\alpha}})\Rightarrow t^{-\frac{d}{\alpha}}\leq(1+D)^{d+\alpha}t|x-y|^{-d-\alpha}$, and so the inequality $t^{-\frac{d}{\alpha}}\leq k_0(1+D)^{d+\alpha}e^{-tA}(x,y)$ is valid. By (NIE_w) (Theorem 2), $e^{-t\Lambda^*}(x,y)\leq c_{N,w}t^{-\frac{d}{\alpha}}\psi_t(x)$ for all $t>0$, $x,y\in\mathbb{R}^d$. Therefore,

$$e^{-t\Lambda^*}(x,y)\leq k_0c_{N,w}(1+D)^{d+\alpha}e^{-tA}(x,y)\psi_t(x).$$

□

In what follows, we will need the following estimates.

Lemma 7. *Set $E^t(x,y)=t\big(|x-y|^{-d-\alpha-1}\wedge t^{-\frac{d+\alpha+1}{\alpha}}\big)$, $E^tf(x):=\langle E^t(x,\cdot)f(\cdot)\rangle$, $t>0$.*
Then there exist constants k_i $(i=1,2,3)$ such that for all $0<t<\infty$, x, $y\in\mathbb{R}^d$
(i) $|\nabla_xe^{-tA}(x,y)|\leq k_1E^t(x,y)$;
(ii) $\int_0^t\langle e^{-(t-\tau)A}(x,\cdot)E^\tau(\cdot,y)\rangle d\tau\leq k_2t^{\frac{\alpha-1}{\alpha}}e^{-tA}(x,y)$;
(iii) $\int_0^t\langle E^{t-\tau}(x,\cdot)E^\tau(\cdot,y)\rangle d\tau\leq k_3t^{\frac{\alpha-1}{\alpha}}E^t(x,y)$.

Proof. For the proof of (i), (ii) see e.g. [BJ]. Essentially the same argument yields (iii), see e.g. [KSS, sect. 5] for details. □

Step 2: *Fix $\delta\in]0,2^{-1}[$. Set $C_g:=\kappa k_1(2k_2+k_3)$, $R:=(C_g\delta^{-1})^{\frac{1}{\alpha-1}}$ and $m=1+2k_0k_1$.*
If $D\geq Rm$, then the following bound

$$e^{-t(\Lambda^\varepsilon)^*}(x,y)\leq(1+\delta)e^{-tA}(x,y),\quad x\in\mathbb{R}^d,\quad |y|>Dt^{\frac{1}{\alpha}},\quad t>0 \tag{4}$$

is valid.

We use the Duhamel formula

$$\begin{aligned}e^{-t(\Lambda^\varepsilon)^*}&=e^{-tA}+\int_0^te^{-\tau(\Lambda^\varepsilon)^*}(B^t_{\varepsilon,R}+B^{t,c}_{\varepsilon,R})e^{-(t-\tau)A}d\tau\\&=:e^{-tA}+K^t_R+K^{t,c}_R,\quad R:=(C_g\delta^{-1})^{\frac{1}{\alpha-1}},\end{aligned} \tag{5}$$

where

$$B^t_{\varepsilon,R}:=\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}B_\varepsilon,\quad B^{t,c}_{\varepsilon,R}:=\mathbf{1}_{B^c(0,Rt^{\frac{1}{\alpha}})}B_\varepsilon,\quad B_\varepsilon:=-b_\varepsilon\cdot\nabla-W_\varepsilon,$$

where $W_\varepsilon(x) := \kappa(d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2)$.

Set

$$M_R^t(x,y) := (d-\alpha)\kappa \int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|\cdot|_\varepsilon^{-\alpha} e^{-(t-\tau)A}(\cdot,y)\rangle d\tau.$$

Claim 3. *For every $D \geq Rm$ and all $|y| > Dt^{\frac{1}{\alpha}}$, $x \in \mathbb{R}^d$, we have*

$$K_R^t(x,y) \leq -\frac{1}{2}M_R^t(x,y).$$

Proof of Claim 3. Using Lemma 7(i), we obtain

$$\begin{aligned} K_R^t(x,y) &\equiv \int_0^t \big\langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)B_{\varepsilon,R}^t(\cdot)e^{-(t-\tau)A}(\cdot,y)\big\rangle d\tau \\ &\leq k_1 \int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|b_\varepsilon(\cdot)|E^{t-\tau}(\cdot,y)\rangle d\tau \\ &- \int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)W_\varepsilon(\cdot)e^{-(t-\tau)A}(\cdot,y)\rangle d\tau =: I_1 + I_2, \end{aligned}$$

where $|b_\varepsilon(x)| = \kappa|x|_\varepsilon^{-\alpha}|x|$.

Using $E^{t-\tau}(z,y) \leq k_0 e^{-(t-\tau)A}(z,y)|z-y|^{-1}$, we obtain

$$\begin{aligned} I_1 &\leq k_0k_1 \int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|b_\varepsilon(\cdot)|e^{-(t-\tau)A}(\cdot,y)|\cdot-y|^{-1}\rangle d\tau \\ &\quad (\text{we are using } \mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|b_\varepsilon(\cdot)||\cdot-y|^{-1} \leq \mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)R(D-R)^{-1}\kappa|\cdot|_\varepsilon^{-\alpha}) \\ &\leq k_0k_1R(D-R)^{-1}\kappa \int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|\cdot|_\varepsilon^{-\alpha}e^{-(t-\tau)A}(\cdot,y)\rangle d\tau \\ &= k_0k_1R(D-R)^{-1}(d-\alpha)^{-1}M_R^t(x,y). \end{aligned}$$

We now compare the RHS of the last estimate with I_2. Since $W_\varepsilon(\cdot) \geq \kappa(d-\alpha)|\cdot|_\varepsilon^{-\alpha}$, we have

$$K_R^t(x,y) \leq \big(k_0k_1R(D-R)^{-1}(d-\alpha)^{-1} - 1\big)M_R^t(x,y).$$

Since $k_0k_1R(D-R)^{-1} \leq \frac{k_0k_1}{m-1} \leq \frac{1}{2}$ and $d-\alpha > 1$ by our assumptions, we end the proof of Claim 3. □

Claim 4. *For every $D \geq Rm$ and all $|y| > Dt^{\frac{1}{\alpha}}$, $x \in \mathbb{R}^d$, we have*

$$K_R^{t,c}(x,y) \leq \delta(M_R^t(x,y) + e^{-tA}(x,y)).$$

Proof of Claim 4. Recall that

$$K_R^{t,c}(x,y) \equiv \int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)B_{\varepsilon,R}^{t,c}(\cdot)e^{-(t-\tau)A}(\cdot,y)\rangle d\tau,$$

where $B_{\varepsilon,R}^{t,c} = \mathbf{1}_{B^c(0,Rt^{\frac{1}{\alpha}})}(-b_\varepsilon\cdot\nabla - W_\varepsilon)$. Thus, discarding in $K_R^{t,c}$ the term containing $-W_\varepsilon$ and using Lemma 7(i), we obtain

$$K_R^{t,c}(x,y) \leq k_1\kappa R^{1-\alpha}t^{-\frac{\alpha-1}{\alpha}} \int_0^t \big\langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)E^{t-\tau}(\cdot,y)\big\rangle d\tau. \tag{$*$}$$

We will have to estimate the integral in the RHS of $(*)$.

By the Duhamel formula

$$\begin{aligned}
&\int_0^t \big(e^{-\tau(\Lambda^\varepsilon)^*}E^{t-\tau}\big)(x,y)d\tau \\
&=\int_0^t \big(e^{-\tau A}E^{t-\tau}\big)(x,y)d\tau + \int_0^t\int_0^\tau \big(e^{-\tau'(\Lambda^\varepsilon)^*}(B^t_{\varepsilon,R}+B^{t,c}_{\varepsilon,R})e^{-(\tau-\tau')A}d\tau' E^{t-\tau}\big)(x,y)d\tau \\
&\equiv \int_0^t \big(e^{-\tau A}E^{t-\tau}\big)(x,y)d\tau + J_R(x,y)+J_R^c(x,y),
\end{aligned}$$

where, by Lemma 7(ii), $\int_0^t\langle\big(e^{-\tau A}(x,\cdot)E^{t-\tau}(\cdot,y)\rangle\big)(x,y)d\tau \le k_2 t^{\frac{\alpha-1}{\alpha}}e^{-tA}(x,y)$. Let us estimate $J_R(x,y)$ and $J_R^c(x,y)$.

In $J_R(x,y)$, discarding the term containing $-W_\varepsilon$ and applying Lemma 7(i), we obtain

$$\begin{aligned}
J_R(x,y) &\le k_1\int_0^t\int_0^\tau \big(e^{-\tau'(\Lambda^\varepsilon)^*}\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b_\varepsilon|E^{\tau-\tau'}d\tau' E^{t-\tau}\big)(x,y)d\tau \\
&\qquad \text{(we are changing the order of integration and applying Lemma 7}(iii)) \\
&\le k_1k_3\int_0^t \big(e^{-\tau'(\Lambda^\varepsilon)^*}\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b_\varepsilon|(t-\tau')^{\frac{\alpha-1}{\alpha}}E^{t-\tau'}\big)(x,y)d\tau' \\
&\le k_1k_3 t^{\frac{\alpha-1}{\alpha}}\int_0^t \big(e^{-\tau'(\Lambda^\varepsilon)^*}\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b_\varepsilon|E^{t-\tau'}\big)(x,y)d\tau'.
\end{aligned}$$

Now, repeating the corresponding argument in the proof of Claim 3, we obtain

$$J_R(x,y)\le C_2 t^{\frac{\alpha-1}{\alpha}}M_R^t(x,y), \quad C_2=k_0k_1k_3R(D-R)^{-1}(d-\alpha)^{-1}\le \frac{k_3}{2}.$$

($C_2 \le \frac{k_0k_1k_3}{m-1}(d-\alpha)^{-1}\le \frac{k_3}{2}(d-\alpha)^{-1}\le\frac{k_3}{2}$.)

In turn, $J_R^c=\int_0^t (J_R^c)^\tau E^{t-\tau}d\tau$, where

$$(J_R^c)^\tau := \int_0^\tau e^{-\tau'(\Lambda^\varepsilon)^*}B^c_{\varepsilon,R}e^{-(\tau-\tau')A}d\tau'.$$

Again, discarding the $-W_\varepsilon$ term in $B^c_{\varepsilon,R}$ and applying Lemma 7(i), we obtain

$$|(J_R^c)^\tau(x,y)|\le \kappa k_1R^{1-\alpha}t^{-\frac{\alpha-1}{\alpha}}\int_0^\tau \big(e^{-\tau'(\Lambda^\varepsilon)^*}E^{\tau-\tau'}\big)(x,y)d\tau'.$$

Due to Lemma 7(iii),

$$\begin{aligned}
|J_R^c(x,y)| &\le \kappa k_1k_3R^{1-\alpha}t^{-\frac{\alpha-1}{\alpha}}\int_0^t\langle e^{-\tau'(\Lambda^\varepsilon)^*}(x,\cdot)(t-\tau')^{\frac{\alpha-1}{\alpha}}E^{t-\tau'}(\cdot,y)\rangle d\tau' \\
&\le \kappa k_1k_3R^{1-\alpha}\int_0^t\langle e^{-\tau'(\Lambda^\varepsilon)^*}(x,\cdot)E^{t-\tau'}(\cdot,y)\rangle d\tau'.
\end{aligned}$$

Thus, due to $\kappa k_1k_3R^{1-\alpha}\le\delta<\frac{1}{2}$,

$$\begin{aligned}
&\int_0^t\langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)E^{t-\tau}(\cdot,y)\rangle d\tau \\
&\le k_2t^{\frac{\alpha-1}{\alpha}}e^{-tA}(x,y)+\frac{k_3}{2}t^{\frac{\alpha-1}{\alpha}}M_R^t(x,y)+\frac{1}{2}\int_0^t\langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)E^{t-\tau}(\cdot,y)\rangle d\tau.
\end{aligned}$$

Thus, we obtain $\int_0^t \langle e^{-\tau(\Lambda^\varepsilon)^*}(x,\cdot)E^{t-\tau}(\cdot,y)\rangle d\tau \leq 2k_2 t^{\frac{\alpha-1}{\alpha}} e^{-tA}(x,y) + k_3 t^{\frac{\alpha-1}{\alpha}} M_R^t(x,y)$. Substituting the latter in $(*)$, we obtain Claim 4. □

Now, applying Claim 3 and Claim 4 in (5), we have

$$\begin{aligned} e^{-t(\Lambda^\varepsilon)^*}(x,y) &\leq e^{-tA}(x,y) - \frac{1}{2}M_R^t(x,y) + \delta(M_R^t(x,y) + e^{-tA}(x,y)) \\ &\leq (1+\delta)e^{-tA}(x,y), \end{aligned}$$

thus ending the proof of Step 2.

Step 3: *Set $R = 1 \vee (2\kappa k_3)^{\frac{1}{\alpha-1}}$ and let $D \geq 2R$. Then there is a constant $C = C(d,\alpha,\kappa,R)$ such that the following bound*

$$e^{-t(\Lambda^\varepsilon)^*}(x,y) \leq Ce^{-tA}(x,y), \quad |x| > 2Dt^{\frac{1}{\alpha}}, \quad |y| \leq Dt^{\frac{1}{\alpha}}, \quad t > 0.$$

is valid

(See the proof below for explicit formula for $C(d,\alpha,\kappa,R.)$

Using the Duhamel formula and applying Lemma $7(i)$, we have

$$\begin{aligned} e^{-t(\Lambda^\varepsilon)^*}(x,y) &\leq e^{-tA}(x,y) + k_1\int_0^t \big(E^\tau |b_\varepsilon| e^{-(t-\tau)(\Lambda^\varepsilon)^*}\big)(x,y)d\tau \\ &\leq e^{-tA}(x,y) + k_1 L^t_{\varepsilon,R}(x,y) + k_1 L^{t,c}_{\varepsilon,R}(x,y). \end{aligned} \tag{6}$$

where

$$L^t_{\varepsilon,R}(x,y) := \int_0^t \big(E^\tau \mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b_\varepsilon| e^{-(t-\tau)(\Lambda^\varepsilon)^*}\big)(x,y)d\tau,$$

$$L^{t,c}_{\varepsilon,R}(x,y) := \int_0^t \big(E^\tau \mathbf{1}_{B^c(0,Rt^{\frac{1}{\alpha}})}|b_\varepsilon| e^{-(t-\tau)(\Lambda^\varepsilon)^*}\big)(x,y)d\tau.$$

Let us estimate $L^t_{\varepsilon,R}(x,y)$. Recalling that $E^t(x,z) = t\big(|x-z|^{-d-\alpha-1} \wedge t^{-\frac{d+\alpha+1}{\alpha}}\big)$ and taking into account that $|x| \geq 2Dt^{\frac{1}{\alpha}}$, $|z| \leq Rt^{\frac{1}{\alpha}}$, we obtain $E^\tau(x,z) \leq t|x-z|^{-d-\alpha-1} \leq t|x-z|^{-d-\alpha}(3R)^{-1}t^{-\frac{1}{\alpha}}$.

