$W^{1,p}$ REGULARITY OF SOLUTIONS TO KOLMOGOROV EQUATION AND ASSOCIATED FELLER SEMIGROUP

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ABSTRACT. In $\mathbb{R}^d$, $d \geq 3$, consider the divergence and the non-divergence form operators

$$-\nabla \cdot a \cdot \nabla + b \cdot \nabla, \tag{i}$$

$$-a \cdot \nabla^2 + b \cdot \nabla, \tag{ii}$$

where $a = I + c\mathsf{f} \otimes \mathsf{f}$, the vector fields $\nabla_i \mathsf{f}$ $(i = 1, 2, \dots, d)$ and b are form-bounded (this includes the sub-critical class $[L^d + L^\infty]^d$ as well as vector fields having critical-order singularities). We characterize quantitative dependence on c and the values of the form-bounds of the $L^q \to W^{1,qd/(d-2)}$ regularity of the resolvents of the operator realizations of (i), (ii) in L^q, $q \geq 2 \vee (d-2)$ as (minus) generators of positivity preserving L^∞ contraction C_0 semigroups. The latter allows to run an iteration procedure $L^p \to L^\infty$ that yields associated with (i), (ii) L^q-strong Feller semigroups.

1. Consider in $\mathbb{R}^d$, $d \geq 3$, the formal differential operator

$$-\nabla \cdot a \cdot \nabla + b \cdot \nabla \equiv -\sum_{i,j=1}^d \nabla_i a_{ij}(x) \nabla_j + \sum_{j=1}^d b_j(x) \nabla_j, \tag{1}$$

where

$$\begin{aligned} &a = a^* : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \quad \text{is } \mathcal{L}^d \text{ measurable}, \\ &\sigma I \leq a(x) \leq \xi I \quad \text{for } \mathcal{L}^d \text{ a.e. } x \in \mathbb{R}^d \text{ for some } 0 < \sigma \leq \xi < \infty. \end{aligned} \tag{H_u}$$

By the De Giorgi-Nash theory, the solution $u \in W^{1,2}(\mathbb{R}^d)$ to the corresponding elliptic equation $(\mu - \nabla \cdot a \cdot \nabla + b \cdot \nabla)u = f$, $\mu > 0$, $f \in L^p \cap L^2$, $p \in]\frac{d}{2}, \infty[$, is in $C^{0,\gamma}$, where the Hölder continuity exponent $\gamma \in]0,1[$ depends only on d and σ, ξ, provided that $b : \mathbb{R}^d \to \mathbb{R}^d$ is in the Nash class ($\supset [L^p + L^\infty]^d$, $p > d$) [S], but already the class $[L^d + L^\infty]^d$ is not admissible (e.g. it is easy to find $b \in [L^d + L^\infty]^d$ that makes the two-sided Gaussian bounds on the fundamental solution of (1) invalid). On the other hand, for $-\Delta + b \cdot \nabla$, the $C^{0,\gamma}$ regularity of solutions to the corresponding elliptic equations is known to hold for b having much stronger singularities. Recall that a vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ is in the class of form-bounded vector fields $\mathbf{F}_\delta \equiv \mathbf{F}_\delta(-\Delta)$, $\delta > 0$ if $|b| \in L^2_{\rm loc} \equiv L^2_{\rm loc}(\mathbb{R}^d)$ and there exist a constant $\lambda = \lambda_\delta > 0$ such that

$$\| |b| (\lambda - \Delta)^{-\frac{1}{2}} \|_{2 \to 2} \leqslant \sqrt{\delta}.$$

(The class $\mathbf{F}_\delta$ contains $[L^d + L^\infty]^d$ with δ arbitrarily small, as follows by the Sobolev Embedding Theorem, as well as vector fields having critical-order singularities such as $b(x) = \frac{d-2}{2}\sqrt{\delta}|x|^{-2}x$ (by Hardy's inequality) or, more generally, vector fields in $[L^{d,\infty} + L^\infty]^d$, the Campanato-Morrey class or the Chang-Wilson-T. Wolff class, with δ depending on the norm of the vector field in these classes, see

