$\mathcal{W}^{\alpha,p}$ AND $C^{0,\gamma}$ REGULARITY OF SOLUTIONS TO $(\mu-\Delta+b\cdot\nabla)u=f$ WITH FORM-BOUNDED VECTOR FIELDS

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Abstract. We consider the operator $-\Delta+b\cdot\nabla$ with $b:\mathbb{R}^d\to\mathbb{R}^d$ ($d\geq 3$) in the class of form-bounded vector fields (containing vector fields having critical-order singularities), and characterize quantitative dependence of the $\mathcal{W}^{1+\frac{2}{q},p}$ ($2\leq p<q$) and the $C^{0,\gamma}$ regularity of solutions to the corresponding elliptic equation in L^p on the value of the form-bound of b.

Let $d\geq 3$. Consider the formal differential expression

$$-\Delta+b\cdot\nabla,\quad b:\mathbb{R}^d\to\mathbb{R}^d, \tag{1}$$

with b in the class of form-bounded vector fields $\mathbf{F}_\delta$, $\delta>0$, i.e. $|b|\in L^2_{\rm loc}\equiv L^2_{\rm loc}(\mathbb{R}^d,\mathcal{L}^d)$ and there exists a constant $\lambda=\lambda_\delta>0$ such that

$$\||b|(\lambda-\Delta)^{-\frac{1}{2}}\|_{2\to 2}\leq\sqrt{\delta}$$

(see examples below). It has been established in [KS] that if $\delta<1$, then for every $p\in[2,2/\sqrt{\delta}[$ (1) has an operator realization $\Lambda_p(b)$ on L^p as the generator of a positivity preserving, L^∞ contraction, quasi contraction C_0 semigroup $e^{-t\Lambda_p(b)}$ such that $D(\Lambda_p(b))\subset W^{1,p}\cap W^{1,\frac{pd}{d-2}}$. Moreover, there exist constants $\mu_1\equiv\mu_1(d,p,\delta)>0$ and $K_i=K_i(d,p,\delta)>0$, $i=1,2$, such that $u:=(\mu+\Lambda_p(b))^{-1}f$, $f\in L^p$ satisfies for all $\mu>\mu_1$

$$\|\nabla u\|_p\leq K_1(\mu-\mu_1)^{-\frac{1}{2}}\|f\|_p,\qquad \|\nabla|\nabla u|^{\frac{p}{2}}\|_2^{\frac{2}{p}}\leq K_2(\mu-\mu_1)^{\frac{1}{p}-\frac{1}{2}}\|f\|_p.$$

In particular, if $\delta<1\wedge\big(\frac{2}{d-2}\big)^2$, there exists $p>2\vee(d-2)$ such that $u\in C^{0,\gamma}$, $\gamma=1-\frac{d-2}{p}$.

The next theorem improves on the regularity of u under the same constraints on δ:

Theorem 1 (Main result). *Let $d\geq 3$. Assume that $b\in\mathbf{F}_\delta$, $\delta<1$. Then for every $p\in\big[2,\frac{2}{\sqrt{\delta}}\big[$ the formal differential expression $-\Delta+b\cdot\nabla$ has an operator realization $\Lambda_p(b)$ on L^p as the generator of a positivity preserving, L^∞ contraction, quasi contraction C_0 semigroup $e^{-t\Lambda_p(b)}$ such that:*

(i) *The resolvent admits the representation*

$$\big(\mu+\Lambda_p(b)\big)^{-1}=\Theta(\mu,b),\quad \mu>\mu_0,$$

for a $\mu_0\equiv\mu_0(d,p,\delta)>0$, where

$$\Theta(\mu,b):=(\mu-\Delta)^{-1}-Q_p(1+T_p)^{-1}G_p,$$

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the operators $Q_p, G_p, T_p \in \mathcal{B}(L^p)$, $\|G_p\|_{p\to p} \le C_1\mu^{-\frac{1}{2}+\frac{1}{p}}$, $\|Q_p\|_{p\to p} \le C_2\mu^{-\frac{1}{2}-\frac{1}{p}}$, $\|T_p\|_{p\to p} \le c_{\delta,p} < 1$, *where* $c_{\delta,p} := \big(\frac{p}{2}\delta + \frac{p-2}{2}\sqrt{\delta}\big)^{\frac{1}{p}}\big(p-1-(p-1)\frac{p-2}{2}\sqrt{\delta} - \frac{p(p-2)}{4}\delta\big)^{-\frac{1}{p}}$,