Therefore,

$$
\begin{aligned}
L^t_{\varepsilon,R}(x,y) &\leq (3R)^{-1}t^{-\frac{1}{\alpha}}\int_0^t \langle t|x-\cdot|^{-\alpha-d}\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|b_\varepsilon(\cdot)|e^{-(t-\tau)(\Lambda^\varepsilon)^*}(\cdot,y)\rangle d\tau \\
&\quad \text{(we are using that } |x|>2Dt^{\frac{1}{\alpha}},\ |\cdot|\leq Rt^{\frac{1}{\alpha}}) \\
&\leq (3R)^{-1}(4/3)^{d+\alpha}t^{-\frac{1}{\alpha}}t|x|^{-\alpha-d}\int_0^t \langle \mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|b_\varepsilon(\cdot)|e^{-(t-\tau)(\Lambda^\varepsilon)^*}(\cdot,y)\rangle d\tau \\
&\quad \text{(we are using that } |y|\leq Dt^{\frac{1}{\alpha}},\ D\geq 2R \text{ and setting } c=3^{-1}(16/9)^{d+\alpha}) \\
&\leq cR^{-1}t^{-\frac{1}{\alpha}}t|x-y|^{-\alpha-d}\int_0^t \langle \mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}(\cdot)|b_\varepsilon(\cdot)|e^{-(t-\tau)(\Lambda^\varepsilon)^*}(\cdot,y)\rangle d\tau \\
&\quad \text{(we are using } t|x-y|^{-\alpha-d}=t(|x-y|^{-\alpha-d}\wedge t^{-\frac{d+\alpha}{\alpha}}) \\
&\quad \text{since } |x-y|^{-\alpha-d}\leq (2R)^{-d-\alpha}t^{-\frac{d+\alpha}{\alpha}}<t^{-\frac{d+\alpha}{\alpha}}\text{, and are re-denoting } t-\tau \text{ by } \tau) \\
&\leq k_0cR^{-1}t^{-\frac{1}{\alpha}}e^{-tA}(x,y)\int_0^t \|e^{-\tau\Lambda^\varepsilon}\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b|\|_\infty d\tau \\
&\quad \text{(we are applying Proposition 8)} \\
&\leq k_0cR^{-1}t^{-\frac{1}{\alpha}}e^{-tA}(x,y)c_N\int_0^t \tau^{-\frac{d}{\alpha p}}d\tau\,\|\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b|\|_p \qquad \big(p=\frac{d}{\alpha-\frac{1}{2}}\big).
\end{aligned}
$$

Since $\int_0^t \tau^{-\frac{d}{\alpha p}}d\tau = 2\alpha t^{\frac{1}{2\alpha}}$ and $\|\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b|\|_p = \kappa R^{\frac{1}{2}}t^{\frac{1}{2\alpha}}\tilde{c}$, $\tilde{c}=\tilde{c}(d)<\infty$, we have

$$L^t_{\varepsilon,R}(x,y)\leq C'R^{-\frac{1}{2}}e^{-tA}(x,y), \quad C'=2\kappa\alpha k_0cc_N\tilde{c}$$

or, for convenience,

$$L^t_{\varepsilon,R}(x,y)\leq C'e^{-tA}(x,y). \tag{7}$$

In turn, clearly,

$$L^{t,c}_{\varepsilon,R}(x,y)\leq \kappa R^{1-\alpha}t^{-\frac{\alpha-1}{\alpha}}\int_0^t E^\tau e^{-(t-\tau)(\Lambda^\varepsilon)^*}d\tau.$$

Let us estimate the integral in the RHS. Using the Duhamel formula, we obtain

$$\int_0^t \big(E^\tau e^{-(t-\tau)(\Lambda^\varepsilon)^*}\big)(x,y)d\tau$$

$$\leq \int_0^t \big(E^\tau e^{-(t-\tau)A}\big)(x,y)d\tau + \int_0^t \big(E^\tau \int_0^{t-\tau} E^{t-\tau-s}|b_\varepsilon|e^{-s(\Lambda^\varepsilon)^*}ds\big)(x,y)d\tau$$

(we are applying Lemma 7(ii) and changing the order of integration)

$$\leq k_2 t^{\frac{\alpha-1}{\alpha}}e^{-tA}(x,y) + \int_0^t \int_0^{t-s} \big(E^\tau E^{t-s-\tau}|b_\varepsilon|e^{-s(\Lambda^\varepsilon)^*}\big)(x,y)d\tau ds$$

(we are applying Lemma 7(iii))

$$\leq k_2 t^{\frac{\alpha-1}{\alpha}}e^{-tA}(x,y) + k_3\int_0^t (t-s)^{\frac{\alpha-1}{\alpha}}\big(E^{t-s}|b_\varepsilon|e^{-s(\Lambda^\varepsilon)^*}\big)(x,y)ds$$

$$\leq k_2 t^{\frac{\alpha-1}{\alpha}}e^{-tA}(x,y) + k_3 t^{\frac{\alpha-1}{\alpha}}\int_0^t \big(E^{t-s}\mathbf{1}_{B(0,Rt^{\frac{1}{\alpha}})}|b_\varepsilon|e^{-s(\Lambda^\varepsilon)^*}\big)(x,y)d\tau ds$$

$$+ k_3 t^{\frac{\alpha-1}{\alpha}}\int_0^t \big(E^{t-s}\mathbf{1}_{B^c(0,Rt^{\frac{1}{\alpha}})}|b|e^{-s(\Lambda^\varepsilon)^*}\big)(x,y)ds$$

$$\leq k_2 t^{\frac{\alpha-1}{\alpha}}e^{-tA}(x,y) + k_3 t^{\frac{\alpha-1}{\alpha}}L^t_{\varepsilon,R}(x,y) + k_3\kappa R^{1-\alpha}\int_0^t \big(E^{t-s}e^{-s(\Lambda^\varepsilon)^*}\big)(x,y)ds$$

(we are applying (7) to the second term, and note that $k_3\kappa R^{1-\alpha}\leq \dfrac{1}{2}$)

$$\leq (k_2+k_3C')t^{\frac{\alpha-1}{\alpha}}e^{-tA}(x,y) + \frac{1}{2}\int_0^t \big(E^{t-s}e^{-s(\Lambda^\varepsilon)^*}\big)(x,y)ds.$$

Therefore,

$$\int_0^t E^\tau\big(e^{-(t-\tau)(\Lambda^\varepsilon)*}\big)(x,y)d\tau \leq 2(k_2+k_3C')t^{\frac{\alpha-1}{\alpha}}e^{-tA}(x,y),$$

and so

$$L^{c,t}_{\varepsilon,R}(x,y) \leq 2\kappa(k_2+k_3C')R^{1-\alpha}e^{-tA}(x,y). \tag{8}$$

Applying (7) and (8) in (6), we obtain the desired bound

$$e^{-t(\Lambda^\varepsilon)^*}(x,y) \leq Ce^{-tA}(x,y), \quad |x|>2Dt^{\frac{1}{\alpha}}, \quad |y|\leq Dt^{\frac{1}{\alpha}},$$

for all $R>1$ such that $k_3\kappa R^{1-\alpha}\leq \frac{1}{2}$, $D\geq 2R$, where $C:=1+k_1C'+k_1 2\kappa(k_2+k_3C')R^{1-\alpha}$. The assertion of Step 3 follows.

We are in position to complete the proof of Theorem 3(i), i.e. to prove the bound

$$e^{-t(\Lambda^\varepsilon)^*}(x,y) \leq C_1 e^{-tA}(x,y), \quad x,y\in\mathbb{R}^d, \quad t>0, \tag{9}$$

for appropriate constant $C_1 = C_1(d,\alpha,\kappa)$.

To prove (9), we combine Steps 1-3 as follows. *Fix* D large enough so that the assertions of both Step 2 and Step 3 hold.

Without loss of generality, the assertion of Step 3 holds for all $|x|>Dt^{\frac{1}{\alpha}}$, $|y|\leq Dt^{\frac{1}{\alpha}}$ (indeed, by Step 1, (9) is true for all $|x|\leq 2Dt^{\frac{1}{\alpha}}$, $|y|\leq 2Dt^{\frac{1}{\alpha}}$ (with $C_1 = C_0'(4D)^{d+\alpha}$) and so, in particular, for all $Dt^{\frac{1}{\alpha}} < |x| \leq 2Dt^{\frac{1}{\alpha}}$, $|y|\leq Dt^{\frac{1}{\alpha}}$; the rest follows from the assertion of Step 3 as stated). Thus, the desired bound (9) is true for all $|x|>Dt^{\frac{1}{\alpha}}$, $|y|\leq Dt^{\frac{1}{\alpha}}$ and, by Step 2, for all $x\in\mathbb{R}^d$, $|y|>Dt^{\frac{1}{\alpha}}$.

It remains to prove (9) in the case $|x| \le Dt^{\frac{1}{\alpha}}$, $|y| \le Dt^{\frac{1}{\alpha}}$. But this is the assertion of Step 1.

Thus, (9) is true, with constant C_1 equal to the maximum of the constants in Step 1 (with $2D$ in place of D) and in Steps 2, 3.

(ii) The result follows immediately from Step 2 in the proof of (i) upon taking $\varepsilon \downarrow 0$ (cf. Proposition 12).

The proof of Theorem 3 is completed. □

6. Proof of Theorem 4: The weighted upper bound

Recall $A \equiv (-\Delta)^{\frac{\alpha}{2}}$. We are going to prove that there is a constant $C < \infty$ such that

$$e^{-t\Lambda}(x,y) \le Ce^{-tA}(x,y)\psi_t(y), \quad t>0, \quad x,y \in \mathbb{R}^d. \tag{10}$$

Clearly, Theorem 2 and Theorem 3(i) combined, yield

$$e^{-t\Lambda}(x,y) \le C_1 c_{N,w}\biggl(e^{-tA}(x,y) \wedge \bigl(t^{-\frac{d}{\alpha}}\psi_t(y)\bigr)\biggr), \quad t>0, \quad x,y \in \mathbb{R}^d. \tag{11}$$

1. If $|y| \ge t^{\frac{1}{\alpha}}$, then $\psi_t(y) \ge 1$. Then, by (11),

$$e^{-t\Lambda}(x,y) \le C_1 c_{N,w} e^{-tA}(x,y) \le C_1 c_{N,w} e^{-tA}(x,y)\psi_t(y),$$

i.e. (10) holds.

2. If $|x| \le Dt^{\frac{1}{\alpha}}$, $|y| < t^{\frac{1}{\alpha}}$ for some constant $D>1$, then by (11) (cf. Lemma 6(i))

$$e^{-t\Lambda}(x,y) \le C_1 c_{N,w} t^{-\frac{d}{\alpha}}\psi_t(y) \le C_1 c_{N,w} k_0^{-1}(D+1)^{d+\alpha} e^{-tA}(x,y)\psi_t(y),$$

i.e. (10) holds.

3. It remains therefore to consider the case $|x| > Dt^{\frac{1}{\alpha}}$, $|y| < t^{\frac{1}{\alpha}}$.

By duality (cf. Proposition 12), it suffices to prove the estimate

$$e^{-t\Lambda^*}(x,y) \le Ce^{-tA}(x,y)\psi_t(x) \tag{12}$$

for all $|x| < t^{\frac{1}{\alpha}}$, $|y| > Dt^{\frac{1}{\alpha}}$, $t>0$, for some $D>1$.

We will use Corollary 2,

$$\langle e^{-t\Lambda^*}(x,\cdot)\rangle \le C_2\psi_t(x) \quad \text{for all } x \in \mathbb{R}^d, \quad t>0,$$

the "standard" upper bound (Theorem 3(i))

$$e^{-t\Lambda^*}(x,y) \le C_1 e^{-tA}(x,y), \quad \text{for all } x,y \in \mathbb{R}^d, \quad t>0,$$

and its partial improvement (Theorem 3(ii)): For every $\delta > 0$ there exists a sufficiently large D such that for all $|x| < t^{\frac{1}{\alpha}}$, $|y| > Dt^{\frac{1}{\alpha}}$ and all $z \in B(y, \frac{|y-x|}{2})$

$$e^{-t\Lambda^*}(x,z) \le C_\delta e^{-tA}(x,z), \qquad e^{-t\Lambda^*}(z,y) \le C_\delta e^{-tA}(z,y), \qquad C_\delta := 1+\delta. \tag{13}$$

We will need the following elementary inequality:

$$2\left\langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot)e^{-\frac{t}{2}A}(x,\cdot)e^{-\frac{t}{2}A}(\cdot,y)\right\rangle \le e^{-tA}(x,y). \tag{14}$$

Indeed, by symmetry, the LHS of (14) coincides with

$$\begin{aligned}\big\langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot)e^{-\frac{t}{2}A}(x,\cdot)e^{-\frac{t}{2}A}(\cdot,y)\big\rangle + \big\langle \mathbf{1}_{B(x,\frac{|x-y|}{2})}(\cdot)e^{-\frac{t}{2}A}(x,\cdot)e^{-\frac{t}{2}A}(\cdot,y)\big\rangle \\ \leq \langle e^{-\frac{t}{2}A}(x,\cdot)e^{-\frac{t}{2}A}(\cdot,y)\rangle = e^{-tA}(x,y),\end{aligned}$$

i.e. (14) follows.

Proposition 3. (i) *There exists a constant* c_5 *such that*

$$e^{-t\Lambda^*}(x,y) \leq \big\langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot)e^{-\frac{t}{2}\Lambda^*}(x,\cdot)e^{-\frac{t}{2}\Lambda^*}(\cdot,y)\big\rangle + c_5 e^{-tA}(x,y)\psi_t(x)$$

(ii) *If* $|x| < t^{\frac{1}{\alpha}}$, $|y| > Dt^{\frac{1}{\alpha}}$ *with* $D > 1$ *sufficiently large, then*

$$e^{-t\Lambda^*}(x,y) \leq \bigg(\frac{C_\delta^2}{2} + c_5\psi_t(x)\bigg)e^{-tA}(x,y).$$

Proof. We have

$$\begin{aligned}e^{-t\Lambda^*}(x,y) &= \big\langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot)e^{-\frac{t}{2}\Lambda^*}(x,\cdot)e^{-\frac{t}{2}\Lambda^*}(\cdot,y)\big\rangle + \big\langle \mathbf{1}_{B^c(y,\frac{|x-y|}{2})}e^{-\frac{t}{2}\Lambda^*}(x,\cdot)e^{-\frac{t}{2}\Lambda^*}(\cdot,y)\big\rangle \\ &=: J_1 + J_2.\end{aligned}$$

(i) For $z \in B^c(y,\frac{|x-y|}{2})$, $e^{-\frac{t}{2}\Lambda^*}(z,y) \leq C_1 e^{-\frac{t}{2}A}(z,y) \leq k_1 e^{-tA}(x,y)$. Thus,

$$\begin{aligned}J_2 &\leq k_1 e^{-tA}(x,y)\big\langle \mathbf{1}_{B^c(y,\frac{|x-y|}{2})}(\cdot)e^{-\frac{t}{2}\Lambda^*}(x,\cdot)\big\rangle \\ &\quad \text{(we are applying Corollary 2)} \\ &\leq k_1 C_2 e^{-tA}(x,y)\psi_{\frac{t}{2}}(x) \leq c_5 e^{-tA}(x,y)\psi_t(x),\end{aligned}$$

and so (i) follows.

(ii) Using (i), it remains to estimate J_1. Applying (13), we have

$$J_1 \leq C_\delta^2 \big\langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot)e^{-\frac{t}{2}A}(x,\cdot)e^{-\frac{t}{2}A}(\cdot,y)\big\rangle$$

Finally, we use (14). □

Let us complete the proof of Theorem 4.