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e.g. [KiS] for details.) It has been established in [KS] that if $b \in \mathbf{F}_\delta$, $\delta < 1$, then for every $q \in [2, 2/\sqrt{\delta}[$ $-\Delta + b \cdot \nabla$ has an operator realization $\Lambda_q(b)$ on L^q as the generator of a positivity preserving, L^∞ contraction, quasi contraction C_0 semigroup $e^{-t\Lambda_q(b)}$ such that $u := (\mu + \Lambda_q(b))^{-1} f$, $f \in L^q$ satisfies

$$\|\nabla u\|_q \leq K_1(\mu - \mu_0)^{-\frac{1}{2}} \|f\|_q, \qquad \|\nabla |\nabla u|^{\frac{q}{2}}\|_2^{\frac{2}{q}} \leq K_2(\mu - \mu_0)^{\frac{1}{q} - \frac{1}{2}} \|f\|_q, \quad \mu > \mu_0,$$

for some constants $\mu_0 \equiv \mu_0(d, q, \delta) > 0$ and $K_i = K_i(d, q, \delta) > 0$, $i = 1, 2$. In particular, if $\delta < 1 \wedge \big(\frac{2}{d-2}\big)^2$, there exists $q > 2 \vee (d-2)$ such that $u \in C^{0,\gamma}$, $\gamma = 1 - \frac{d-2}{q}$. The explicit dependence of the regularity properties of u on δ (which effectively plays the role of a "coupling constant") is a crucial feature of the result in [KS].

In the present paper our concern is: to find a class of matrices $a \in (H_u)$ such that the operator (1) with $b \in \mathbf{F}_\delta$ admits a $W^{1,p}$ and $C^{0,\gamma}$ regularity theory. Below we consider

$$a = I + c\,\mathsf{f} \otimes \mathsf{f}, \quad c > -1, \quad \mathsf{f} \in \big[L^\infty \cap W^{1,2}_{\rm loc}\big]^d, \quad \|\mathsf{f}\|_\infty = 1, \tag{$\star$}$$

$$\nabla_i \mathsf{f} \in \mathbf{F}_{\delta^i}, \ \ \delta^i > 0, \ \ i = 1, 2, \dots, d, \quad \delta_{\mathsf{f}} := \textstyle\sum_{i=1}^d \delta^i. \tag{$\mathbf{C}_{\delta_{\mathsf{f}}}$}$$

The model example of such a is the matrix

$$a(x) = I + c|x|^{-2} x \otimes x, \quad x \in \mathbb{R}^d \tag{2}$$

having critical discontinuity at the origin, see [GS, GrS, KiS2, OGr] and references therein. (Replacing the requirement $\nabla_i \mathsf{f} \in \mathbf{F}_{\delta^i}$ by a more restrictive $\nabla_i \mathsf{f} \in [L^p + L^\infty]^d$, $p > d$, forces a to be Hölder continuous. On the other hand, a weaker condition $\nabla_i \mathsf{f} \in [L^p + L^\infty]^d$, $p < d$, is incompatible with the uniform ellipticity of a. The condition $(\mathbf{C}_{\delta_{\mathsf{f}}})$ ($\supsetneq \nabla_i \mathsf{f} \in [L^d + L^\infty]^d$) seems to be rather natural. We also note that the operator $-a \cdot \nabla^2$ with $\nabla_k a_{ij} \in L^{d,\infty}$ has been studied earlier in [AT], cf. the discussion below concerning the non-divergence form operators.)

In Theorems 1, 2 below we characterize quantitative dependence on c, δ, δ_{f} of the $L^q \to W^{1, qd/(d-2)}$ regularity of the resolvent of the operator realization of (1) as (minus) generator of positivity preserving L^∞ contraction C_0 semigroups in L^q, $q \geq 2 \vee (d-2)$.

Consider the non-divergence form operator

$$-a \cdot \nabla^2 + b \cdot \nabla \equiv -\sum_{i,j=1}^d a_{ij}(x) \nabla_i \nabla_j + \sum_{j=1}^d b_j(x) \nabla_j. \tag{3}$$

Write $-a \cdot \nabla^2 + b \cdot \nabla \equiv -\nabla \cdot a \cdot \nabla + (\nabla a + b) \cdot \nabla$, where $(\nabla a)_k := \sum_{i=1}^d (\nabla_i a_{ik})$, $k = 1, 2, \dots, d$. Then

$$\nabla a = c\big[({\rm div}\mathsf{f})\mathsf{f} + \mathsf{f} \cdot \nabla \mathsf{f}\big].$$

It is easily seen that the condition $(\mathbf{C}_{\delta_{\mathsf{f}}})$ yields $\nabla a \in \mathbf{F}_{\delta_a}$ with $\delta_a \leq |c|^2 (\sqrt{d} + 1)^2 \delta_{\mathsf{f}}$. The latter yields an analogue of Theorem 2 for (3) (Corollary 1 below).