$$G_p := b^{\frac{2}{p}}\cdot\nabla(\mu-\Delta)^{-1}, \quad b^{\frac{2}{p}} := |b|^{\frac{2}{p}-1}b,$$

and Q_p, T_p *are the extensions by continuity of densely defined (on* $\mathcal{E} := \bigcup_{\varepsilon>0} e^{-\varepsilon|b|}L^p$*) operators*

$$Q_p \restriction \mathcal{E} := (\mu-\Delta)^{-1}|b|^{1-\frac{2}{p}}, \quad T_p \restriction \mathcal{E} := b^{\frac{2}{p}}\cdot\nabla(\mu-\Delta)^{-1}|b|^{1-\frac{2}{p}}.$$

(ii) For each $2 \le r < p < q < \infty$ *and* $\mu > \mu_0$, *define*

$$G_p(r) := b^{\frac{2}{p}}\cdot\nabla(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{r}} \in \mathcal{B}(L^p), \qquad Q_p(q) := (\mu-\Delta)^{-\frac{1}{2}+\frac{1}{q}}|b|^{1-\frac{2}{p}} \quad \text{on } \mathcal{E}.$$

The extension of $Q_p(q)$ *by continuity we denote again by* $Q_p(q)$. *Then* $Q_p(q) \in \mathcal{B}(L^p)$ *and*

$$\Theta_p(\mu,b) = (\mu-\Delta)^{-1} - (\mu-\Delta)^{-\frac{1}{2}-\frac{1}{q}}Q_p(q)(1+T_p)^{-1}G_p(r)(\mu-\Delta)^{-\frac{1}{2}+\frac{1}{r}}, \qquad \mu > \mu_0.$$

Thus,

$$\big(\mu + \Lambda_p(b)\big)^{-1} \in \mathcal{B}\big(\mathcal{W}^{-1+\frac{2}{r},p},\ \mathcal{W}^{1+\frac{2}{q},p}\big) \tag{$\star$}$$

($\mathcal{W}^{\alpha,p}$ *is the Bessel potential space).*

(iii) By (i) and (ii), $D\big(\Lambda_p(b)\big) \subset \mathcal{W}^{1+\frac{2}{q},p}$ $(q > p)$. *In particular, by the Sobolev Embedding Theorem, for* $d \ge 4$, *if* $\delta < \big(\frac{2}{d-2}\big)^2$ *then there exists* $p > d-2$ *such that* $D\big(\Lambda_p(b)\big) \subset C^{0,\gamma}$, $\gamma < 1-\frac{d-2}{p}$. *(For* $d = 3$ *the corresponding inclusion can be improved, see remarks below.)*

(iv) $e^{-t\Lambda_p(b_n)} \to e^{-t\Lambda_p(b)}$ *strongly in* L^p *locally uniformly in* $t \ge 0$,

where $b_n := e^{\epsilon_n\Delta}(\mathbf{1}_n b)$, $\epsilon_n \downarrow 0$, $n \ge 1$, $\mathbf{1}_n$ *is the indicator of* $\{x \in \mathbb{R}^d \mid |x| \le n, |b(x)| \le n\}$, *and* $\Lambda_p(b_n) := -\Delta + b_n\cdot\nabla$, $D(\Lambda_p(b_n)) = \mathcal{W}^{2,p}$.

Remarks. 1. For $d = 3$, by the Miyadera Perturbation Theorem, the assumption $b \in \mathbf{F}_\delta$, $\delta < 1$ implies that $-\Lambda_2(b) = \Delta - b\cdot\nabla$ of domain $W^{2,2}$ is the generator of a C_0 semigroup in L^2, and hence, for $\mu > \lambda\delta$, $(\mu + \Lambda_2(b))^{-1} : L^2 \to W^{1,6}$. In particular, $D(\Lambda_2(b)) \subset C^{0,\gamma}$ with $\gamma = \frac{1}{2}$.

2. The class $\mathbf{F}_\delta$ contains a sub-critical class $[L^d + L^\infty]^d$ (with arbitrarily small form-bound δ) as well as vector fields having critical-order singularities, e.g. in the weak L^d class or the Campanato-Morrey class etc. See e.g. [KiS, sect. 4].