By Proposition 3(ii),

$$e^{-t\Lambda^*}(x,y) \leq \bigg(\frac{C_\delta^2}{2} + c_5\psi_t(x)\bigg)e^{-tA}(x,y).$$

Set $\nu := \frac{C_\delta}{2}2^{\frac{\beta}{\alpha}}$, so that $\frac{C_\delta}{2}\psi_{t/2} = \nu\psi_t$. Fix $\delta \in \big]0, (\sqrt{2}-1) \wedge (2^{1-\frac{\alpha}{\beta}}-1)\big[$. Then $\frac{C_\delta^2}{2} < 1$ and $\nu < 1$. Now, suppose that, for $n = 2, 3, \dots$,

$$e^{-t\Lambda^*}(x,y) \leq \bigg(\frac{C_\delta^{n+1}}{2^n} + c_5(1+\nu+\dots+\nu^{n-1})\psi_t(x)\bigg)e^{-tA}(x,y), \tag{15}$$

Then, using Proposition 3(i), we have

$$\begin{aligned}
e^{-t\Lambda^*}(x,y) &\leq \langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot)e^{-\frac{t}{2}\Lambda^*}(x,\cdot)C_\delta e^{-\frac{t}{2}A}(\cdot,y)\rangle + c_5 e^{-tA}(x,y)\psi_t(x) \\
&\leq \Big\langle \mathbf{1}_{B(y,\frac{|x-y|}{2})}(\cdot)C_\delta\bigg(\frac{C_\delta^{n+1}}{2^n} + c_5(1+\nu+\dots+\nu^{n-1})\psi_{\frac{t}{2}}(x)\bigg)e^{-\frac{t}{2}A}(x,\cdot)e^{-\frac{t}{2}A}(\cdot,y)\Big\rangle \\
&\quad + c_5 e^{-tA}(x,y)\psi_t(x) \\
&\quad \text{(we are applying (14))} \\
&\leq \bigg(\frac{C_\delta^{n+2}}{2^{n+1}} + c_5(\nu+\nu^2+\dots+\nu^n)\psi_t(x)\bigg)e^{-tA}(x,y) + c_5 e^{-tA}(x,y)\psi_t(x) \\
&= \bigg(\frac{C_\delta^{n+2}}{2^{n+1}} + c_5(1+\nu+\nu^2+\dots+\nu^n)\psi_t(x)\bigg)e^{-tA}(x,y).
\end{aligned}$$

Thus by induction, (15) holds for $n+1$. Sending $n \to \infty$ there, we obtain

$$e^{-t\Lambda^*}(x,y) \leq c_5(1-\nu)^{-1}e^{-tA}(x,y)\psi_t(x),$$

as needed. The proof of (12) is completed. The proof of Theorem 4 is completed.

7. Proof of Theorem 5: The weighted lower bound

Recall that

$$k_0^{-1}t\big(|x-y|^{-d-\alpha} \wedge t^{-\frac{d+\alpha}{\alpha}}\big) \leq e^{-tA}(x,y) \leq k_0 t\big(|x-y|^{-d-\alpha} \wedge t^{-\frac{d+\alpha}{\alpha}}\big) \tag{16}$$

for all $x,y \in \mathbb{R}^d$, $x \neq y$, $t>0$, for a constant $k_0 = k_0(d,\alpha) > 1$.

1. First, we prove the "standard" lower bound away from the origin.

Lemma 8. *There exists a generic constant $0<\gamma<\frac{1}{2}$ such that, for all $r \geq \gamma^{-2}$ and $t>0$,*

$$e^{-t\Lambda^*}(x,y) \geq \frac{1}{2}e^{-tA}(x,y)$$

whenever $|x| \geq rt^{\frac{1}{\alpha}}$, $|y| \geq rt^{\frac{1}{\alpha}}$.

Proof. In view of Proposition 10 it suffices to prove the inequality $e^{-t(\Lambda^\varepsilon)^*}(x,y) \geq \frac{1}{2}e^{-tA}(x,y)$.

By the Duhamel formula,

$$e^{-t(\Lambda^\varepsilon)^*}(x,y) \geq e^{-tA}(x,y) - |M_t(x,y)|, \qquad M_t(x,y) := \int_0^t e^{-(t-\tau)A}\nabla\cdot b_\varepsilon\, e^{-\tau(\Lambda^\varepsilon)^*}d\tau.$$

Using Lemma 7(i), we have

$$\begin{aligned}
|M_t(x,y)| &\leq k_1\kappa\int_0^t \langle E^{t-\tau}(x,\cdot)|\cdot|^{-\alpha+1}e^{-\tau(\Lambda^\varepsilon)^*}(\cdot,y)\rangle d\tau \\
&\quad \text{(we are using Theorem 3(i) – the standard upper bound)} \\
&\leq k_1\kappa C_1\int_0^t \langle E^{t-\tau}(x,\cdot)|\cdot|^{-\alpha+1}e^{-\tau A}(\cdot,y)\rangle d\tau.
\end{aligned}$$

Set

$$J(\mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}(|\cdot|^{1-\alpha}) := \int_0^t \langle \mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}(\cdot) E^{t-\tau}(x,\cdot)|\cdot|^{-\alpha+1} e^{-\tau A}(\cdot,y)\rangle d\tau,$$
$$J(\mathbf{1}_{B^c(0,\gamma r t^{\frac{1}{\alpha}})}(|\cdot|^{1-\alpha}) := \int_0^t \langle \mathbf{1}_{B^c(0,\gamma r t^{\frac{1}{\alpha}})}(\cdot) E^{t-\tau}(x,\cdot)|\cdot|^{-\alpha+1} e^{-\tau A}(\cdot,y)\rangle d\tau,$$

where $0<\gamma<2^{-1}$.

Note that if $|x| \geq r t^{\frac{1}{\alpha}}$,then

$$E^{t-\tau}(x,z) \leq C_5 e^{-(t-\tau)A}(x,z)|x-z|^{-1} \leq C_5 2 r^{-1} t^{-\frac{1}{\alpha}} e^{-(t-\tau)A}(x,z) \quad z \in B(0,\gamma r t^{\frac{1}{\alpha}}).$$

Thus, using the inequality

$$e^{-tA}(x,z)e^{-sA}(z,y) \leq K e^{-(t+s)A}(x,y)\big(e^{-tA}(x,z)+e^{-sA}(z,y)\big), \tag{17}$$

which holds for a constant $K=K(d,\alpha)$, all $x,z,y\in\mathbb{R}^d$ and $t,s>0$ (see e.g. [BJ]), we have

$$J(\mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}|\cdot|^{1-\alpha}) \leq C_5 2 r^{-1} t^{-\frac{1}{\alpha}} K e^{-tA}(x,y) \int_0^t \langle \mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}(\cdot)|\cdot|^{1-\alpha}(e^{-(t-\tau)A}(x,\cdot)+e^{-\tau A}(\cdot,y))\rangle d\tau.$$

Next, for all $0<\tau<t$, $|x| \geq r t^{\frac{1}{\alpha}}$, $|y| \geq r t^{\frac{1}{\alpha}}$,

$$\mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}(\cdot)e^{-\tau A}(\cdot,y) \leq C_6 t^{-\frac{d}{\alpha}} r^{-d-\alpha} \quad \text{if } (1-\gamma)r>1,$$
$$\mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}(\cdot)e^{-(t-\tau)A}(x,\cdot) \leq C_7 t^{-\frac{d}{\alpha}} r^{-d-\alpha}, \quad \text{if } (1-\gamma)r>1,$$

and so

$$\begin{aligned} J(\mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}|\cdot|^{1-\alpha}) &\leq C_8 t^{-\frac{d+1}{\alpha}} r^{-d-\alpha-1} e^{-tA}(x,y) \int_0^t \langle \mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}(\cdot)|\cdot|^{1-\alpha}\rangle d\tau \\ &\leq C_9 r^{-2\alpha}\gamma^{d-\alpha+1} e^{-tA}(x,y) \\ &\leq C_9 2^{2\alpha}\gamma^{d-\alpha+1} e^{-tA}(x,y) \quad \text{if } r>(1-\gamma)^{-1}. \end{aligned}$$

Therefore,

$$J(\mathbf{1}_{B(0,\gamma r t^{\frac{1}{\alpha}})}|\cdot|^{1-\alpha}) \leq C_{10}\gamma^{d-\alpha+1} e^{-tA}(x,y) \quad \text{if} \quad r>(1-\gamma)^{-1}, \quad 0<\gamma<2^{-1}. \tag{$*$}$$

In turn,

$$J(\mathbf{1}_{B^c(0,\gamma r t^{\frac{1}{\alpha}})}|\cdot|^{1-\alpha}) \leq \frac{c_1 C}{2} C_0 (\gamma r t^{\frac{1}{\alpha}})^{1-\alpha} t^{1-\frac{1}{\alpha}} e^{-tA}(x,y) = C_{11}(\gamma r)^{1-\alpha} e^{-tA}(x,y)$$

as follows immediately from Lemma 7(ii):

$$\int_0^t \langle e^{-(t-\tau)A}(x,\cdot)E^\tau(\cdot,y)\rangle d\tau \leq C_0 t^{1-\frac{1}{\alpha}} e^{-tA}(x,y).$$

Thus, if $r \geq \gamma^{-2}$, then

$$J(\mathbf{1}_{B^c(0,\gamma r t^{\frac{1}{\alpha}})}|\cdot|^{1-\alpha}) \leq C_{11}\gamma^{1-\alpha} e^{-tA}(x,y). \tag{$**$}$$

Finally, selecting $\gamma>0$ sufficiently small: $k_1 \kappa C(C_{10} \vee C_{11})\gamma^{\alpha-1} \leq \frac{1}{4}$, and using $(*)$, $(**)$, we have

$$|M_t(x,y)| \leq \frac{1}{2} e^{-tA}(x,y),$$

which ends the proof. □

Corollary 3. *For every $r>0$, there is a constant $c(r)>0$ such that*

$$e^{-t\Lambda^*}(x,y) \geq c(r) e^{-tA}(x,y)$$

whenever $|x| \geq r t^{\frac{1}{\alpha}}$, $|y| \geq r t^{\frac{1}{\alpha}}$, $t>0$.

Proof. In Lemma 8, fix some $r \geq \gamma^{-2}$, so that

$$e^{-t\Lambda^*}(x,y) \geq 2^{-1} e^{-tA}(x,y), \quad |x| \geq r t^{\frac{1}{\alpha}}, \quad |y| \geq r t^{\frac{1}{\alpha}}, \tag{18}$$

$$e^{-t\frac{1}{2}\Lambda^*}(x,y) \geq 2^{-1} e^{-\frac{t}{2}A}(x,y), \quad |x| \geq r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}, \quad |y| \geq r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}. \tag{19}$$

We now extend (18), by proving existence of a constant $0<c_1<2^{-1}$ such that

$$e^{-t\Lambda^*}(x,y) \geq c_1 e^{-tA}(x,y), \quad |x| \geq r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}, \quad |y| \geq r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}. \tag{18$'$}$$

Clearly, we need to consider only the case $rt^{\frac{1}{\alpha}} \geq |x| \geq r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}$, $r \geq |y| \geq r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}$. By the reproduction property,

$$\begin{aligned}
e^{-t\Lambda^*}(x,y) &\geq \langle e^{-\frac{1}{2}t\Lambda^*}(x,\cdot)\mathbf{1}_{B^c\left(0,r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}\right)}(\cdot) e^{-\frac{1}{2}t\Lambda^*}(\cdot,y)\rangle \\
&\text{(we are applying (19))} \\
&\geq 2^{-2}\langle e^{-\frac{1}{2}tA}(x,\cdot)\mathbf{1}_{B^c\left(0,r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}\right)}(\cdot) e^{-\frac{1}{2}tA}(\cdot,y)\rangle \\
&> 2^{-2}\langle e^{-\frac{1}{2}tA}(x,\cdot)\mathbf{1}_{B\left(0,(r+1)\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}\right)-B\left(0,r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}\right)}(\cdot) e^{-\frac{1}{2}tA}(\cdot,y)\rangle \\
&\text{(we are using the lower bound in (16))} \\
&\geq 2^{-2}\tilde{c}t^{-\frac{d}{\alpha}} \qquad (\tilde{c}=\tilde{c}(r)>0) \\
&\text{(we are using the upper bound in (16))} \\
&\geq c_1 e^{-tA}(x,y) \qquad \text{for appropriate } 0<c_1=c_1(r)<2^{-1},
\end{aligned}$$

i.e. we have proved (18$'$).

The same argument yields

$$e^{-\frac{1}{2}t\Lambda^*}(x,y) \geq c_1 e^{-\frac{1}{2}tA}(x,y), \quad |x| \geq r\left(\frac{t}{2^2}\right)^{\frac{1}{\alpha}}, \quad |y| \geq r\left(\frac{t}{2^2}\right)^{\frac{1}{\alpha}}. \tag{19$'$}$$

Thus, we can repeat the above procedure $m-1$ times obtaining

$$e^{-t\Lambda^*}(x,y) \geq c_m e^{-tA}(x,y), \quad |x| \geq r\left(\frac{t}{2^m}\right)^{\frac{1}{\alpha}}, \quad |y| \geq r\left(\frac{t}{2^m}\right)^{\frac{1}{\alpha}}$$

for appropriate $c_m>0$, from which the assertion of Corollary 3 follows. □

2. Next, in Proposition 4 we will prove an "integral lower bound". We need

Lemma 9. *For every* $0 \le h \in L^1$, $t > 0$

$$t^{-1}\int_0^t \|\psi_\tau h\|_1 d\tau \le \hat{C}\|\psi_t h\|_1$$

for a constant $\hat{C} = \hat{C}(\alpha, \beta)$.

Proof. Define $\psi_{0,t}(y) = \eta_0(t^{-\frac{1}{\alpha}}|y|)$, where

$$\eta_0(u) = \begin{cases} u^\beta, & 0 < u < 1, \\ 1, & u \ge 1. \end{cases}$$

Since $c^{-1}\psi_t \le \psi_{0,t} \le c\psi_t$, $c > 1$, it suffices to prove Lemma 9 for weight $\psi_{0,t}$.

For brevity, write $\psi_t := \psi_{0,t}$. We have

$$\|\psi_\tau h\|_1 = \langle \mathbf{1}_{B(0,\tau^{\frac{1}{\alpha}})}(\tau^{-\frac{1}{\alpha}}|x|)^\beta h\rangle + \langle \mathbf{1}_{B^c(0,\tau^{\frac{1}{\alpha}})}h\rangle,$$

and so

$$\int_0^t \|\psi_\tau h\|_1 d\tau = \langle\bigg(\int_0^t \mathbf{1}_{B(0,\tau^{\frac{1}{\alpha}})}\tau^{-\frac{\beta}{\alpha}}d\tau\bigg)|x|^\beta h\rangle + \langle\bigg(\int_0^t \mathbf{1}_{B^c(0,\tau^{\frac{1}{\alpha}})}d\tau\bigg)h\rangle.$$

If $|x| \le t^{\frac{1}{\alpha}}$, then

$$\int_0^t \mathbf{1}_{B(0,\tau^{\frac{1}{\alpha}})}(x)\tau^{-\frac{\beta}{\alpha}}d\tau = \int_{|x|^\alpha}^t \tau^{-\frac{\beta}{\alpha}}d\tau = \frac{1}{1-\frac{\beta}{\alpha}}(t^{-\frac{\beta}{\alpha}+1} - |x|^{-\beta+\alpha})$$

and

$$\int_0^t \mathbf{1}_{B^c(0,\tau^{\frac{1}{\alpha}})}(x)d\tau = \int_0^{|x|^\alpha} d\tau = |x|^\alpha.$$

If $|x| > t^{\frac{1}{\alpha}}$, then

$$\int_0^t \mathbf{1}_{B(0,\tau^{\frac{1}{\alpha}})}(x)\tau^{-\frac{\beta}{\alpha}}d\tau = 0, \qquad \int_0^t \mathbf{1}_{B^c(0,\tau^{\frac{1}{\alpha}})}(x)d\tau = t.$$

Thus,

$$\begin{aligned}\int_0^t \|\psi_\tau h\|_1 d\tau =& \langle \mathbf{1}_{B(0,t^{\frac{1}{\alpha}})}\frac{\alpha}{\alpha-\beta}(t^{-\frac{\beta}{\alpha}+1} - |x|^{-\beta+\alpha})|x|^\beta h\rangle + \langle \mathbf{1}_{B(0,t^{\frac{1}{\alpha}})}|x|^\alpha h\rangle + t\langle \mathbf{1}_{B^c(0,t^{\frac{1}{\alpha}})}h\rangle \\ =& t\frac{\alpha}{\alpha-\beta}\langle \mathbf{1}_{B(0,t^{\frac{1}{\alpha}})}\psi_t h\rangle - \frac{\beta}{\alpha-\beta}\langle \mathbf{1}_{B(0,t^{\frac{1}{\alpha}})}|x|^\alpha h\rangle + t\langle \mathbf{1}_{B^c(0,t^{\frac{1}{\alpha}})}\psi_t h\rangle \\ \le& t\frac{2\alpha-\beta}{\alpha-\beta}\langle \psi_t h\rangle.\end{aligned}$$

□

Proposition 4. *Define* $g_t = \psi_t h$, $0 \le h \in \mathcal{S}$*-the L. Schwartz space of test functions. Then, there exists generic constant* $\nu > 0$ *such that, for all* $t > 0$,

$$\langle \psi_t e^{-t\Lambda}\psi_t^{-1}g_t\rangle \ge \nu\langle g_t\rangle.$$

Proof. Recall that both $e^{-t\Lambda^\varepsilon}$, $e^{-t(\Lambda^\varepsilon)^*}$ are holomorphic in L^1 and C_u due to Hille's Perturbation Theorem. We have $\psi = \psi_{(1)} + \psi_{(u)}$, where

$$\psi_{(1)} \in D((-\Delta)_1^{\frac{\alpha}{2}})\ \big(= D((\Lambda^\varepsilon)_1^*) = D(\Lambda_1^\varepsilon)\big),$$

$$\psi_{(u)} \in D((-\Delta)_{C_u}^{\frac{\alpha}{2}})\ \big(= D((\Lambda^\varepsilon)_{C_u}^*) = D(\Lambda_{C_u}^\varepsilon)\big)$$

(see the proof of Proposition 2 for details), so $(\Lambda^\varepsilon)^*\psi \ \big(= \Lambda^\varepsilon)^*_{L^1}\psi_{(1)} + (\Lambda^\varepsilon)^*_{C_u}\psi_{(u)}\big)$ and belongs to $\in L^1 + C_u$.