Theorem 2 and Corollary 1 are needed to run an iteration procedure $L^p \to L^\infty$ that yields associated with (1), (3) Feller semigroups on $C_\infty = C_\infty(\mathbb{R}^d)$ (the space of all continuous functions vanishing at infinity endowed with the sup-norm), see Theorem 3 and Corollary 2 below.

In the same manner as it was done in [KiS3] for the operator $-\Delta + b \cdot \nabla$, the Feller process constructed in Corollary 2 admits a characterization as a weak solution to the stochastic differential equation

$$dX_t = -b(X_t)dt + \sqrt{2a(X_t)}\,dW_t, \quad X_0 = x_0 \in \mathbb{R}^d.$$

We plan to address this matter in another paper.

All the proofs below work for

$$a = I + \sum_{j=1}^{\infty} c_j \mathsf{f}_j \otimes \mathsf{f}_j, \quad \|\mathsf{f}_j\|_\infty = 1, \tag{4}$$

with f_j satisfying ($\mathbf{C}_{\delta_\mathsf{f}}$), and $c_+ := \sum_{c_j>0} c_j < \infty$, $c_- := \sum_{c_j<0} c_j > -1$. (A decomposition (4) can be obtained from the spectral decomposition of a general uniformly elliptic a.)

2. We now state our results in full.

Theorem 1 ($-\nabla \cdot a \cdot \nabla$)**.** *Let $d \geq 3$. Let $a = I + c\,\mathsf{f} \otimes \mathsf{f}$ be given by ($\star$).*

(i) The formal differential expression $-\nabla \cdot a \cdot \nabla$ has an operator realization A_q in L^q for all $q \in [1, \infty[$ as the (minus) generator of a positivity preserving L^∞ contraction C_0 semigroup.

(ii) Assume that ($\mathbf{C}_{\delta_\mathsf{f}}$) holds with δ_f, c and $q \geq 2 \vee (d-2)$ satisfying the following constraint:

$$-\big(1 + q\sqrt{\delta_\mathsf{f}}\big)^{-1} < c < \begin{cases} 16\big[q\sqrt{\delta_\mathsf{f}}\big(8 + q\sqrt{\delta_\mathsf{f}}\big)\big]^{-1} & \text{if } q\sqrt{\delta_\mathsf{f}} \leq 4, \\ \big(q\sqrt{\delta_\mathsf{f}} - 1\big)^{-1} & \text{if } q\sqrt{\delta_\mathsf{f}} \geq 4. \end{cases}$$

Then, for each $\mu > 0$ and $f \in L^q$, $u := (\mu + A_q)^{-1} f$ belongs to $W^{1,q} \cap W^{1,\frac{qd}{d-2}}$. Moreover, there exist constants $\mu_0 = \mu_0(d, q, c, \delta_\mathsf{f}) > 0$ and $K_l = K_l(d, q, c, \delta_\mathsf{f})$, $l = 1, 2$, such that, for all $\mu > \mu_0$,

$$\begin{aligned} \|\nabla u\|_q &\leq K_1 (\mu - \mu_0)^{-\frac{1}{2}} \|f\|_q, \\ \|\nabla u\|_{\frac{qd}{d-2}} &\leq K_2 (\mu - \mu_0)^{\frac{1}{q} - \frac{1}{2}} \|f\|_q. \end{aligned} \tag{$\star\star$}$$

Remarks. 1. δ_f effectively estimates from above the "size" of the discontinuities of a.

2. For the matrix (2), the constraints on c in Theorem 1 (and in other results below) can be substantially relaxed, see [KiS2].

Theorem 2 ($-\nabla \cdot a \cdot \nabla + b \cdot \nabla$)**.** *Let $d \geq 3$. Let $a = I + c\,\mathsf{f} \otimes \mathsf{f}$ be given by ($\star$). Let $b \in \mathbf{F}_\delta$.*

(i) If $\delta_1 := [1 \vee (1+c)^{-2}]\,\delta < 4$, then $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ has an operator realization $\Lambda_q(a, b)$ in L^q for all $q \in [\frac{2}{2-\sqrt{\delta_1}}, \infty[$ as the (minus) generator of a positivity preserving L^∞ contraction C_0 semigroup.