3. We say that $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $\mathbf{F}_\delta^{1/2}$, the class of *weakly* form-bounded vector fields, and write $b \in \mathbf{F}_\delta^{1/2}$, if $|b| \in L^1_{\rm loc}$ and there exists $\lambda = \lambda_\delta > 0$ such that

$$\||b|^{\frac{1}{2}}(\lambda-\Delta)^{-\frac{1}{4}}\|_{2\to 2} \le \sqrt{\delta}.$$

In [Ki, Theorem 1.3], [KiS, Theorem 4.3], we have constructed an operator realization $\Lambda_p(b)$ of $-\Delta + b\cdot\nabla$, $b \in \mathbf{F}_\delta^{1/2}$, $m_d\delta < 1$, $m_d := \pi^{\frac{1}{2}}(2e)^{-\frac{1}{2}}d^{\frac{d}{2}}(d-1)^{-\frac{d-1}{2}}$ as the generator of a positivity preserving, L^∞ contraction, holomorphic semigroup on L^p, $p \in]p_-, p_+[$, $p_\mp := \frac{2}{1\pm\sqrt{1-m_d\delta}}$, such that for all $1 \le r < p < q$

$$\big(\zeta + \Lambda_p(b)\big)^{-1} \in \mathcal{B}(\mathcal{W}^{-1+\frac{1}{r},p}, \mathcal{W}^{1+\frac{1}{q},p}) \tag{$\star\star$}$$

(cf. ($\star$)). In particular, if $m_d\delta < 4\frac{d-2}{(d-1)^2}$, then there exists a $p > d-1$ such that $D(\Lambda_p(b)) \subset C^{0,\gamma}$, $\gamma < 1 - \frac{d-1}{p}$.

(Despite the inclusion $\mathbf{F}_{\delta^2}\subsetneq\mathbf{F}_\delta^{1/2}$, see [KiS, sect. 4], these two classes should be viewed as essentially incomparable, for the corresponding regularity results hold under different assumptions on the value of δ.)

The proof of Theorem 1 follows the Hille-Trotter approach of [Ki], [KiS]. However, the proof of the crucial estimates in Proposition 1 below is based on [KS].

4. For $|b|\in L^{d,\infty}$, one can extract additional information about the regularity of $D(\Lambda_p(b))$ arguing as in remark 4 in [KiS, sect. 4.4].

5. Let $C_\infty:=\{f\in C(\mathbb{R}^d):\lim_{x\to\infty}f(x)=0\}$ (with the sup-norm). Theorem 1 allows to construct the generator $\Lambda_{C_\infty}(b)$ of an associated with $-\Delta+b\cdot\nabla$, $b\in\mathbf{F}_\delta$, $\delta<1\wedge\big(\frac{2}{d-2}\big)^2$ Feller semigroup as $(\mu+\Lambda_{C_\infty}(b))^{-1}:=\big(\Theta_p(\mu,b)\upharpoonright L^p\cap C_\infty\big)^{\text{clos}}_{C_\infty\to C_\infty}$, $p>2\vee(d-2)$, by repeating [KiS, proof of Theorem 4.4] (for $-\Delta+b\cdot\nabla$, $b\in\mathbf{F}_\delta^{1/2}$). Thus, we restore the result of [KS, Theorem 2]. This proof doesn't require the Moser-type iteration procedure $L^p\to L^\infty$ of [KS].

6. The proof of Theorem 1 extends directly to the operator studied in [KiS2]:

$$-\nabla\cdot a\cdot\nabla+b\cdot\nabla\equiv-\sum_{i,j=1}^d\nabla_i a_{ij}(x)\nabla_j+\sum_{k=1}^d b_k(x)\nabla_k,\qquad b\in\mathbf{F}_\delta,$$

with

$$a=I+c\,\mathsf{f}\otimes\mathsf{f}\qquad\text{where}\qquad c>-1,\quad \mathsf{f}\in\big[L^\infty\cap W^{1,2}_{\text{loc}}\big]^d,\quad \|\mathsf{f}\|_\infty=1,$$

$$\nabla_i\mathsf{f}\in\mathbf{F}_{\eta^i},\;\; i=1,2,\dots,d,\quad \eta:=\sum_{i=1}^d\eta^i$$

for all $p\geq 2$, c, η and δ satisfying the assumptions of [KiS2, Theorem 2]. (More generally, with $a=I+\sum_{\ell=1}^\infty c_\ell\,\mathsf{f}_\ell\otimes\mathsf{f}_\ell$, $\mathsf{f}_\ell\in\big[L^\infty\cap W^{1,2}_{\text{loc}}\big]^d$, $\|\mathsf{f}_\ell\|_\infty=1$ such that $\sum_{c_\ell<0}c_\ell>-1$, $\sum_{c_\ell>0}c_\ell<\infty$ and $\nabla_i\mathsf{f}_\ell\in\mathbf{F}_{\eta^i_\ell}$, $i=1,2,\dots,d$, for appropriate $\eta^i_\ell>0$.)