Now, set $g_{s,n} = \phi_{s,n}h$, $\phi_{s,n}(x) = (e^{-\frac{(\Lambda^\varepsilon)^*}{n}}\psi_s)(x)$. We have, for $s > t > 0$,

$$\begin{aligned}\langle g_{s,n}\rangle - \langle \phi_{s,n}e^{-t\Lambda^\varepsilon}h\rangle &= \int_0^t \langle \psi_s, \Lambda^\varepsilon e^{-\tau\Lambda^\varepsilon}e^{-\frac{\Lambda^\varepsilon}{n}}h\rangle d\tau \\ &= \lim_{r\downarrow 0} r^{-1}\int_0^t \langle \psi_s, (1-e^{-r\Lambda^\varepsilon})e^{-\tau\Lambda^\varepsilon}e^{-\frac{\Lambda^\varepsilon}{n}}h\rangle d\tau \\ &= \lim_{r\downarrow 0} r^{-1}\int_0^t \langle (1-e^{-r(\Lambda^\varepsilon)^*})\psi_s, e^{-\tau\Lambda^\varepsilon}e^{-\frac{\Lambda^\varepsilon}{n}}h\rangle d\tau \\ &= \int_0^t \langle (\Lambda^\varepsilon)^*\psi_s, e^{-\tau\Lambda^\varepsilon}e^{-\frac{\Lambda^\varepsilon}{n}}h\rangle d\tau.\end{aligned}$$

Arguing as in the proof of Proposition 2, we represent

$$(\Lambda^\varepsilon)^*\psi_s = \mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}W_\varepsilon\psi_s + v_\varepsilon,$$

where $W_\varepsilon(x) = \kappa(|x|_\varepsilon^{-\alpha} - |x|^{-\alpha})\beta + \kappa\big[d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2 - (d-\alpha)|x|^{-\alpha}\big]$ and $0 \leq v_\varepsilon \in L^\infty$, $\|v_\varepsilon\|_\infty \leq \frac{c'}{s}$, $c' \neq c'(\varepsilon)$ (see Remark 7 below for detailed calculation).

Then

$$\langle g_{s,n}\rangle - \langle \phi_{s,n}e^{-t\Lambda^\varepsilon}h\rangle \leq \int_0^t \langle \mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}W_\varepsilon\psi_s, e^{-(\tau+\frac{1}{n})\Lambda^\varepsilon}h\rangle d\tau + \int_0^t \langle v_\varepsilon, e^{-\tau\Lambda^\varepsilon}e^{-\frac{\Lambda^\varepsilon}{n}}h\rangle d\tau$$

or, sending $n \to \infty$,

$$\begin{aligned}\langle g_s\rangle - \langle \psi_s e^{-t\Lambda^\varepsilon}h\rangle &\leq \int_0^t \langle \mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}W_\varepsilon\psi_s, e^{-\tau\Lambda^\varepsilon}h\rangle d\tau + \int_0^t \langle v_\varepsilon, e^{-\tau\Lambda^\varepsilon}h\rangle d\tau \\ &\leq \int_0^t \langle \mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}W_\varepsilon\psi_s, e^{-\tau\Lambda^\varepsilon}h\rangle d\tau + c's^{-1}\int_0^t \|e^{-\tau\Lambda^\varepsilon}h\|_1 d\tau.\end{aligned}$$

Next, we pass to the limit $\varepsilon \downarrow 0$:

$$\langle g_s\rangle - \langle \psi_s e^{-t\Lambda}h\rangle \leq c's^{-1}\int_0^t \|e^{-\tau\Lambda}h\|_1 d\tau. \tag{$\star$}$$

We estimate the RHS of $(\star)$ using the upper bound:

$$\begin{aligned}c's^{-1}\int_0^t \|e^{-\tau\Lambda}h\|_1 d\tau &\leq c's^{-1}C\int_0^t \|e^{-\tau A}\psi_\tau h\|_1 d\tau \leq c's^{-1}C\int_0^t \|\psi_\tau h\|_1 d\tau \\ &\text{(we are applying Lemma 9)} \\ &\leq c'C\hat{C}\frac{t}{s}\|\psi_t h\|_1,\end{aligned}$$

Therefore, using $\psi_s \geq \big(\frac{t}{s}\big)^{\frac{\beta}{\alpha}}\psi_t$, we obtain

$$c's^{-1}\int_0^t \|e^{-\tau\Lambda}h\|_1 d\tau \leq c'C\hat{C}\frac{t}{s}\left(\frac{t}{s}\right)^{-\frac{\beta}{\alpha}}\|g_s\|_1.$$

Thus, by $(\star)$, $\big(1 - c'C\hat{C}\left(\frac{t}{s}\right)^{\frac{\alpha-\beta}{\alpha}}\big)\langle g_s\rangle \leq \langle \psi_s e^{-t\Lambda} h\rangle$. Since $\beta < \alpha$, we can select $s > t$ such that $c'C\hat{C}\left(\frac{t}{s}\right)^{\frac{\alpha-\beta}{\alpha}} = \frac{1}{2}$, which yields the bound

$$\langle \psi_s e^{-t\Lambda} \psi_s^{-1} g_s\rangle \geq \frac{1}{2}\langle g_s\rangle.$$

Finally, using $\psi_t \geq \psi_s \geq \left(\frac{t}{s}\right)^{\frac{\beta}{\alpha}}\psi_t$ and setting $2\nu := \left(\frac{t}{s}\right)^{\frac{\beta}{\alpha}} = \big(2c'C\hat{C}\big)^{-\frac{\beta}{\alpha-\beta}}$, we have

$$\langle \psi_t e^{-t\Lambda}\psi_t^{-1} g_t\rangle = \langle \psi_t e^{-t\Lambda}\psi_s^{-1} g_s\rangle \geq \langle \psi_s e^{-t\Lambda}\psi_s^{-1} g_s\rangle \geq \frac{1}{2}\langle g_s\rangle \geq \frac{1}{2}\left(\frac{t}{s}\right)^{\frac{\beta}{\alpha}}\langle g_t\rangle = \nu\langle g_t\rangle.$$

□

Remark 7. In the proof of Proposition 4, we calculate $(\Lambda^\varepsilon)^*\psi_s$ arguing as in the proof of Proposition 2:

$$(\Lambda^\varepsilon)^*\psi = (-\Delta)^{\frac{\alpha}{2}}\psi + \operatorname{div}(b_\varepsilon\psi), \quad \psi = \psi_s,$$

where

$$(-\Delta)^{\frac{\alpha}{2}}\psi = -s^{-\frac{\beta}{\alpha}}\beta(d+\beta-2)\frac{\gamma(d+\beta-2)}{\gamma(d+\beta-\alpha)}|x|^{\beta-\alpha} + h_0$$

for $h_0 := -I_{2-\alpha}\Delta(\psi - \tilde{\psi}) \in L^\infty$, $\|h_0\|_\infty \leq c_0 s^{-1}$. In turn,

$$\operatorname{div}(b_\varepsilon\psi) = \operatorname{div}(b\tilde{\psi}) + W_\varepsilon + h_1 + h_2 + h_3$$

where $\|h_i\|_\infty \leq c_i s^{-1}$, $i = 1, 2, 3$. Since, by the choice of β, $-\beta(d+\beta-2)\frac{\gamma(d+\beta-2)}{\gamma(d+\beta-\alpha)}|x|^{-\alpha}\tilde{\psi} + \operatorname{div}(b\tilde{\psi}) = 0$, we have

$$(\Lambda^\varepsilon)^*\psi = \mathbf{1}_{B(0,s^{\frac{1}{\alpha}})}W_\varepsilon + v_\varepsilon, \quad v_\varepsilon := \mathbf{1}_{B^c(0,s^{\frac{1}{\alpha}})}W_\varepsilon + h_0 + h_1 + h_2 + h_3,$$

where, it easily seen, $\|v_\varepsilon\|_\infty \leq c' s^{-1}$, as claimed.

Proposition 5. *For every $R_0 > 0$ there exist constants $0 < r < R_0 < R$ such that for all $t > 0$*

$$\frac{\nu}{2}\psi_t(x) \leq e^{-t\Lambda^*}\psi_t \mathbf{1}_{R_t,r_t}(x) \quad \textit{for all } x \in B(0, R_{0,t}), \quad x \neq 0.$$

where $r_t := rt^{\frac{1}{\alpha}}$, $R_{0,t} := R_0 t^{\frac{1}{\alpha}}$, $R_t := Rt^{\frac{1}{\alpha}}$, $\mathbf{1}_{R_t,r_t} := \mathbf{1}_{B(0,R_t)} - \mathbf{1}_{B(0,r_t)}$.

Proof. It suffices to prove that, for all $g := \psi_t h$, $0 \leq h \in \mathcal{S}$ with $\operatorname{sprt} h \subset B(0, R_{0,t})$,

$$\frac{\nu}{2}\langle g\rangle \leq \langle \mathbf{1}_{R_t,r_t}\psi_t e^{-t\Lambda}\psi_t^{-1} g\rangle.$$

By the upper bound,

$$\begin{aligned}\langle \mathbf{1}_{B(0,r_t)}\psi_t e^{-t\Lambda}\psi_t^{-1} g\rangle &\leq C\langle \mathbf{1}_{B(0,r_t)}\psi_t, e^{-tA} g\rangle \\ &\leq CC_1 t^{-\frac{d}{\alpha}}\|\mathbf{1}_{B(0,r_t)}\psi_t\|_1\|g\|_1 \\ &= CC_1\|\mathbf{1}_{B(0,r)}\psi_1\|_1\|g\|_1, \quad \|\mathbf{1}_{B(0,r)}\psi_1\|_1 \to 0 \text{ as } r \downarrow 0.\end{aligned}$$

$$\begin{aligned}\langle \mathbf{1}_{B^c(0,R_t)}\psi_t e^{-t\Lambda}\psi_t^{-1}g\rangle &\leq C\langle \mathbf{1}_{B^c(0,R_t)}\psi_t, e^{-tA}g\rangle \\ &\leq C\langle e^{-tA}\mathbf{1}_{B^c(0,R_t)}, g\mathbf{1}_{B(0,R_{0,t})}\rangle \\ &\leq 2C \sup_{x\in B(0,R_{0,t})} e^{-tA}\mathbf{1}_{B^c(0,R_t)}(x)\|g\|_1 \\ &\leq C(R_0,R)\|g\|_1, \quad C(R_0,R)\to 0 \text{ as } R-R_0\uparrow\infty\end{aligned}$$

where at the last step we have used, for $x \in B(0,R_{0,t})$, $y \in B^c(0,R_t)$ and $\tilde{x} = R_0^{-1}t^{-\frac{1}{\alpha}}x \in B(0,1)$, $\tilde{y} = R^{-1}t^{-\frac{1}{\alpha}}y \in B^c(0,1)$,

$$e^{-tA}(x,y) \leq k_0 t|x-y|^{-d-\alpha} \leq k_0 t|R_0 t^{\frac{1}{\alpha}}\tilde{x} - Rt^{\frac{1}{\alpha}}\tilde{y}|^{-d-\alpha} < 2k_0 t^{-\frac{d}{\alpha}}(R-R_0)^{-d-\alpha}|\tilde{y}|^{-d-\alpha}.$$

It remains to apply Proposition 4 to obtain $\frac{\nu}{2}\langle g\rangle \leq \langle \mathbf{1}_{R_t,r_t}\psi_t e^{-t\Lambda}\psi_t^{-1}g\rangle$. □

Proposition 6. $\langle h\rangle = \langle e^{-t\Lambda^*}h\rangle$ *for every* $h \in L^1$, $t > 0$.

Proof. Proposition 6 follows from $\langle h\rangle = \langle e^{-t(\Lambda^\varepsilon)^*}h\rangle$ and Proposition 10. □

Proposition 7. *For every* $R_0 > 0$ *there exist constants* $0 < r < R_0 < R$ *such that for all* $t > 0$

$$\frac{1}{2} \leq e^{-t\Lambda}\mathbf{1}_{R_t,r_t}(x) \quad \text{for all } x \in B(0,R_{0,t}),$$

where $r_t := rt^{\frac{1}{\alpha}}$, $R_{0,t} := R_0 t^{\frac{1}{\alpha}}$, $R_t := Rt^{\frac{1}{\alpha}}$, $\mathbf{1}_{R_t,r_t} := \mathbf{1}_{B(0,R_t)} - \mathbf{1}_{B(0,r_t)}$.

Proof. We essentially repeat the proof of Proposition 5. It suffices to prove that, for all $0 \leq h \in \mathcal{S}$ with $\operatorname{sprt} h \subset B(0,R_{0,t})$,

$$\frac{1}{2}\langle h\rangle \leq \langle \mathbf{1}_{R_t,r_t}e^{-t\Lambda^*}h\rangle.$$

By the upper bound,

$$\begin{aligned}\langle \mathbf{1}_{B(0,r_t)}e^{-t\Lambda^*}h\rangle &\leq C\langle \mathbf{1}_{B(0,r_t)}\psi_t, e^{-tA}h\rangle \\ &\leq CC_1 t^{-\frac{d}{\alpha}}\|\mathbf{1}_{B(0,r_t)}\psi_t\|_1\|h\|_1 \\ &= o(r)\|h\|_1, \quad o(r)\to 0 \text{ as } r\downarrow 0;\end{aligned}$$

$$\begin{aligned}\langle \mathbf{1}_{B^c(0,R_t)}e^{-t\Lambda^*}h\rangle &\leq C\langle \mathbf{1}_{B^c(0,R_t)}\psi_t, e^{-tA}h\rangle \\ &\leq C\langle e^{-tA}\mathbf{1}_{B^c(0,R_t)}, h\mathbf{1}_{B(0,R_{0,t})}\rangle \\ &\leq C \sup_{x\in B(0,R_{0,t})} e^{-tA}\mathbf{1}_{B^c(0,R_t)}(x)\|h\|_1 \\ &= C(R_0,R)\|h\|_1, \quad C(R_0,R)\to 0 \text{ as } R-R_0\uparrow\infty.\end{aligned}$$

The last two estimates and Proposition 6 yield $\frac{1}{2}\langle h\rangle \leq \langle \mathbf{1}_{R_t,r_t}e^{-t\Lambda^*}h\rangle$. □

3. We are in position to complete the proof of the lower bound using the so-called $3q$ argument. Set $q_t(x,y) := \psi_t^{-1}(x)e^{-t\Lambda^*}(x,y)$, $x \neq 0$.
(a) Let $x, y \in B^c(0,t^{\frac{1}{\alpha}})$, $x \neq y$. Then, using that $\psi_{3t}^{-1} \geq 1$, we have by Corollary 3,

$$q_{3t}(x,y) \geq e^{-3t\Lambda^*}(x,y) \geq ce^{-3tA}(x,y).$$

Let $r_t = rt^{\frac{1}{\alpha}}$, $R_t = Rt^{\frac{1}{\alpha}}$ be as in Proposition 5 and Proposition 7, where we fix $R_0 = 1$ (hence $r < 1$).