(ii) Assume that ($\mathbf{C}_{\delta_\mathsf{f}}$) holds, $\nabla a \in \mathbf{F}_{\delta_a}$, $\delta < 1 \wedge \big(\frac{2}{d-2}\big)^2$, δ_a, δ_f, c and $q \geq 2 \vee (d-2)$ satisfy the constraints:

$$0 < c < (q - 1 - Q) \begin{cases} \big[(q-1)\frac{q\sqrt{\delta_\mathsf{f}}}{2} + \frac{q^2(\sqrt{\delta_\mathsf{f}} + \sqrt{\delta})^2}{16} + (q-2)\frac{q^2\delta_\mathsf{f}}{16}\big]^{-1} & \text{if} \quad 1 - \frac{q\sqrt{\delta_\mathsf{f}}}{4} - \frac{q\sqrt{\delta}}{4} \geq 0, \\ \big(\frac{q^2\sqrt{\delta_\mathsf{f}}}{2} + (q-2)\frac{q^2\delta_\mathsf{f}}{16} + \frac{q\sqrt{\delta}}{2} - 1\big)^{-1} & \text{if} \quad 0 \leq 1 - \frac{q\sqrt{\delta_\mathsf{f}}}{4} < \frac{q\sqrt{\delta}}{4}, \\ \big[(q-1)\big(q\sqrt{\delta_\mathsf{f}} - 1\big) + \frac{q\sqrt{\delta}}{2}\big]^{-1} & \text{if} \quad 1 - \frac{q\sqrt{\delta_\mathsf{f}}}{4} < 0, \end{cases}$$

where $Q := \frac{q\sqrt{\delta}}{2}\big[q - 2 + \big(\sqrt{\delta_a} + \sqrt{\delta}\big)\frac{q}{2}\big]$, or

$$-(q - 1 - Q)\left[(q-1)\big(1 + q\sqrt{\delta_\mathsf{f}}\big) + \frac{q\sqrt{\delta}}{2}\right]^{-1} < c < 0.$$

Then there exist constants $\mu_0 = \mu_0(d, q, c, \delta, \delta_a, \delta_\mathsf{f}) > 0$ and $K_l = K_l(d, q, c, \delta, \delta_a, \delta_\mathsf{f})$, $l = 1, 2$, such that ($\star\star$) hold for $u := (\mu + \Lambda_q(a, b))^{-1} f$, $\mu > \mu_0$, $f \in L^q$.

Remarks. 1. Taking $c=0$ (then $\delta_a=0$), we recover in Theorem 2(ii) the result of [KS, Lemma 5]: $\delta<1\wedge\big(\frac{2}{d-2}\big)^2$.

2. Theorem 2(i) is an immediate consequence of the following general result. Let a be an $\mathcal{L}^d$ measurable uniformly elliptic matrix on $\mathbb{R}^d$. Set $A\equiv A_2:=[-\nabla\cdot a\cdot\nabla\upharpoonright C_c^\infty]^{\rm clos}_{2\to 2}$. A vector field $b:\mathbb{R}^d\to\mathbb{R}^d$ belongs to $\mathbf{F}_{\delta_1}(A)$, $\delta_1>0$, the class of form-bounded vector fields (with respect to A), if $b_a^2:=b\cdot a^{-1}\cdot b\in L^1_{\rm loc}$ and there exists a constant $\lambda=\lambda_{\delta_1}>0$ such that

$$\|b_a(\lambda+A)^{-\frac{1}{2}}\|_{2\to 2}\leq\sqrt{\delta_1}.$$

If $b\in\mathbf{F}_{\delta_1}(A)$, $\delta_1<4$, then $-\nabla\cdot a\cdot\nabla+b\cdot\nabla$ has an operator realization $\Lambda_q(a,b)$ in L^q for all $q\in\big[\frac{2}{2-\sqrt{\delta_1}},\infty\big[$ as the (minus) generator of a positivity preserving L^∞ contraction C_0 semigroup, see [KiS, Theorem 3.2].