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1. Proof of Theorem 1

The following proposition is a new element in the Hille-Trotter approach of [Ki], [KiS].

Proposition 1. *(j) Set $G_p=b^{\frac{2}{p}}\cdot\nabla(\mu-\Delta)^{-1}$, $Q_p=(\mu-\Delta)^{-1}|b|^{1-\frac{2}{p}}$, $T_p=b^{\frac{2}{p}}\cdot\nabla(\mu-\Delta)^{-1}|b|^{1-\frac{2}{p}}$. Q_p, T_p are densely defined (on $\mathcal{E}$) operators. Then there exists $\mu_0\equiv\mu_0(d,p,\delta)>0$ such that*

$$\|G_p\|_{p\to p}\leq C_1\mu^{-\frac{1}{2}+\frac{1}{p}},\quad \|Q_p\|_{p\to p}\leq C_2\mu^{-\frac{1}{2}-\frac{1}{p}},\quad \|T_p\|_{p\to p}\leq c_{\delta,p}<1,\qquad \mu>\mu_0,$$

where $c_{\delta,p}:=\left(\frac{p}{2}\delta+\frac{p-2}{2}\sqrt{\delta}\right)^{\frac{1}{p}}\left(p-1-(p-1)\frac{p-2}{2}\sqrt{\delta}-\frac{p(p-2)}{4}\delta\right)^{-\frac{1}{p}}$.

(jj) Set $G_p(r)=b^{\frac{2}{p}}\cdot\nabla(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{r}}$, $Q_p(q)=(\mu-\Delta)^{-\frac{1}{2}+\frac{1}{q}}|b|^{1-\frac{2}{p}}$, where $2\leq r<p<q<\infty$. $Q_p(q)$ is a densely defined (on $\mathcal{E}$) operator. Then for $\mu>\mu_0$

$$\|G_p(r)\|_{p\to p}\leq K_{1,r},\qquad \|Q_p(q)\|_{p\to p}\leq K_{2,q}.$$

The extension of $Q_p(q)$ by continuity we denote again by $Q_p(q)$.

Proof. In what follows, we use notation

$$\langle h\rangle := \int_{\mathbb{R}^d} h(x)d\mathcal{L}^d, \quad \langle h,g\rangle := \langle h\bar{g}\rangle.$$

It suffices to consider the case $p>2$.

(j) (**a**) Set $u := (\mu-\Delta)^{-1}|b|^{1-\frac{2}{p}}f$, $0\le f\in L^p$. Then

$$\begin{aligned}\|T_pf\|_p^p &= \|b^{\frac{2}{p}}\nabla u\|_p^p = \langle |b|^2|\nabla u|^p\rangle \\ &= \||b|(\lambda-\Delta)^{-\frac12}(\lambda-\Delta)^{\frac12}|\nabla u|^{\frac{p}{2}}\|_2^2 \qquad (\lambda=\lambda_\delta) \\ &\le \||b|(\lambda-\Delta)^{-\frac12}\|_{2\to2}^2\|(\lambda-\Delta)^{\frac12}|\nabla u|^{\frac{p}{2}}\|_2^2 \\ &= \delta\|(\lambda-\Delta)^{\frac12}|\nabla u|^{\frac{p}{2}}\|_2^2 = \delta\big(\lambda\|\nabla u\|_p^p + \|\nabla|\nabla u|^{\frac{p}{2}}\|_2^2\big).\end{aligned}$$

It remains to prove the principal inequality

$$\delta\big(\lambda\|\nabla u\|_p^p + \|\nabla|\nabla u|^{\frac{p}{2}}\|_2^2\big) \le c_{\delta,p}^p\|f\|_p^p, \tag{$*$}$$

and conclude that $\|T_p\|_{p\to p}\le c_{\delta,p}$.