(b) Let $x \in B(0, t^{\frac{1}{\alpha}})$, $|y| \geq rt^{\frac{1}{\alpha}}$, $x \neq y$. By the reproduction property,

$$
\begin{aligned}
q_{2t}(x,y) &\geq \psi_{2t}^{-1}(x)\langle e^{-t\Lambda^*}(x,\cdot)\psi_t^{-1}(\cdot)\psi_t(\cdot)e^{-t\Lambda^*}(\cdot,y)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\
&\geq \psi_{2t}^{-1}(x)\psi_t^{-1}(R_t)\langle e^{-t\Lambda^*}(x,\cdot)\psi_t(\cdot)e^{-t\Lambda^*}(\cdot,y)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\
&\geq \psi_{2t}^{-1}(x)\psi_t^{-1}(R_t)\big(e^{-t\Lambda^*}\psi_t\mathbf{1}_{R_t,r_t}\big)(x) \inf_{r_t \leq |z| \leq R_t} e^{-t\Lambda^*}(z,y) \\
&\quad \text{(we are applying Corollary 3, Proposition 5 and using } \psi_t^{-1}(R_t) = 1) \\
&\geq \frac{\nu}{2}\psi_{2t}^{-1}(x)\psi_t(x)c(r) \inf_{r_t \leq |z| \leq R_t} e^{-tA}(z,y) \\
&\quad \text{(we are using } \psi_t \geq \psi_{2t}) \\
&\geq C_1 e^{-2tA}(x,y).
\end{aligned}
$$

(b') Let $x \in B(0, t^{\frac{1}{\alpha}}), |y| \geq t^{\frac{1}{\alpha}}$, $x \neq y$. Arguing as in (b), we obtain

$$
q_{3t}(x,y) \geq C_2 e^{-3tA}(x,y).
$$

(c) Let $|x| \geq rt^{\frac{1}{\alpha}}$, $y \in B(0, t^{\frac{1}{\alpha}})$, $x \neq y$. We have

$$
\begin{aligned}
q_{2t}(x,y) &\geq \psi_{2t}^{-1}(x)\langle e^{-t\Lambda^*}(x,\cdot)e^{-t\Lambda^*}(\cdot,y)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\
&= \psi_{2t}^{-1}(x)\langle e^{-t\Lambda^*}(x,\cdot)e^{-t\Lambda}(y,\cdot)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\
&\quad \text{(we are using } \psi_{2t}^{-1} \geq 1 \text{ and applying Corollary 3)} \\
&\geq c(r)\langle e^{-tA}(x,\cdot)e^{-t\Lambda}(y,\cdot)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\
&\quad \text{(we are applying (16))} \\
&\geq C_3(r)t(Rt^{\frac{1}{\alpha}} + |x|)^{-d-\alpha}\langle e^{-\Lambda}(y,\cdot)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\
&\quad \text{(we are applying Proposition 7)} \\
&\geq C_3(r)2^{-1}t(Rt^{\frac{1}{\alpha}} + |x|)^{-d-\alpha} \geq C_4(r)e^{-2tA}(x,y).
\end{aligned}
$$

(c') Let $|x| \geq t^{\frac{1}{\alpha}}$, $y \in B(0, t^{\frac{1}{\alpha}})$, $x \neq y$. Arguing as in (c), we obtain

$$
q_{3t}(x,y) \geq C_5(r)e^{-3tA}(x,y).
$$

(d) Let $x,y \in B(0,t^{\frac{1}{\alpha}})$, $x \neq y$. By the reproduction property,

$$
\begin{aligned}
q_{3t}(x,y) &\geq \psi_{3t}^{-1}(x)\langle e^{-t\Lambda^*}(x,\cdot)e^{-2t\Lambda^*}(\cdot,y)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\
&\quad \text{(we are using (c))} \\
&\geq C_4(r)\psi_{3t}^{-1}(x)\langle e^{-t\Lambda^*}(x,\cdot)\psi_{2t}(\cdot)e^{-2tA}(\cdot,y)\mathbf{1}_{R_t,r_t}(\cdot)\rangle \\
&\quad \text{(we are using } \psi_{2t} \geq 2^{\frac{\beta}{\alpha}}\psi_t \text{ and } e^{-2tA}(z,y) \geq c(r,R)t^{-\frac{d}{\alpha}} > 0 \text{ for } r_t \leq |z| \leq R_t,\ |y| \leq t^{\frac{1}{\alpha}}) \\
&\geq c(r,R)C_4 2^{\frac{\beta}{\alpha}}\psi_{3t}^{-1}(x)t^{-\frac{d}{\alpha}}\langle e^{-t\Lambda^*}(x,\cdot)\mathbf{1}_{R_t,r_t}(\cdot)\psi_t(\cdot)\rangle \\
&\quad \text{(we are applying Proposition 5 and using } \psi_t \geq \psi_{3t}) \\
&\geq c(r,R)C_4 2^{\frac{\beta}{\alpha}}\frac{\nu}{2}t^{-\frac{d}{\alpha}} \\
&\quad \text{(we are applying (16))} \\
&\geq C_5(r,R)e^{-3tA}(x,y).
\end{aligned}
$$

By (a), (b'), (c'), (d), $q_{3t}(x,y) \geq Ce^{-3tA}(x,y)$ for all $x,y \in \mathbb{R}^d$, $x \neq y$, $x \neq 0$, and so

$$e^{-3t\Lambda^*}(x,y) \geq Ce^{-3tA}(x,y)\psi_{3t}(x), \quad t > 0.$$

The lower bound is proved.

8. Construction of the semigroup $e^{-t\Lambda_r}$, $\Lambda_r = (-\Delta)^{\frac{\alpha}{2}} - b\cdot\nabla$ in L^r, $1 \leq r < \infty$

Set $b_\varepsilon(x) := \kappa|x|_\varepsilon^{-\alpha}x$, $\kappa > 0$, $|x|_\varepsilon := \sqrt{|x|^2+\varepsilon}$, $\varepsilon > 0$,

$$\Lambda_r^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon\cdot\nabla, \quad D(\Lambda_r^\varepsilon) = \mathcal{W}^{\alpha,r} := \big(1+(-\Delta)^{\frac{\alpha}{2}}\big)^{-1}L^r.$$

To prove that $-\Lambda^\varepsilon \equiv -\Lambda_r^\varepsilon$ is the generator of a holomorphic semigroup in L^r, $1 \leq r < \infty$, we appeal to the Hille Perturbation Theorem [Ka, Ch. IX, sect. 2.2]. To verify its assumptions, we use a well known estimate

$$|\nabla\big(\zeta+A\big)^{-1}(x,y)| \leq C\big(\mathrm{Re}\zeta+A\big)^{-\frac{\alpha-1}{\alpha}}(x,y), \quad \mathrm{Re}\zeta > 0, \quad C = C(d,\alpha), \quad A \equiv (-\Delta)^{\frac{\alpha}{2}}.$$

Then for $Y = L^p$

$$\|b_\varepsilon\cdot\nabla\big(\zeta+A\big)^{-1}\|_{Y\to Y} \leq C\|b_\varepsilon\|_\infty\|\big(\mathrm{Re}\zeta+A\big)^{-\frac{\alpha-1}{\alpha}})\|_{Y\to Y} \leq C\|b_\varepsilon\|_\infty(\mathrm{Re}\zeta)^{-\frac{\alpha-1}{\alpha}},$$

and so $\|b_\varepsilon\cdot\nabla\big(\zeta+A\big)^{-1}\|_{Y\to Y}$, $\mathrm{Re}\zeta \geq c_\varepsilon$, can be made arbitrarily small by selecting c_ε sufficiently large. It follows that the Neumann series for

$$(\zeta+\Lambda^\varepsilon)^{-1} = (\zeta+A)^{-1}(1+T)^{-1}, \quad T := -b_\varepsilon\cdot\nabla(\zeta+A)^{-1},$$

converges in L^p and C_u and satisfies $\|(\zeta+\Lambda^\varepsilon)^{-1}\|_{Y\to Y} \leq C_\varepsilon|\zeta|^{-1}$, $\mathrm{Re}\zeta \geq c_\varepsilon$, i.e. $-\Lambda^\varepsilon$ is the generator of a holomorphic semigroup.

The same argument (with $Y = C_u$) shows that $\Lambda^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon\cdot\nabla$ with $D(\Lambda^\varepsilon) := D((-\Delta)_{C_u}^{\frac{\alpha}{2}})$ generates a holomorphic semigroup in C_u.

Proposition 8. *For every $r \in [1,\infty[$ and $\varepsilon > 0$, $e^{-t\Lambda_r^\varepsilon}$ is a contraction C_0 semigroup in L^r. There exists a constant $c \neq c(\varepsilon)$ such that*

$$\|e^{-t\Lambda_r^\varepsilon}\|_{r\to q} \leq c_N t^{-\frac{d}{\alpha}(\frac{1}{r}-\frac{1}{q})}, \quad t > 0,$$

for all $1 \leq r < q \leq \infty$.

In particular, there is a constant $c_S > 0$, $c_S \neq c_S(\varepsilon)$ such that ($\Lambda^\varepsilon \equiv \Lambda_2^\varepsilon$)

$$\mathrm{Re}\langle \Lambda^\varepsilon u, u\rangle \geq c_S \|u\|_{2j}^2, \quad u \in D(\Lambda^\varepsilon).$$

Proof. First, let $1 < r < \infty$. Set $u \equiv u(t) := e^{-t\Lambda_r^\varepsilon} f$, $f \in L^1 \cap L^\infty$, and write $A := (-\Delta)^{\frac{\alpha}{2}}$. Multiplying the equation $\partial_t u + \Lambda_r^\varepsilon u = 0$ by $\bar{u}|u|^{r-2}$ and integrating over the spatial variables we obtain (taking into account that $D(\Lambda_r^\varepsilon) = D(A_r) \subset W^{1,r}$)

$$\frac{1}{r}\partial_t \|u\|_r^r + \mathrm{Re}\langle Au, u|u|^{r-2}\rangle - \mathrm{Re}\langle b_\varepsilon \cdot \nabla u, u|u|^{r-2}\rangle = 0.$$

Note that, since $-A$ is a Markov generator,

$$\mathrm{Re}\langle Au, u|u|^{r-2}\rangle \geq \frac{4}{rr'}\|A^{\frac{1}{2}}|u|^{\frac{r}{2}}\|_2^2$$

(indeed, by [LS, Theorem 2.1] or by Theorem 10 in Appendix A, $\mathrm{Re}\langle Au, u|u|^{r-2}\rangle \geq \frac{4}{rr'}\|A^{\frac{1}{2}}u^{\frac{r}{2}}\|_2^2$, $u^{\frac{r}{2}} := u|u|^{\frac{r}{2}-1}$, and by the Beurling-Deny theory $\|A^{\frac{1}{2}}u^{\frac{r}{2}}\|_2^2 \geq \|A^{\frac{1}{2}}|u|^{\frac{r}{2}}\|_2^2$). Integration by parts yields

$$-\mathrm{Re}\langle b_\varepsilon \cdot \nabla u, u|u|^{r-2}\rangle = \frac{\kappa}{r}\big\langle \big(d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2\big)|u|^r\big\rangle \geq \kappa\frac{d-\alpha}{r}\langle |x|_\varepsilon^{-\alpha}|u|^r\rangle.$$

Thus,

$$-\partial_t\|u\|_r^r \geq \frac{4}{r'}\|A^{\frac{1}{2}}|u|^{\frac{r}{2}}\|_2^2 \tag{20}$$

From (20) we obtain $\|u(t)\|_r \leq \|f\|_r$, $t \geq 0$ and since $L^1 \cap L^\infty$ is dense in L^r, $\|e^{-t\Lambda_r^\varepsilon}\|_{r\to r} \leq 1$ as needed.

Since $e^{-t\Lambda_1^\varepsilon} \upharpoonright L^1 \cap L^r = e^{-t\Lambda_r^\varepsilon} \upharpoonright L^1 \cap L^r$, the latter clearly yields

$$\|e^{-t\Lambda_1^\varepsilon} f\|_r \leq \|f\|_r, \quad f \in L^1 \cap L^\infty.$$

Sending $r \uparrow \infty$, we have $\|e^{-t\Lambda_r^\varepsilon} f\|_\infty \leq \|f\|_\infty$, and sending $r \downarrow 1$, we have $\|e^{-t\Lambda_1^\varepsilon}\|_{1\to 1} \leq 1$.

Let us prove the ultracontractivity of $e^{-t\Lambda_r^\varepsilon}$. By (20),

$$-\partial_t\|u\|_{2r}^{2r} \geq \frac{4}{(2r)'}\|A^{\frac{1}{2}}|u|^r\|_2^2, \quad 1 \leq r < \infty.$$

Using the Nash inequality $\|A^{\frac{1}{2}}h\|_2^2 \geq C_N\|h\|_2^{2+\frac{2\alpha}{d}}\|h\|_1^{-\frac{2\alpha}{d}}$ and $\|u(t)\|_r \leq \|f\|_r$, we have, setting $v := \|u\|_{2r}^{2r}$,

$$\partial_t v^{-\frac{\alpha}{d}} \geq c_1\|f\|_r^{-\frac{2r\alpha}{d}},$$

where $c_1 = C_N\frac{\alpha}{d}\frac{4}{(2r)'}$. Integrating this inequality yields

$$\|e^{-t\Lambda_r^\varepsilon}\|_{r\to 2r} \leq c_1^{-\frac{d}{2\alpha r}} t^{-\frac{d}{\alpha}(\frac{1}{r}-\frac{1}{2r})}, \quad t > 0, \tag{$*$}$$

and so, by semigroup property,

$$\|e^{-t\Lambda_r^\varepsilon}\|_{1\to 2^m} \leq c_N t^{-\frac{d}{\alpha}(1-\frac{1}{2^m})}, \quad t > 0, \quad m \geq 1,$$

where the constant $c_N \neq c_N(m)$. Thus, sending m to infinity we arrive at $\|e^{-t\Lambda_r^\varepsilon}\|_{1\to\infty} \leq c_N t^{-\frac{d}{\alpha}}$, $t > 0$. The latter and the contractivity of $e^{-t\Lambda_r^\varepsilon}$ in all L^q, $1 \leq q \leq \infty$ yield via interpolation the desired bound $\|e^{-t\Lambda_p^\varepsilon}\|_{p\to q} \leq c_N t^{-\frac{d}{\alpha}(\frac{1}{p}-\frac{1}{q})}$, $t > 0$, for all $1 \leq p < q \leq \infty$.

Finally, since $D(\Lambda^\varepsilon) = D(A)$, we have, for $u \in D(A)$, $\mathrm{Re}\langle \Lambda^\varepsilon u, u\rangle \geq \|A^{\frac{1}{2}}u\|_2^2 \geq c_S\|u\|_{2j}^2$ □

8.1. **Case $d \geq 4$.** We will first provide an elementary argument that allows to treat all $d = 4, 5, \dots$ but the main case $d = 3$.

Proposition 9. *For every $r \in [1, \infty[$ the limit*

$$s\text{-}L^r\text{-}\lim_{\varepsilon\downarrow 0} e^{-t\Lambda_r^\varepsilon} \quad (\textit{loc. uniformly in } t \geq 0)$$

exists and determines a contraction C_0 semigroup on L^r, say $e^{-t\Lambda_r}$.