Corollary 1 $(-a\cdot\nabla^2+b\cdot\nabla)$**.** *Let $d\geq 3$. Let $a=I+c\,\mathsf{f}\otimes\mathsf{f}$ be given by $(\star)$. Let $b\in\mathbf{F}_\delta$, $\nabla a\in\mathbf{F}_{\delta_a}$. Then $\nabla a+b\in\mathbf{F}_{\delta_2}$, $\sqrt{\delta_2}:=\sqrt{\delta_a}+\sqrt{\delta}$.*

(i) If $\delta_1:=\big[1\vee(1+c)^{-2}\big]\,\delta_2<4$, then $-a\cdot\nabla^2+b\cdot\nabla$ has an operator realization $\Lambda_q(a,\nabla a+b)$ in L^q for all $q\in\big[\frac{2}{2-\sqrt{\delta_1}},\infty\big[$ as the (minus) generator of a positivity preserving L^∞ contraction C_0 semigroup.

(ii) Assume that $(\mathbf{C}_{\delta_\mathsf{f}})$ holds, and $\delta_2<1\wedge\big(\frac{2}{d-2}\big)^2$, δ_a, δ_f, c, $q\geq 2\vee(d-2)$ satisfy the constraints:

$$0<c<(q-1-Q)\begin{cases}\Big[(q-1)\frac{q\sqrt{\delta_\mathsf{f}}}{2}+\frac{q^2(\sqrt{\delta_\mathsf{f}}+\sqrt{\delta_2})^2}{16}+(q-2)\frac{q^2\delta_\mathsf{f}}{16}\Big]^{-1} & \text{if}\quad 1-\frac{q\sqrt{\delta_\mathsf{f}}}{4}-\frac{q\sqrt{\delta_2}}{4}\geq 0,\\[2mm] \Big(\frac{q^2\sqrt{\delta_\mathsf{f}}}{2}+(q-2)\frac{q^2\delta_\mathsf{f}}{16}+\frac{q\sqrt{\delta_2}}{2}-1\Big)^{-1} & \text{if}\quad 0\leq 1-\frac{q\sqrt{\delta_\mathsf{f}}}{4}<\frac{q\sqrt{\delta_2}}{4},\\[2mm] \Big[(q-1)\big(q\sqrt{\delta_\mathsf{f}}-1\big)+\frac{q\sqrt{\delta_2}}{2}\Big]^{-1} & \text{if}\quad 1-\frac{q\sqrt{\delta_\mathsf{f}}}{4}<0,\end{cases}$$

where $Q:=\frac{q\sqrt{\delta_2}}{2}\big[q-2+\big(\sqrt{\delta_a}+\sqrt{\delta_2}\big)\frac{q}{2}\big]$, or

$$-\big(q-1-Q\big)\bigg[(q-1)\big(1+q\sqrt{\delta_\mathsf{f}}\big)+\frac{q\sqrt{\delta_2}}{2}\bigg]^{-1}<c<0.$$

Then there exist constants $\mu_0=\mu_0(d,q,c,\delta_2,\delta_a,\delta_\mathsf{f})>0$ and $K_l=K_l(d,q,c,\delta_2,\delta_a,\delta_\mathsf{f})$, $l=1,2$, such that the estimates $(\star\star)$ hold for $u=(\mu+\Lambda_q(a,\nabla a+b))^{-1}f$, $\mu>\mu_0$, $f\in L^q$.

Set $b_n:=e^{\epsilon_n\Delta}(\mathbf{1}_nb)$, $\epsilon_n\downarrow 0$, $n\geq 1$, where $\mathbf{1}_n$ is the indicator of $\{x\in\mathbb{R}^d\mid\ |x|\leq n,|b(x)|\leq n\}$. Also, set $\mathsf{f}_n:=(\mathsf{f}_n^i)_{i=1}^d$, $\mathsf{f}^i_n:=e^{\epsilon_n\Delta}(\eta_n\mathsf{f}^i)$, $\epsilon_n\downarrow 0$, $n\geq 1$, where

$$\eta_n(x):=\begin{cases}1, & \text{if } |x|<n,\\ n+1-|x|, & \text{if } n\leq|x|\leq n+1,\\ 0, & \text{if } |x|>n+1.\end{cases}\qquad(x\in\mathbb{R}^d)$$