First, we prove an a priori variant of $(*)$, i.e. for $u := (\mu-\Delta)^{-1}|b|^{1-\frac{2}{p}}f$ with $b=b_n$. Since our assumptions on δ involve only strict inequalities, we may assume, upon selecting appropriate $\varepsilon_n\downarrow 0$, that $b_n\in\mathbf{F}_\delta$ with the same $\lambda=\lambda_\delta$ for all n.

Set

$$w := \nabla u, \quad I_q := \sum_{r=1}^d\langle(\nabla_r w)^2|w|^{p-2}\rangle, \quad J_q := \langle(\nabla|w|)^2|w|^{p-2}\rangle.$$

We multiply $(\mu-\Delta)u = |b|^{1-\frac{2}{p}}f$ by $\phi := -\nabla\cdot(w|w|^{p-2})$ and integrate by parts to obtain

$$\mu\|w\|_p^p + I_p + (p-2)J_p = \langle |b|^{1-\frac{2}{p}}f, -\nabla\cdot(w|w|^{p-2})\rangle, \tag{2}$$

where

$$\begin{aligned}&\langle |b|^{1-\frac{2}{p}}f, -\nabla\cdot(w|w|^{p-2})\rangle = \langle |b|^{1-\frac{2}{p}}f, (-\Delta u)|w|^{p-2} - (p-2)|w|^{p-3}w\cdot\nabla|w|\rangle \\ &(\text{use the equation } -\Delta u = -\mu u + |b|^{1-\frac{2}{p}}f) \\ &= \langle |b|^{1-\frac{2}{p}}f, \big(-\mu u + |b|^{1-\frac{2}{p}}f\big)|w|^{p-2}\rangle - (p-2)\langle |b|^{1-\frac{2}{p}}f, |w|^{p-3}w\cdot\nabla|w|\rangle.\end{aligned}$$

We have

1) $\langle |b|^{1-\frac{2}{p}}f, (-\mu u)|w|^{p-2}\rangle \le 0$,

2) $|\langle |b|^{1-\frac{2}{p}}f, |w|^{p-3}w\cdot\nabla|w|\rangle| \le \alpha J_p + \frac{1}{4\alpha}N_p$ $(\alpha>0)$, where $N_p := \langle |b|^{1-\frac{2}{p}}f, |b|^{1-\frac{2}{p}}f|w|^{p-2}\rangle$,

so, the RHS of (2) $\le (p-2)\alpha J_p + \big(1+\frac{p-2}{4\alpha}\big)N_p$, where, in turn,

$$\begin{aligned}N_p &\le \langle |b|^2|w|^p\rangle^{\frac{p-2}{p}}\langle f^p\rangle^{\frac{2}{p}} \\ &\le \frac{p-2}{p}\langle |b|^2|w|^p\rangle + \frac{2}{p}\|f\|_p^p \qquad (\text{use } b\in\mathbf{F}_\delta \Leftrightarrow \|b\varphi\|_2^2\le\delta\|\nabla\varphi\|_2^2+\lambda\delta\|\varphi\|_2^2,\ \varphi\in W^{1,2}) \\ &\le \frac{p-2}{p}\left(\frac{p^2}{4}\delta J_q + \lambda\delta\|w\|_p^p\right) + \frac{2}{p}\|f\|_p^p.\end{aligned}$$

Thus, applying $I_q\geq J_q$ in the LHS of (2), we obtain

$$\big(\mu-c_0\big)\|w\|_p^p+\left[p-1-(p-2)\left(\alpha+\frac{1}{4\alpha}\frac{p(p-2)}{4}\delta\right)-\frac{p(p-2)}{4}\delta\right]\frac{4}{p^2}\|\nabla|\nabla u|^{\frac{p}{2}}\|_2^2\leq\left(1+\frac{p-2}{4\alpha}\right)\frac{2}{p}\|f\|_p^p,$$

where $c_0=\frac{p-2}{p}\lambda\delta\big(1+\frac{p-2}{4\alpha}\big)$. It is now clear that one can find a sufficiently large $\mu_0\equiv\mu_0(d,p,\delta)>0$ so that, for all $\mu>\mu_0$, $(*)$ (with $b=b_n$) holds with

$$c_{\delta,p}^p=\delta\frac{p^2}{4}\frac{\left(1+\frac{p-2}{4\alpha}\right)\frac{2}{p}}{p-1-(p-2)\left(\alpha+\frac{1}{4\alpha}\frac{p(p-2)}{4}\delta\right)-\frac{p(p-2)}{4}\delta}\qquad\Big(\text{we select }\alpha=\frac{p}{4}\sqrt{\delta}\Big)$$
$$=\frac{\frac{p}{2}\delta+\frac{p-2}{2}\sqrt{\delta}}{p-1-(p-1)\frac{p-2}{2}\sqrt{\delta}-\frac{p(p-2)}{4}\delta},$$

as claimed. Finally, we pass to the limit $n\to\infty$ using Fatou's Lemma. The proof of $(*)$ is completed.