For all $1 \leq r < q \leq \infty$,

$$\|e^{-t\Lambda_r}\|_{r\to q} \leq c_N t^{-\frac{d}{\alpha}(\frac{1}{r}-\frac{1}{q})}, \quad t > 0$$

with c_N from Proposition 8

Proof of Proposition 9. First, let $r = 2$. Set $u^\varepsilon(t) := e^{-t\Lambda^\varepsilon} f$, $f \in C_c^\infty$.

Claim 5. $\|\nabla u^\varepsilon(t)\|_2 \leq \|\nabla f\|_2$, $t \geq 0$.

Proof of Claim 5. Denote $u \equiv u^\varepsilon$, $w := \nabla u$, $w_i := \nabla_i u$. Due to $f \in C_c^\infty$ and $\nabla_i^n b_\varepsilon^i \in C^\infty \cap L^\infty$, $i = 1, \dots d$, $n \geq 1$ we can and will differentiate the equation $\partial_t u + \Lambda^\varepsilon u = 0$ in x_i, obtaining

$$\partial_t w_i + (-\Delta)^{\frac{\alpha}{2}} w_i - b_\varepsilon \cdot \nabla w_i - (\nabla_i b_\varepsilon) \cdot w = 0.$$

Multiplying the latter by $\bar{w}_i$, integrating by parts and summing up in $i = 1, \dots, d$ we have

$$\frac{1}{2}\partial_t\|w\|_2^2 + \sum_{i=1}^d \|(-\Delta)^{\frac{\alpha}{4}} w_i\|_2^2 - \mathrm{Re}\sum_{i=1}^d \langle b_\varepsilon \cdot \nabla w_i, w_i\rangle - \mathrm{Re}\sum_{i=1}^d \langle (\nabla_i b_\varepsilon) \cdot w, w_i\rangle = 0,$$

$$-\mathrm{Re}\langle b_\varepsilon \cdot \nabla w_i, w_i\rangle = \frac{\kappa}{2}\langle (d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2) w_i, w_i\rangle,$$

$$-\langle (\nabla_i b_\varepsilon) \cdot w, w_i\rangle = -\kappa\langle |x|_\varepsilon^{-\alpha} w_i, w_i\rangle + \kappa\alpha\langle |x|_\varepsilon^{-\alpha-2} x_i \bar{w}_i (x \cdot w)\rangle.$$

Thus,

$$\frac{1}{2}\partial_t\|w\|_2^2 + \sum_{i=1}^d \|(-\Delta)^{\frac{\alpha}{4}} w_i\|_2^2 + \kappa\frac{d-\alpha}{2}\langle |x|_\varepsilon^{-\alpha}|w|^2\rangle + \frac{\kappa\alpha\varepsilon}{2}\langle |x|_\varepsilon^{-\alpha-2}|w|^2\rangle$$
$$- \kappa\langle |x|_\varepsilon^{-\alpha}|w|^2\rangle + \kappa\alpha\langle |x|_\varepsilon^{-\alpha-2}|x \cdot w|^2\rangle = 0,$$

and so, since $\kappa > 0$,

$$\frac{1}{2}\partial_t\|w\|_2^2 + \sum_{i=1}^d \|(-\Delta)^{\frac{\alpha}{4}} w_i\|_2^2 + \kappa\frac{d-\alpha-2}{2}\langle |x|_\varepsilon^{-\alpha}|w|^2\rangle + \kappa\alpha\langle |x|_\varepsilon^{-\alpha-2}|x \cdot w|^2\rangle \leq 0.$$

Since $d \geq 4$, $\alpha < 2$, we have $d - \alpha - 2 > 0$. Thus, integrating in t, we obtain $\|w(t)\|_2^2 \leq \|\nabla f\|_2^2$, $t \geq 0$, as needed. □

Next, set $u_n := u^{\varepsilon_n}$, $u_m := u^{\varepsilon_m}$ and $g(t) := u_n(t) - u_m(t)$, $\quad t \geq 0$.

Claim 6. $\|g(t)\|_2 \to 0$ *uniformly in* $t \in [0,1]$ *as* $n,m \to \infty$.

Proof of Claim 6. We subtract the equations for u_n and u_m and obtain

$$\partial_t g + (-\Delta)^{\frac{\alpha}{2}} g - b_n \cdot \nabla g - (b_n - b_m) \cdot \nabla u_m = 0,$$

$$\partial_t \|g\|_2^2 + \|(-\Delta)^{\frac{\alpha}{4}} g\|_2^2 - \mathrm{Re}\langle b_n \cdot \nabla g, g\rangle - \mathrm{Re}\langle (b_n - b_m)\cdot \nabla u_m, g\rangle = 0. \tag{21}$$

Concerning the last two terms, we have:

$$-\mathrm{Re}\langle b_n \cdot \nabla g, g\rangle = \frac{\kappa}{2}\langle (d|x|_\varepsilon^{-\alpha} - \alpha |x|_\varepsilon^{-\alpha-2}|x|^2 g, g\rangle \geq \kappa \frac{d-\alpha}{2}\langle |x|_\varepsilon^{-\alpha}, |g|^2\rangle,$$

$$\begin{aligned}
|\langle (b_n - b_m)\cdot \nabla u_m, g\rangle| &\leq |\langle \mathbf{1}_{B(0,1)}(b_n - b_m)\cdot \nabla u_m, g\rangle| + |\langle \mathbf{1}^c_{B(0,1)}(b_n - b_m)\cdot \nabla u_m, g\rangle| \\
&\quad \text{(we are using } \|g\|_\infty \leq 2\|f\|_\infty,\ \|g\|_2 \leq 2\|f\|_2) \\
&\leq \|\mathbf{1}_{B(0,1)}(b_n - b_m)\|_2 \|\nabla u_m\|_2 2\|f\|_\infty + \|\mathbf{1}^c_{B(0,1)}(b_n - b_m)\|_\infty \|\nabla u_m\|_2 2\|f\|_2 \\
&\quad \text{(we are using Claim 5)} \\
&\leq \|\mathbf{1}_{B(0,1)}(b_n - b_m)\|_2 \|\nabla f\|_2 2\|f\|_\infty + \|\mathbf{1}^c_{B(0,1)}(b_n - b_m)\|_\infty \|\nabla f\|_2 2\|f\|_2 \\
&\to 0 \quad \text{as } n,m \to \infty.
\end{aligned}$$

Thus, integrating (21) in t and using the last two observations, we end the proof of Claim 6. □

By Claim 6, $\{e^{-t\Lambda^{\varepsilon_n}} f\}_{n=1}^\infty$, $f \in C_c^\infty$ is a Cauchy sequence in $L^\infty([0,1], L^2)$. Set

$$T_2^t f := s\text{-}L^2\text{-}\lim_n e^{-t\Lambda^{\varepsilon_n}} f \text{ uniformly in } 0 \leq t \leq 1. \tag{22}$$

(Clearly, the limit does not depend on the choice of $\{\varepsilon_n\} \downarrow 0$.) Since $e^{-t\Lambda^{\varepsilon_n}}$ are contractions in L^2, we have $\|T_2^t f\|_2 \leq \|f\|_2$, $t \in [0,1]$. Extending T_2^t by continuity to L^2, we obtain that T_2^t is strongly continuous. Furthermore,

$$T_2^t f = \lim_n e^{-t\Lambda^{\varepsilon_n}} f \text{ in } L^2 \text{ for all } f \in L^2, \quad 0 \leq t \leq 1.$$

Finally, extending T_2^t to all $t \geq 0$ using the reproduction property, we obtain a contraction C_0 semigroup $T_2^t =: e^{-t\Lambda}$, $t \geq 0$.

Now, let $1 \leq r < \infty$. Since $e^{-t\Lambda^\varepsilon}$ is a contraction in L^r, we obtain, by construction (22) of $e^{-t\Lambda} f$, $f \in C_c^\infty$, appealing e.g. to Fatou's Lemma, that

$$\|e^{-t\Lambda} f\|_r \leq \|f\|_r, \quad t \geq 0.$$

Thus, extending $e^{-t\Lambda}$ by continuity to L^r, we can define contraction semigroups $T_r^t := [e^{-t\Lambda}]^{\mathrm{clos}}_{L^r \to L^r}$, $t \geq 0$. The strong continuity of T_r^t in L^r is a consequence of strong continuity of $e^{-t\Lambda}$, contractivity of T_r^t and Fatou's Lemma. Write $T_r^t =: e^{-t\Lambda_r}$. Clearly,

$$e^{-t\Lambda_r} = s\text{-}L^r\text{-}\lim_n e^{-t\Lambda_r^{\varepsilon_n}}, \quad t \geq 0.$$

The latter and Proposition 8 complete the proof of Proposition 9.

□

8.2. **Case** $d = 3$**.** The proof of the next proposition works in all dimensions $d \geq 3$.

Proposition 10. *For every* $r \in [1, \infty[$ *the limit*

$$s\text{-}L^r\text{-}\lim_{\varepsilon \downarrow 0} e^{-t\Lambda_r^\varepsilon} \quad (\textit{loc. uniformly in } t \geq 0)$$

exists and determines a contraction C_0 *semigroup on* L^r*, say,* $e^{-t\Lambda_r}$*. There exists a constant* $c_N \neq c_N(\varepsilon)$ *such that*

$$\|e^{-t\Lambda_r}\|_{r \to q} \leq c_N t^{-\frac{d}{\alpha}(\frac{1}{r} - \frac{1}{q})}, \quad t > 0,$$

for all $1 \leq r \leq q \leq \infty$*.*

Proof of Proposition 10. Denote $u^\varepsilon(t) := e^{-t\Lambda_r^\varepsilon} f$, $f \in C_c^\infty$. For brevity, write $u \equiv u^\varepsilon$ and $w := \nabla u$.

Claim 7. For every $r \in]1, \infty[$,

$$\frac{1}{r}\|w(t_1)\|_r^r + \frac{4}{rr'}\int_0^{t_1} \sum_{i=1}^d \|(-\Delta)^{\frac{\alpha}{4}}(w_i|w|^{\frac{r-2}{2}})\|_2^2 dt$$
$$+ \kappa \frac{d - \alpha - r}{r} \int_0^{t_1} \langle |x|_\varepsilon^{-\alpha}|w|^r\rangle dt + \alpha\kappa \int_0^{t_1} \langle |x|_\varepsilon^{\alpha-2}|x \cdot w|^2|w|^{r-2}\rangle dt \leq \frac{1}{r}\|\nabla f\|_r^r, \quad t_1 > 0.$$

In particular, for $1 < r < d - \alpha$,

$$\|w(t_1)\|_r^r + \frac{4}{r'}c_S d^{-\frac{\alpha}{d}} \int_0^{t_1} \|w\|_{rj}^r dt \leq \|\nabla f\|_r^r, \quad t_1 > 0, \quad j := \frac{d}{d - \alpha}.$$

Proof of Claim 7. Set $w_i := \nabla_i u$. We differentiate $\partial_t u + \Lambda_r^\varepsilon u = 0$ in x_i, obtaining identity

$$\partial_t w_i + (-\Delta)^{\frac{\alpha}{2}} w_i - b_\varepsilon \cdot \nabla w_i - (\nabla_i b_\varepsilon) \cdot w = 0,$$

which we multiply by $\bar{w}_i|w|^{r-2}$, integrate over the spatial variables and then sum in $1 \leq i \leq d$ to obtain

$$\frac{1}{r}\partial_t\|w\|_r^r + \mathrm{Re}\langle(-\Delta)^{\frac{\alpha}{2}} w, w|w|^{r-2}\rangle - \mathrm{Re}\sum_{i=1}^d \langle b_\varepsilon \cdot \nabla w_i, w_i|w|^{r-2}\rangle - \mathrm{Re}\sum_{i=1}^d \langle(\nabla_i b_\varepsilon) \cdot w, w_i|w|^{r-2}\rangle = 0.$$

By Theorem 10 (Appendix A),

$$\mathrm{Re}\langle(-\Delta)^{\frac{\alpha}{2}} w, w|w|^{r-2}\rangle \geq \frac{4}{rr'}\langle(-\Delta)^{\frac{\alpha}{4}}(w|w|^{\frac{r-2}{2}}), (-\Delta)^{\frac{\alpha}{4}}(w|w|^{\frac{r-2}{2}})\rangle \equiv \frac{4}{rr'}\sum_{i=1}^d \|(-\Delta)^{\frac{\alpha}{4}}(w_i|w|^{\frac{r-2}{2}})\|_2^2.$$

Next, integrating by parts, we obtain

$$-\mathrm{Re}\sum_{i=1}^d \langle b_\varepsilon \cdot \nabla w_i, w_i|w|^{r-2}\rangle = \frac{\kappa}{r}\langle(d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2)|w|^r\rangle \geq \kappa\frac{d-\alpha}{r}\langle|x|_\varepsilon^{-\alpha}|w|^r\rangle,$$

and

$$\mathrm{Re}\sum_{i=1}^d \langle(\nabla_i b_\varepsilon) \cdot w, w_i|w|^{r-2}\rangle = \kappa\langle|x|_\varepsilon^{-\alpha}|w|^r\rangle - \alpha\kappa\langle|x|_\varepsilon^{-\alpha-2}(x \cdot w)^2|w|^{r-2}\rangle.$$

The first required inequality follows.

Now, let $1 < r < d-\alpha$. Note that

$$\sum_{i=1}^d \|(-\Delta)^{\frac{\alpha}{4}}(w_i|w|^{\frac{r-2}{2}})\|_2^2 \geq c_S \sum_{i=1}^d \|w_i|w|^{\frac{r-2}{2}}\|_{2j}^2 = c_S \sum_{i=1}^d \langle |w_i|^{2j}|w|^{(r-2)j}\rangle^{\frac{1}{j}}$$
$$\geq c_S \biggl(\langle |w|^{(r-2)j}\sum_{i=1}^d |w_i|^{2j}\rangle\biggr)^{\frac{1}{j}}$$
$$\biggl(\text{we use } \bigl(\sum_{i=1}^d |w|^{2j}\bigr)^{1/j} \geq \bigl(\sum_{i=1}^d |w_i|^2\bigr)d^{-1/j'} = |w|^2 d^{-1/j'}\biggr)$$
$$\geq c_S d^{-1/j'}\langle |w|^{rj}\rangle^{\frac{1}{j}} = c_S d^{-\frac{\alpha}{d}}\|w\|_{rj}^r.$$

The second required inequality follows. □

Next, set $u_n := u^{\varepsilon_n}$, $u_m := u^{\varepsilon_m}$. Let $g(t) := u_n(t) - u_m(t)$, $t \geq 0$.