Theorem 3 $(-\nabla\cdot a\cdot\nabla+b\cdot\nabla)$**.** *$(i)$ In the assumptions of Theorem 2(ii), the formal differential operator $-\nabla\cdot a\cdot\nabla+b\cdot\nabla$ has an operator realization $-\Lambda_{C_\infty}(a,b)$ as the generator of a positivity preserving contraction C_0 semigroup in C_∞ defined by*

$$e^{-t\Lambda_{C_\infty}(a,b)}:=s\text{-}C_\infty\text{-}\lim_n e^{-t\Lambda_{C_\infty}(a_n,b_n)}\quad(\textit{loc. uniformly in } t\geq 0),$$

where $a_n:=I+c\,\mathsf{f}_n\otimes\mathsf{f}_n\subset[C^\infty]^{d\times d}$, $\Lambda_{C_\infty}(a_n,b_n):=-\nabla\cdot a_n\cdot\nabla+b_n\cdot\nabla$, $D(\Lambda_{C_\infty}(a_n,b_n))=(1-\Delta)^{-1}C_\infty$.

(*ii*) [*The L^r-strong Feller property*] $\big((\mu+\Lambda_{C_\infty}(a,b))^{-1}\upharpoonright L^r\cap C_\infty\big)^{\rm clos}_{L^r\to C_\infty}\in\mathcal{B}(L^r,C^{0,1-\frac{d}{rj}})$ *for some* $r>d-2$ *and all* $\mu>\mu_0$.

(*iii*) *The integral kernel of* $e^{-t\Lambda_{C_\infty}(a,b)}$ *determines the transition probability function of a Feller process.*

Corollary 2 ($-a\cdot\nabla^2+b\cdot\nabla$)**.** (*i*) *In the assumptions of Corollary 1(ii), the formal differential operator* $-a\cdot\nabla^2+b\cdot\nabla$ *has an operator realization* $-\Lambda_{C_\infty}(a,\nabla a+b)$ *as the generator of a positivity preserving contraction* C_0 *semigroup in* C_∞ *defined by*

$$e^{-t\Lambda_{C_\infty}(a,\nabla a+b)}:=s\text{-}C_\infty\text{-}\lim_n e^{-t\Lambda_{C_\infty}(a_n,\nabla a_n+b_n)}\quad (\textit{loc. uniformly in } t\geq 0),$$

where $a_n=I+c\,\mathsf{f}_n\otimes\mathsf{f}_n\subset[C^\infty]^{d\times d}$, $\Lambda_{C_\infty}(a_n,\nabla a_n+b_n):=-a_n\cdot\nabla^2+b_n\cdot\nabla$, $D(\Lambda_{C_\infty}(a_n,\nabla a_n+b_n))=(1-\Delta)^{-1}C_\infty$.

(*ii*) [*The L^r-strong Feller property*] $\big((\mu+\Lambda_{C_\infty}(a,\nabla a+b))^{-1}\upharpoonright L^r\cap C_\infty\big)^{\rm clos}_{L^r\to C_\infty}\in\mathcal{B}(L^r,C^{0,1-\frac{d}{rj}})$ *for some* $r>d-2$ *and all* $\mu>\mu_0$.

(*iii*) *The integral kernel of* $e^{-t\Lambda_{C_\infty}(a,\nabla a+b)}$ *determines the transition probability function of a Feller process.*

Remarks. Since our assumptions on δ_{f}, δ_a and δ involve only strict inequalities, we may assume that

$$(\mathbf{C}_{\delta_{\mathsf{f}}})\text{ holds for } \mathsf{f}_n,\quad \nabla a_n\in\mathbf{F}_{\delta_a},\quad b_n\in\mathbf{F}_\delta\quad\text{ with } \lambda\neq\lambda(n) \tag{5}$$

for appropriate $\epsilon_n\downarrow 0$. In fact, the proofs work for any approximations $\{\mathsf{f}_n\}$, $\{b_n\}\subset[C^\infty]^d$ such that $\|\mathsf{f}_n\|_\infty=1$, (5) holds, and

$$\mathsf{f}_n\to\mathsf{f},\nabla_i\mathsf{f}_n\to\nabla_i\mathsf{f}\text{ strongly in }[L^2_{\rm loc}]^d,\quad i=1,2,\dots,d,$$
$$b_n\to b\text{ strongly in }[L^2_{\rm loc}]^d.$$

1. Proof of Theorem 1

Proof of (i). In what follows, we use notation

$$\langle h\rangle:=\int_{\mathbb{R}^d}h(x)dx,\quad \langle h,g\rangle:=\langle h\bar g\rangle.$$