Remark 1. It is seen that $\sqrt{\delta}<\frac{2}{p}\Rightarrow c_{\delta,p}<1$. We also note that the above choice of α is the best possible.

(**b**) Set $u=(\mu-\Delta)^{-1}f$, $0\leq f\in L^p$. Then

$$\begin{aligned}\|G_pf\|_p^p&=\|b^{\frac{2}{p}}\cdot\nabla u\|_p^p\\ &(\text{we argue as in }(\mathbf{a}))\\ &\leq\delta\big(\lambda\|\nabla u\|_p^p+\|\nabla|\nabla u|^{\frac{p}{2}}\|_2^2\big),\end{aligned}$$

where, clearly, $\|\nabla u\|_p^p\leq\mu^{-\frac{p}{2}}\|f\|_p^p$. In turn, arguing as in (**a**), we arrive at $\mu\|w\|_p^p+I_p+(p-2)J_p=\langle f,-\nabla\cdot(w|w|^{p-2})$ $(w=\nabla u)$,

$$\mu\|w\|_p^p+(p-1)J_p\leq\langle f^2,|w|^{p-2}\rangle+(p-2)\langle f,|w|^{p-3}w\cdot\nabla|w|\rangle),$$

$$\mu\|w\|_p^p+(p-1)J_p\leq\langle f^2,|w|^{p-2}\rangle+(p-2)\big(\varepsilon J_p+\frac{1}{4\varepsilon}\langle f^2,|w|^{p-2}\rangle\big),\quad\varepsilon>0.$$

Selecting ε sufficiently small, we obtain

$$J_p\leq C_0\|w\|_p^{p-2}\|f\|_p^2.$$

Now, applying $\|w\|_p\leq\mu^{-\frac{1}{2}}\|f\|_p$, we arrive at $\|\nabla|\nabla u|^{\frac{p}{2}}\|_2^2\leq C\mu^{-\frac{p}{2}+1}\|f\|_p^p$. Hence, $\|G_pf\|_p\leq C_1\mu^{-\frac{1}{2}+\frac{1}{p}}\|f\|_p$ for all $\mu>\mu_0$.

(**c**) Set $u=(\mu-\Delta)^{-1}|b|^{1-\frac{2}{p}}f$ $(=Q_pf)$, $0\leq f\in L^p$. Then, multiplying $(\mu-\Delta)u=|b|^{1-\frac{2}{p}}f$ by u^{p-1}, we obtain

$$\mu\|u\|_p^p+\frac{4(p-1)}{p^2}\|\nabla u^{\frac{p}{2}}\|_2^2=\langle|b|^{1-\frac{2}{p}}f,u^{p-1}\rangle,$$

where we estimate the RHS using Young's inequality:

$$\langle|b|^{1-\frac{2}{p}}u^{\frac{p}{2}-1},fu^{\frac{p}{2}}\rangle\leq\varepsilon^{\frac{2p}{p-2}}\frac{p-2}{2p}\langle|b|^2u^p\rangle+\varepsilon^{-\frac{2p}{p+2}}\frac{p+2}{2p}\langle f^{\frac{2p}{p+2}}u^{\frac{p^2}{p+2}}\rangle\quad\varepsilon>0.$$

Using $b \in \mathbf{F}_\delta$ and selecting $\varepsilon > 0$ sufficiently small, we obtain that for any $\mu_1 > 0$ there exists $C > 0$ such that

$$(\mu - \mu_1)\|u\|_p^p \leq C\langle f^{\frac{2p}{p+2}} u^{\frac{p^2}{p+2}}\rangle, \qquad \mu > \mu_1.$$

Therefore, $(\mu - \mu_1)\|u\|_p^p \leq C\langle f^p\rangle^{\frac{2}{p+2}}\langle u^p\rangle^{\frac{p}{p+2}}$, so $\|u\|_p \leq C_2\mu^{-\frac{1}{2}-\frac{1}{p}}\|f\|_p$. The proof of (j) is completed.