Claim 8. *$\|g(t)\|_2 \to 0$ uniformly in $t \in [0,1]$ as $n, m \to \infty$.*

Proof of Claim 8. We subtract the equations for u_n and u_m:

$$\partial_t g + (-\Delta)^{\frac{\alpha}{2}} g - b_n \cdot \nabla g - (b_n - b_m)\cdot \nabla u_m = 0.$$

Multiplying the latter by $\bar{g}$ and integrating, we obtain

$$\|g(t_1)\|_2^2 + \int_0^{t_1} \|(-\Delta)^{\frac{\alpha}{4}} g\|_2^2 dt - \text{Re}\int_0^{t_1}\langle b_n\cdot\nabla g, g\rangle dt - \text{Re}\int_0^{t_1}\langle (b_n-b_m)\cdot\nabla u_m, g\rangle dt = 0$$

for every $t_1 > 0$. Since

$$-\text{Re}\langle b_n\cdot\nabla g, g\rangle = \frac{\kappa}{2}\langle (d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2 g, g\rangle \geq \kappa\frac{d-\alpha}{2}\langle |x|_\varepsilon^{-\alpha}, |g|^2\rangle,$$

we have

$$\|g(t_1)\|_2^2 + \int_0^{t_1}\|(-\Delta)^{\frac{\alpha}{4}} g\|_2^2 dt + \kappa\frac{d-\alpha}{2}\int_0^{t_1}\langle |x|^{-\alpha}, |g|^2\rangle dt \leq \Bigl|\int_0^{t_1}\langle (b_n-b_m)\cdot\nabla u_m, g\rangle dt\Bigr|. \tag{23}$$

Let us estimate the RHS of (10). Fix $1 < r < d-\alpha$ (as in the second assertion of Claim 7). Then

$$|\langle (b_n-b_m)\cdot\nabla u_m, g\rangle| \leq |\langle \mathbf{1}_{B(0,1)}(b_n-b_m)\cdot\nabla u_m, g\rangle| + |\langle \mathbf{1}_{B^c(0,1)}(b_n-b_m)\cdot\nabla u_m, g\rangle|$$
$$(\text{we apply estimates } \|g\|_\infty \leq 2\|f\|_\infty,\ \|g\|_{(rj)'} \leq 2\|f\|_{(rj)'})$$
$$\leq \|\mathbf{1}_{B(0,1)}(b_n-b_m)\|_{(rj)'}\|\nabla u_m\|_{rj}2\|f\|_\infty + \|\mathbf{1}_{B^c(0,1)}(b_n-b_m)\|_\infty\|\nabla u_m\|_{rj}2\|f\|_{(rj)'}.$$

Clearly $\|\mathbf{1}_{B^c(0,1)}(b_n-b_m)\|_\infty \to 0$ as $n, m \to \infty$. The same is true for $\|\mathbf{1}_{B(0,1)}(b_n-b_m)\|_{(rj)'}$ since $(rj)' = \frac{rd}{rd-d+\alpha} < \frac{d}{\alpha-1}$. Thus, in view of Claim 7,

$$\int_0^{t_1}|\langle (b_n-b_m)\cdot\nabla u_m, g\rangle| dt$$
$$\leq \biggl(\|\mathbf{1}_{B(0,1)}(b_n-b_m)\|_{(rj)'}\|f\|_\infty + \|\mathbf{1}_{B^c(0,1)}(b_n-b_m)\|_\infty\|f\|_{(rj)'}\biggr)2\int_0^{t_1}\|\nabla u_m\|_{rj}dt \to 0$$

as $n, m \to \infty$. □

Now, we argue as in the proof of Proposition 9 to obtain that for every $r \in [1,\infty[$ the limit $s\text{-}L^r\text{-}\lim_n e^{-t\Lambda_r^{\varepsilon_n}}$, $t \geq 0$ exists and determines a contraction C_0 semigroup on L^r. It is easily seen that the limit does not depend on the choice of ε_n.

The last assertion follows now from Proposition 8.

The proof of Proposition 10 is completed. ☐

9. Construction of the semigroup $e^{-t\Lambda_r^*}$, $\Lambda_r^* = (-\Delta)^{\frac{\alpha}{2}} + \nabla \cdot b$ in L^r, $1 \leq r < \infty$

Set $(\Lambda^\varepsilon)_r^* := (-\Delta)^{\frac{\alpha}{2}} + \nabla \cdot b_\varepsilon$, $D\big((\Lambda^\varepsilon)_r^*\big) = \mathcal{W}^{\alpha,r}$. By the Hille Perturbation Theorem, $-(\Lambda^\varepsilon)_r^*$ is the generator of a holomorphic C_0 semigroup in L^r (arguing as in Section 8; the argument there also shows that $(\Lambda^\varepsilon)^* := (-\Delta)^{\frac{\alpha}{2}} + \nabla \cdot b_\varepsilon$, $D\big((\Lambda^\varepsilon)^*\big) = D((-\Delta)^{\frac{\alpha}{2}}_{C_u})$ is the generator of a holomorphic semigroup in C_u).

Proposition 11. *For every $r \in [1,\infty[$ and $\varepsilon > 0$, $e^{-t(\Lambda^\varepsilon)_r^*}$ is a contraction C_0 semigroup. There exists a constant $c_N \neq c_N(\varepsilon)$ such that*

$$\|e^{-t(\Lambda^\varepsilon)_r^*}\|_{r\to q} \leq c_N t^{-\frac{d}{\alpha}(\frac{1}{r}-\frac{1}{q})}, \quad t > 0,$$

for all $1 \leq r \leq q \leq \infty$.

Proof. The semigroup $e^{-t(\Lambda^\varepsilon)_r^*}$ is constructed in L^r repeating the argument in Section 8. The ultra contractivity estimate for $1 < r \leq q < \infty$ follows from Proposition 8 by duality, and for all $1 \leq r \leq q \leq \infty$ upon taking limits $r \downarrow 1$, $q \uparrow \infty$. ☐

Proposition 12. *For every $r \in [1,\infty[$ the limit*

$$s\text{-}L^r\text{-}\lim_{\varepsilon \downarrow 0} e^{-t(\Lambda^\varepsilon)_r^*} \quad (\textit{loc. uniformly in } t \geq 0)$$

exists and determines a contraction C_0 semigroup in L^r, say, $e^{-t\Lambda_r^*}$. There exists a constant c_N such that

$$\|e^{-t\Lambda_r^*}\|_{r\to q} \leq c_N t^{-\frac{d}{\alpha}(\frac{1}{r}-\frac{1}{q})}, \quad t > 0,$$

for all $1 \leq r \leq q \leq \infty$.

We have for $1 < r < \infty$

$$\langle e^{-t\Lambda_{r'}(b)} f, g\rangle = \langle f, e^{-t\Lambda_r^*(b)} g\rangle, \quad t > 0, \quad f \in L^{r'}, \quad r' = \frac{r}{r-1}, \quad g \in L^r.$$

Proof. First, let $r = 2$. In view of Proposition 11, we can argue as in the proof of [KSS, Prop. 10], appealing to the Rellich-Kondrashov Theorem, to obtain: For every sequence $\varepsilon_n \downarrow 0$ there exists a subsequence ε_{n_m} such that the limit

$$s\text{-}L^2\text{-}\lim_m e^{-t(\Lambda^{\varepsilon_{n_m}})^*} \quad (\text{loc. uniformly in } t \geq 0) \tag{24}$$

exists and determines a C_0 semigroup in L^2.

On the other hand, since

$$\langle e^{-t\Lambda^\varepsilon} f, g\rangle = \langle f, e^{-t(\Lambda^\varepsilon)^*} g\rangle, \quad t > 0, \quad f, g \in L^2,$$

it follows from Proposition 10 that for every $g \in L^2$ $e^{-t(\Lambda^\varepsilon)^*} g$ converge weakly in L^2 as $\varepsilon \downarrow 0$. Thus, the limit in (24) does not depend on the choice of ε_{n_m} and ε_n.

For $1 \leq r < \infty$, we repeat the argument in the end of the proof of Proposition 9, appealing to Proposition 11.

The last assertion follows from the analogous property of $e^{-t\Lambda^\varepsilon_{r'}}$, $e^{-t(\Lambda^\varepsilon)^*_r}$, $\varepsilon > 0$ and Propositions 10, 12. $\square$

Appendix A. L^r (vector) inequalities for symmetric Markov generators

Let X be a set and μ a σ-finite measure on X. Let $T^t = e^{-tA}$, $t \geq 0$, be a symmetric Markov semigroup in $L^2(X,\mu)$. Let

$$T^t_r := \big[T^t \upharpoonright L^2 \cap L^r\big]_{L^r \to L^r}, \quad t \geq 0,$$

a contraction C_0 semigroup on L^r, $r \in [1,\infty[$. Put $T^t_r =: e^{-tA_r}$.

Theorem 10. *Let $f_i \in D(A_r)$ $(1 \leq i \leq m)$, $r \in]1,\infty[$. Set $f := (f_i)_{i=1}^m$, $f_{(r)} := f|f|^{\frac{r-2}{2}}$. Then $f_i|f|^{\frac{r-2}{2}} \in D(A^{\frac{1}{2}})$ $(1 \leq i \leq m)$ and, applying the operators coordinate-wise, we have*

$$\frac{4}{rr'}\langle A^{\frac{1}{2}} f_{(r)}, A^{\frac{1}{2}} f_{(r)}\rangle \leq \operatorname{Re}\langle A_r f, f|f|^{r-2}\rangle \leq \varkappa(r)\langle A^{\frac{1}{2}} f_{(r)}, A^{\frac{1}{2}} f_{(r)}\rangle, \qquad (i)$$

where $\varkappa(r) := \sup_{s\in]0,1[}\big[(1+s^{\frac{1}{r}})(1+s^{\frac{1}{r'}})(1+s^{\frac{1}{2}})^{-2}\big]$, $r' = \frac{r}{r-1}$,

$$\big|\operatorname{Im}\langle A_r f, f|f|^{r-2}\rangle\big| \leq \frac{|r-2|}{2\sqrt{r-1}}\operatorname{Re}\langle A_r f, f|f|^{r-2}\rangle, \qquad (ii)$$

where

$$\langle A^{\frac{1}{2}} f_{(r)}, A^{\frac{1}{2}} f_{(r)}\rangle = \sum_{i=1}^m \|A^{\frac{1}{2}}(f_i|f|^{\frac{r-2}{2}})\|_2^2, \qquad \langle A_r f, f|f|^{r-2}\rangle = \sum_{i=1}^m \langle A_r f_i, f_i|f|^{r-2}\rangle.$$

Theorem 10 is a prompt but useful modification of [LS, Theorem 2.1] (corresponding to the case $m=1$): it allows us to control higher-order derivatives of $u(t) = e^{-t\Lambda}f$, $\Lambda \supset (-\Delta)^{\frac{\alpha}{2}} - b\cdot\nabla$, $f \in C_c^\infty$ in the proof of Proposition 10 (see Claim 7 there).

For the sake of completeness, we included the detailed proof below.

1. We will need

Claim 9. *There exists a finitely additive measure μ_t on $X \times X$, symmetric in the sense that $\mu_t(A \times B) = \mu_t(B \times A)$ on any μ-measurable sets of finite measure A and B, and satisfying*

$$\langle T^t f, g\rangle = \int_{X\times X} f(x)\overline{g(x)}d\mu_t(x,y) \quad (f,g \in L^1 \cap L^\infty).$$

In order to justify the claim, let us introduce the Banach space $\mathcal{L}^\infty = \mathcal{L}^\infty(X,\mathcal{M}_\mu)$, the Banach space of all bounded μ-measurable functions, endowed with the norm $|||f||| := \sup\{|f(x)| \mid x \in X\}$.

Let $N^\infty \equiv \mathcal{N}^\infty(X,\mathcal{M}_\mu)$ be the set of all μ-negligible functions, so that $L^\infty = \mathcal{L}^\infty/\mathcal{N}^\infty$. Denoting by $\pi : f \to \widetilde{f}$ the canonical mapping of $\mathcal{L}^\infty$ onto L^∞, we can identify L^∞ with $\pi(\mathcal{L}^\infty)$. Since μ is σ-finite, there exists a lifting $\rho : L^\infty \to \mathcal{L}^\infty$, a linear multiplicative positivity preserving map such that

$$\rho(\mathbf{1}_G) = \mathbf{1}_G \text{ for all } G \in \mathcal{M}_\mu \text{ with } \mu(G) < \infty.$$

Given $t > 0$ define $T^t_\rho : \mathcal{L}^\infty \to \mathcal{L}^\infty$ by

$$T^t_\rho f := \rho(T^t_\infty f),$$

and so T^t_ρ is a positivity preserving semigroup, and

$$\langle T^t_\rho f, g\rangle = \langle T^t \widetilde{f}, \widetilde{g}\rangle \quad (\widetilde{f}, \widetilde{g} \in L^\infty \cap L^1).$$

The following set function is associated with the semigroup T^t_∞:

$$P(t,x,G) := (T^t_\rho \mathbf{1}_G)(x) \quad (t>0, x \in X, G \in \mathcal{M}_\mu).$$

This function satisfies the following evident properties:

(1) $P(t,x,G)$ $(G \in \mathcal{M}_\mu)$ is finitely additive.
(2) $P(t,x,X) \leq 1$.
(3) $\int f(y)P(t,\cdot,dy)$ exists and equals to $T^t_\rho f(\cdot)$ $(f \in \mathcal{L}^\infty)$.

Set by definition

$$\mu_t(A \times B) = \int_A P(t,x,B)d\mu(x) \quad (A, B \in \mathcal{M}_\mu).$$

The claimed symmetry of μ_t is a direct consequence of the self-adjointness of T^t and the fact that we can identify $T^t_\infty \mathbf{1}_G$ and $T^t \mathbf{1}_G$ for every $G \in \mathcal{M}_\mu$ of finite measure.

2. We are in position to complete the proof of Theorem 10.

Proof of Theorem 10. We will need the following elementary estimates: for all $s,t \in [0,\infty[$, $r \in [1,\infty[$,

$$\begin{aligned}&\frac{4}{rr'}(s^r + t^r - 2b(st)^{\frac{r}{2}})\\ &\leq s^r + t^r - b(st^{r-1} + ts^{r-1})\\ &\leq \varkappa(r)(s^r + t^r - 2b(st)^{\frac{r}{2}}), \qquad b \in [-1,1]\end{aligned} \tag{$*$}$$

(Lemma 12(l_3), (l_5) below)

$$|a||st^{r-1} - ts^{r-1}| \leq \frac{|r-2|}{2\sqrt{r-1}}\big[s^r + t^r - \sqrt{1-a^2}(st^{r-1} + ts^{r-1})\big], \qquad a \in [-1,1] \tag{$**$}$$

(Lemma 12(l_4) below).

We are going to establish the following inequalities: for all $f \in L^r$

$$\frac{4}{rr'}\langle (1-T^t_2)f_{(r)}, f_{(r)}\rangle \leq \mathrm{Re}\langle (1-T^t_r)f, f|f|^{r-2}\rangle \leq \varkappa(r)\langle (1-T^t_2)f_{(r)}, f_{(r)}\rangle, \tag{25}$$

$$\big|\mathrm{Im}\langle (1-T^t_r)f, f|f|^{r-2}\rangle\big| \leq \frac{|r-2|}{2\sqrt{r-1}}\mathrm{Re}\langle (1-T^t_r)f, f|f|^{r-2}\rangle. \tag{26}$$

The the required estimates would follow from the definitions of A_r and $A^{\frac{1}{2}}$. Indeed, for $f \in D(A_r)$,

$$s\text{-}L^p\text{-}\lim_{t\downarrow 0}\frac{1}{t}(1-T^t_r)f \text{ exists and equals to } A_r f.$$

Combining the LHS of (25) and Fatou's Lemma, it is seen that $\mathcal{J} := \lim_{t\downarrow 0}\frac{1}{t}\langle (1-T^t)f_{(r)}, f_{(r)}\rangle$ exists and is finite. By the spectral theorem for self-adjoint operators, the latter means that $f_{(r)} \in D(A^{\frac{1}{2}})$ and $\mathcal{J} = \|A^{\frac{1}{2}}f_{(r)}\|_2^2$.

First, let $f \in L^1 \cap L^\infty$ with $\operatorname{sprt} f \subset G$, $G \in \mathcal{M}_\mu$, $\mu(G) < \infty$. Using Claim 9, we have

$$\begin{aligned}\langle T^t f, f|f|^{r-2}\rangle &= \frac{1}{2}\langle T^t f, f|f|^{r-2}\rangle + \frac{1}{2}\langle f, T^t(f|f|^{r-2})\rangle \\ &= \frac{1}{2}\int [f(x)\cdot \bar{f}(y)|f(y)|^{r-2} + f(y)\cdot \bar{f}(x)|f(x)|^{r-2}]d\mu_t(x,y),\end{aligned}$$

$$\langle T^t f_{(r)}, f_{(r)}\rangle = \frac{1}{2}\int f_{(r)}(x)\cdot \bar{f}_{(r)}(y)d\mu_t(x,y) + \frac{1}{2}\int \bar{f}_{(r)}(x)\cdot f_{(r)}(y)d\mu_t(x,y),$$

$$\begin{aligned}\langle T^t \mathbf{1}_G, |f|^r\rangle &= \langle \mathbf{1}_G, T^t|f|^r\rangle \\ &= \frac{1}{2}\langle P(t,\cdot,G)|f(\cdot)|^r\rangle + \frac{1}{2}\langle \mathbf{1}_G(\cdot)\int |f(y)|^r P(t,\cdot,dy)\rangle \\ &= \frac{1}{2}\int [|f(x)|^r + |f(y)|^r]d\mu_t(x,y),\end{aligned}$$

$$\|f\|_r^r = \langle T^t \mathbf{1}_G, |f|^r\rangle + \langle (1 - T^t\mathbf{1}_G), |f|^r\rangle.$$

Setting $s := |f(x)|$, $l := |f(y)|$, $\beta := \frac{f(x)\cdot\bar{f}(y)}{|f(x)||f(y)|}$, $b := \operatorname{Re}\beta$, $a := \operatorname{Im}\beta$, we obtain

$$\langle (1-T^t)f, f|f|^{r-2}\rangle = \langle (1-T^t\mathbf{1}_G), |f|^r\rangle + \frac{1}{2}\int [s^r + l^r - \beta s l^{r-1} - \bar{\beta} l s^{r-1})]d\mu_t,$$

$$\operatorname{Re}\langle (1-T^t)f, f|f|^{r-2}\rangle = \langle (1-T^t\mathbf{1}_G), |f|^r\rangle + \frac{1}{2}\int [s^r + l^r - b(s l^{r-1} + l s^{r-1})]d\mu_t,$$

$$\langle (1-T^t)f_{(r)}, f_{(r)}\rangle = \langle (1-T^t\mathbf{1}_G), |f|^r\rangle + \frac{1}{2}\int [s^r + l^r - 2b(st)^{\frac{r}{2}}]d\mu_t,$$

$$\operatorname{Im}\langle (1-T^t)f, f|f|^{r-2}\rangle = \frac{1}{2}\int a(s l^{r-1} - l s^{r-1})d\mu_t.$$

Next, employing $(*)$, $(**)$, we obtain (25), (26) but for $f \in L^1 \cap L^\infty$ with $\operatorname{sprt} f \in G$, $\mu(G) < \infty$.

To end the proof, we note that μ is a σ-finite measure, and so we can first get rid of the condition "$\operatorname{sprt} f \in G$, $\mu(G) < \infty$", and then, using the truncated functions

$$g_n = \begin{cases} g, & \text{if } |g| \le n, \\ 0, & \text{if } |g| > n, \end{cases} \qquad n = 1, 2, \dots$$

and the Dominated Convergence Theorem, to get rid of "$f \in L^1 \cap L^\infty$". □

For the sake of completeness, we also include the following result concerning the scalar case.

Theorem 11. *If $0 \le f \in D(A_r)$, then*

$$\frac{4}{rr'}\|A^{\frac{1}{2}} f^{\frac{r}{2}}\|_2^2 \le \langle A_r f, f^{r-1}\rangle \le \|A^{\frac{1}{2}} f^{\frac{r}{2}}\|_2^2; \tag{iii}$$

Moreover, if $r \in [2, \infty[$ and $f \in D(A) \cap L^\infty$, then $f_{(r)} := |f|^{\frac{r}{2}}\operatorname{sgn} f \in D(A^{\frac{1}{2}})$ and

$$\frac{4}{rr'}\|A^{\frac{1}{2}} f_{(r)}\|_2^2 \le \operatorname{Re}\langle Af, f^{r-1}\operatorname{sgn} f\rangle \le \varkappa(r)\|A^{\frac{1}{2}} f_{(r)}\|_2^2, \qquad \operatorname{sgn} f := \frac{f}{|f|} \tag{i'}$$

If $r \in [2,\infty[$ *and* $0 \leq f \in D(A) \cap L^\infty$, *then* $f^{\frac{r}{2}} \in D(A^{\frac{1}{2}})$ *and*

$$\frac{4}{rr'}\|A^{\frac{1}{2}}f^{\frac{r}{2}}\|_2^2 \leq \langle Af, f^{r-1}\rangle \leq \|A^{\frac{1}{2}}f^{\frac{r}{2}}\|_2^2. \qquad (iii')$$

Proof. Follows closely the proof of Theorem 10 where, instead of inequalities (25), (26), we use

$$\frac{4}{rr'}\langle(1-T^t)f^{\frac{r}{2}}, f^{\frac{r}{2}}\rangle \leq \langle(1-T^t)f, f^{r-1}\rangle \leq \langle(1-T^t)f^{\frac{r}{2}}, f^{\frac{r}{2}}\rangle \quad (f \in L^r_+).$$

□

In the proof of Theorem 10 we use

Lemma 12. *Let* $s,t \in [0,\infty[$, $r \in [1,\infty[$ *and* $b \in [-1,1]$. *Then*

$$\frac{4}{rr'}(s^{\frac{r}{2}} - t^{\frac{r}{2}})^2 \leq (s-t)(s^{r-1} - t^{r-1}) \leq (s^{\frac{r}{2}} - t^{\frac{r}{2}})^2. \qquad (l_1)$$

$$(s^{\frac{r}{2}} + t^{\frac{r}{2}})^2 \leq (s+t)(s^{r-1} + t^{r-1}) \leq \varkappa(r)(s^{\frac{r}{2}} + t^{\frac{r}{2}})^2 \qquad (l_2)$$

$$\frac{4}{rr'}(s^{\frac{r}{2}} + t^{\frac{r}{2}} + 2b(st)^{\frac{r}{2}}) \leq s^r + t^r + b(st^{r-1} + ts^{r-1}). \qquad (l_3)$$

$$|b||st^{r-1} - ts^{r-1}| \leq \frac{|r-2|}{2\sqrt{r-1}}\big[s^r + t^r - \sqrt{1-b^2}(st^{r-1} + ts^{r-1})\big]. \qquad (l_4)$$

$$s^r + t^r + b(st^{r-1} + ts^{r-1}) \leq \varkappa(r)(s^r + t^r + 2b(st)^{\frac{r}{2}}). \qquad (l_5)$$

Proof. The RHS of (l_1) and the LHS of (l_2) are consequences of the inequality $2|\alpha||\beta| \leq \alpha^2 + \beta^2$.

The RHS of (l_2) follows from the definition of $\varkappa(r)$.

The LHS of (l_1) follows from

$$\frac{4}{r^2}(s^{\frac{r}{2}} - t^{\frac{r}{2}})^2 = (\int_t^s z^{\frac{r}{2}-1}dz)^2 \leq \int_t^s dz \cdot \int_t^s z^{r-2}dz.$$

(l_3) is a consequence of the LHS of (l_1).

To derive (l_4) set

$$A = st^{r-1} - ts^{r-1}, B = \frac{|r-2|}{2\sqrt{r-1}}(st^{r-1} + ts^{r-1}), C = \frac{|r-2|}{2\sqrt{r-1}}(s^r + t^r),$$

and note that $A^2 + B^2 \leq C^2 \Rightarrow |A\sin\theta| + |B\cos\theta| \leq C$.

The inequality $A^2 + B^2 \leq C^2$ follows from

$$(st^{r-1} - ts^{r-1})^2 \leq \left(\frac{r-2}{r}\right)^2 (s^r - t^r)^2 \qquad (\star)$$

and the LHS of (l_1) and (l_2).

Setting $v = s/t$, $(\star)$ takes the form

$$|v^{r-1} - v| \leq \frac{|r-2|}{r}|v^r - 1|.$$

All possible cases are reduced to the case where $v > 1$ and $r > 2$.

If $\frac{r-2}{r}v \geq 1$, then the inequality $v^{r-1} - v \leq \frac{r-2}{r}v^r - \frac{r-2}{r}$ is selfevident. If $1 < v < \frac{r}{r-2}$, we set $\psi(v) = \frac{r-2}{r}v^r - v^{r-1} + v - \frac{r-2}{r}$ and note that $\frac{d}{dv}\psi(v) \geq 0$ by Young's inequality.

Finally, (l_5) follows from the RHS of (l_2) and the following elementary inequality:

$$\frac{A+bB}{A+bC} \leq \frac{A+B}{A+C} \quad (b \in [-1,1]), \text{ provided that } A > C \text{ and } B \geq C > 0.$$

□

Appendix B. Extrapolation Theorem

Theorem 13 (T. Coulhon-Y. Raynaud. [VSC, Prop. II.2.1, Prop. II.2.2].). *Let $U^{t,s} : L^1 \cap L^\infty \to L^1 + L^\infty$ be a two-parameter evolution family of operators:*

$$U^{t,s} = U^{t,\tau} U^{\tau,s}, \quad 0 \le s < \tau < t \le \infty.$$

Suppose that, for some $1 \le p < q < r \le \infty$, $\nu > 0$, M_1 and M_2, the inequalities

$$\|U^{t,s} f\|_p \le M_1 \|f\|_p \quad \text{and} \quad \|U^{t,s} f\|_r \le M_2 (t-s)^{-\nu} \|f\|_q$$

are valid for all (t,s) and $f \in L^1 \cap L^\infty$. Then

$$\|U^{t,s} f\|_r \le M (t-s)^{-\nu/(1-\beta)} \|f\|_p,$$

where $\beta = \frac{r}{q}\frac{q-p}{r-p}$ and $M = 2^{\nu/(1-\beta)^2} M_1 M_2^{1/(1-\beta)}$.

Proof. Set $2t_s = t+s$. The hypotheses and Hölder's inequality imply

$$\begin{aligned} \|U^{t,s} f\|_r &\le M_2 (t-t_s)^{-\nu} \|U^{t_s,s} f\|_q \\ &\le M_2 (t-t_s)^{-\nu} \|U^{t_s,s} f\|_r^\beta \, \|U^{t_s,s} f\|_p^{1-\beta} \\ &\le M_2 M_1^{1-\beta} (t-t_s)^{-\nu} \|U^{t_s,s} f\|_r^\beta \, \|f\|_p^{1-\beta}, \end{aligned}$$

and hence

$$(t-s)^{\nu/(1-\beta)} \|U^{t,s} f\|_r / \|f\|_p \le M_2 M_1^{1-\beta} 2^{\nu/(1-\beta)} \big[(t_s - s)^{\nu/(1-\beta)} \|U^{t_s,s} f\|_r \, / \|f\|_p\big]^\beta.$$

Setting $R_{2T} := \sup_{t-s \in]0,T]} \big[(t-s)^{\nu/(1-\beta)} \|U^{t,s} f\|_r / \|f\|_p\big]$, we obtain from the last inequality that $R_{2T} \le M^{1-\beta} (R_T)^\beta$. But $R_T \le R_{2T}$, and so $R_{2T} \le M$. □

Corollary 4. *Let $U^{t,s} : L^1 \cap L^\infty \to L^1 + L^\infty$ be an evolution family of operators. Suppose that, for some $1 < p < q < r \le \infty$, $\nu > 0$, M_1 and M_2, the inequalities*

$$\|U^{t,s} f\|_r \le M_1 \|f\|_r \quad \text{and} \quad \|U^{t,s} f\|_q \le M_2 (t-s)^{-\nu} \|f\|_p$$

are valid for all (t,s) and $f \in L^1 \cap L^\infty$. Then

$$\|U^{t,s} f\|_r \le M (t-s)^{-\nu/(1-\beta)} \|f\|_p,$$

where $\beta = \frac{r}{q}\frac{q-p}{r-p}$ and $M = 2^{\nu/(1-\beta)^2} M_1 M_2^{1/(1-\beta)}$.

Appendix C. The range of an accretive operator

In the proof of Theorem 2 we use the following well known result.

Let P be a closed operator on L^1 such that $\mathrm{Re}\langle(\lambda + P)f, \frac{f}{|f|}\rangle \ge 0$ for all $f \in D(P)$, and $R(\mu + P)$ is dense in L^1 for a $\mu > \lambda$.

Then $R(\mu + P) = L^1$.

Indeed, let $y_n \in R(\mu+P)$, $n=1,2,\dots$, be a Cauchy sequence in L^1; $y_n=(\mu+P)x_n$, $x_n \in D(P)$. Write $[f,g] := \langle f, \frac{g}{|g|}\rangle$. Then

$$\begin{aligned}(\mu-\lambda)\|x_n-x_m\|_1 &= (\mu-\lambda)[x_n-x_m, x_n-x_m] \\ &\leq (\mu-\lambda)[x_n-x_m,x_n-x_m] + [(\lambda+P)(x_n-x_m), x_n-x_m] \\ &= [(\mu+P)(x_n-x_m), x_n-x_m] \leq \|y_n-y_m\|_1.\end{aligned}$$

Thus, $\{x_n\}$ is itself a Cauchy sequence in L^1. Since P is closed, the result follows.

References

[BJ] K. Bogdan and T. Jakubowski, *Estimates of heat kernel of fractional Laplacian perturbed by gradient operators*, Comm. Math. Phys., **271** (2007), p. 179-198.

[CKSV] S. Cho, P. Kim, R. Song and Z. Vondraček, *Factorization and estimates of Dirichlet heat kernels for non-local operators with critical killings*, arXiv:1809.01782 (2018).

[JW] T. Jakubowski and J. Wang. *Heat kernel estimates for fractional Schrödinger operators with negative Hardy potential,* arXiv:1809.02425 (2018).

[Ka] T. Kato. Perturbation Theory for Linear Operators. *Springer-Verlag Berlin Heidelberg*, 1995.

[KM] D. Kinzebulatov and K. R. Madou, *On admissible singular drifts of symmetric α-stable process*, Preprint, arXiv:2002.07001 (2020).

[KS] D. Kinzebulatov and Yu. A. Semënov, *On the theory of the Kolmogorov operator in the spaces L^p and C_∞.* Ann. Sc. Norm. Sup. Pisa (5), to appear.

[KSS] D. Kinzebulatov, Yu. A. Semënov and K. Szczypkowski. *Heat kernel of fractional Laplacian with Hardy drift via desingularizing weights.* arXiv:1904.07363 (2019).

[LS] V. A. Liskevich, Yu. A. Semënov, *Some problems on Markov semigroups*, In: "Schrödinger Operators, Markov Semigroups, Wavelet Analysis, Operator Algebras" M. Demuth et al. (eds.), Mathematical Topics: Advances in Partial Differential Equations, 11, Akademie Verlag, Berlin (1996), 163-217.

[MM] Y. Maekawa, H. Miura, *Upper bounds for fundamental solutions to non-local diffusion equations with divergence free drift*, J. Funct. Anal., **264** (2013), p. 2245-2268.

[MM2] Y. Maekawa, H. Miura, *On fundamental solutions for non-local parabolic equations with divergence free drift*, Adv. Math., **247** (2013), p. 123-191.

[MeSS] G. Metafune, M. Sobajima and C. Spina, *Kernel estimates for elliptic operators with second order discontinuous coefficients*, J. Evol. Equ. 17 (2017), p. 485-522.

[MeSS2] G. Metafune, L. Negro and C. Spina, *Sharp kernel estimates for elliptic operators with second-order discontinuous coefficients*, J. Evol. Equ. 18 (2018), p. 467-514.

[MS0] P. D. Milman and Yu. A. Semënov, *Desingularizing weights and heat kernel bounds*, Preprint (1998).

[MS1] P. D. Milman and Yu. A. Semënov, *Heat kernel bounds and desingularizing weights*, J. Funct. Anal., **202** (2003), p. 1-24.

[MS2] P. D. Milman and Yu. A. Semënov, *Global heat kernel bounds via desingularizing weights*, J. Funct. Anal., **212** (2004), p. 373-398.

[N] J. Nash. *Continuity of solutions of parabolic and elliptic equations*, Amer. Math. J, **80** (1) (1958), p. 931-954.

[VSC] N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon. "Analysis and Geometry on Groups", Cambridge Univ. Press, 1992.