(jj) Below we use the following formula: For every $0 < \alpha < 1$, $\mu > 0$,

$$(\mu - \Delta)^{-\alpha} = \frac{\sin \pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(t + \mu - \Delta)^{-1}dt.$$

We have

$$\begin{aligned} \|Q_p(q)f\|_p \leq\ & \|(\mu - \Delta)^{-\frac{1}{2}+\frac{1}{q}}|b|^{1-\frac{2}{p}}|f|\|_p \\ \leq\ & k_q \int_0^\infty t^{-\frac{1}{2}+\frac{1}{q}}\|(t+\mu-\Delta)^{-1}|b|^{1-\frac{2}{p}}|f|\|_p dt \\ & (\text{we use } (\mathbf{c})) \\ \leq\ & k_q C_2 \int_0^\infty t^{-\frac{1}{2}+\frac{1}{q}}(t+\mu)^{-\frac{1}{2}-\frac{1}{p}}dt\ \|f\|_p = K_{2,q}\|f\|_p, \quad f \in \mathcal{E}, \end{aligned}$$

where, clearly, $K_{2,q} < \infty$ due to $q > p$.

It suffices to consider the case $r > 2$. We have

$$\begin{aligned} \|G_p(r)f\|_p \leq\ & k_r \int_0^\infty t^{-\frac{1}{2}-\frac{1}{r}}\|b^{\frac{2}{p}}\cdot\nabla(t+\mu-\Delta)^{-1}f\|_p dt \\ & (\text{we use } (\mathbf{b})) \\ \leq\ & k_r C_1 \int_0^\infty t^{-\frac{1}{2}-\frac{1}{r}}(t+\mu)^{-\frac{1}{2}+\frac{1}{p}}dt\ \|f\|_p = K_{1,r}\|f\|_p, \quad f \in \mathcal{E}, \end{aligned}$$

where, clearly, $K_{1,r} < \infty$ due to $r < p$.

The proof of (jj) is completed. □

Remark 2. Proposition 1 is valid for b_n, $n = 1, 2, \dots$, with the same constants.

Proposition 2. *The operator-valued function $\Theta_p(\mu, b_n)$ is a pseudo-resolvent on $\mu > \mu_0$, i.e.*

$$\Theta_p(\mu, b_n) - \Theta_p(\nu, b_n) = (\nu - \mu)\Theta_p(\mu, b_n)\Theta_p(\nu, b_n), \quad \mu, \nu > \mu_0.$$

Proof. The proof repeats [Ki, proof of Prop. 2.4]. □

Proposition 3. *For every $n = 1, 2, \dots,$*

$$\mu\Theta_p(\mu, b_n) \to 1 \text{ strongly in } L^p \text{ as } \mu \uparrow \infty \quad (\text{uniformly in } n).$$

Proof. The proof repeats [Ki, proof of Prop. 2.5(ii)]. □

Proposition 4. *We have $\{\mu : \mu > \mu_0\} \subset \rho(-\Lambda_p(b_n))$, the resolvent set of $-\Lambda_p(b_n)$. The operator-valued function $\Theta_p(\mu, b_n)$ is the resolvent of $-\Lambda_p(b_n)$:*

$$\Theta_p(\mu, b_n) = (\mu + \Lambda_p(b_n))^{-1}, \quad \mu > \mu_0.$$

Proof. The proof repeats [Ki, proof of Prop. 2.6]. □

Proposition 5. *We have, for all $n=1,2,\dots$,*

$$\|(\mu+\Lambda_p(b_n))\|_{p\to p}\leq(\mu-\mu_0)^{-1},\quad \mu>\mu_0:=\mu_0\vee\frac{\lambda\delta}{2(p-1)}.$$

Proof. By [KS, Theorem 1]. □

Proposition 6. *For every $\mu>\mu_0$,*

$$\Theta_p(\mu,b_n)\to\Theta_p(\mu,b) \text{ strongly in } L^p.$$

Proof. The proof repeats [Ki, proof of Prop. 2.8]. □

Now, by the Trotter Approximation Theorem [Ka, IX.2.5], $\Theta_p(\mu,b)=(\mu+\Lambda_p(b))^{-1}$, $\mu>\mu_0$, where $\Lambda_p(b)$ is the generator of a quasi contraction C_0 semigroup in L^p. (i) follows. (ii) follows from Proposition 1(jj). $(ii)\Rightarrow(iii)$. (iv) is Proposition 6. The proof of Theorem 1 is completed.

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