Define $t[u,v]:=\langle\nabla u\cdot a\cdot\nabla\bar v\rangle$, $D(t)=W^{1,2}$. There is a unique self-adjoint operator $A\equiv A_2\geq 0$ on L^2 associated with the form t: $D(A)\subset D(t)$, $\langle Au,v\rangle=t[u,v]$, $u\in D(A)$, $v\in D(t)$. $-A$ is the generator of a positivity preserving L^∞ contraction C_0 semigroup $T^t_2\equiv e^{-tA}$, $t\geq 0$, on L^2. Then $T^t_r:=[T_t\upharpoonright L^r\cap L^2]_{L^r\to L^r}$ determines C_0 semigroup on L^r for all $r\in[1,\infty[$. The generator $-A_r$ of $T^t_r\,(\equiv e^{-tA_r})$ is the desired operator realization of $\nabla\cdot a\cdot\nabla$ in L^r, $r\in[1,\infty[$. Moreover, $(\mu+A_r)^{-1}$ is well defined on L^r for all $\mu>0$. This completes the proof of the assertion (i) of the theorem.

Proof of (ii). First, we prove an a priori variant of ($\star\star$). Set $a_n:=I+c\mathsf{f}_n\otimes\mathsf{f}_n$, where f_n have been defined in the beginning of the paper. Since our assumption on δ_{f} is a strict inequality, we may assume that $(\mathbf{C}_{\delta_{\mathsf{f}}})$ holds for f_n for all $n\geq 1$ with $\lambda\neq\lambda(n)$ for appropriate $\epsilon_n\downarrow 0$. We also note that $\|\mathsf{f}_n\|_\infty=1$.

Set $u\equiv u_n:=(\mu+A^n_q)^{-1}f$, $0\leq f\in C^1_c$, where $A^n_q:=-\nabla\cdot a_n\cdot\nabla$, $D(A^n_q)=W^{2,q}$, $n\geq 1$. Clearly, $0\leq u_n\in W^{3,q}$.

Denote $w \equiv w_n := \nabla u_n$. For brevity, below we omit the index n: $\mathsf{f} \equiv \mathsf{f}_n$, $a \equiv a_n$, $A_q \equiv A_q^n$. Set

$$I_q := \sum_{r=1}^d \langle (\nabla_r w)^2 |w|^{q-2} \rangle, \quad J_q := \langle (\nabla |w|)^2 |w|^{q-2} \rangle,$$

$$\bar{I}_q := \langle \big(\mathsf{f} \cdot \nabla w\big)^2 |w|^{q-2} \rangle, \quad \bar{J}_q := \langle (\mathsf{f} \cdot \nabla |w|)^2 |w|^{q-2} \rangle.$$

Set $[F,G]_- := FG - GF$.

1. We multiply $\mu u + A_q u = f$ by $\phi := -\nabla \cdot (w|w|^{q-2})$ and integrate:

$$\mu \langle |w|^q \rangle + \langle A_q w, w|w|^{q-2} \rangle + \langle [\nabla, A_q]_- u, w|w|^{q-2} \rangle = \langle f, \phi \rangle,$$

$$\mu \langle |w|^q \rangle + I_q + c\bar{I}_q + (q-2)(J_q + c\bar{J}_q) + \langle [\nabla, A_q]_- u, w|w|^{q-2} \rangle = \langle f, \phi \rangle.$$

The term to evaluate is this:

$$\langle [\nabla, A_q]_- u, w|w|^{q-2} \rangle := \sum_{r=1}^d \langle [\nabla_r, A_q]_- u, w_r |w|^{q-2} \rangle.$$

From now on, we omit the summation sign in repeated indices. Note that

$$[\nabla_r, A_q]_- = -\nabla \cdot (\nabla_r a) \cdot \nabla, \qquad (\nabla_r a)_{il} = c(\nabla_r \mathsf{f}^i)\mathsf{f}^l + c\mathsf{f}^i \nabla_r \mathsf{f}^l.$$

Thus,

$$\langle [\nabla_r, A_q]_- u, w_r |w|^{q-2} \rangle = c \Big\langle \big[(\nabla_r \mathsf{f}^i)\mathsf{f}^l + \mathsf{f}^i \nabla_r \mathsf{f}^l\big] w_l, \nabla_i (w_r |w|^{q-2}) \Big\rangle =: S_1 + S_2,$$

$$S_1 = c\big\langle (\nabla_r \mathsf{f}) \cdot (\nabla_r w)(\mathsf{f} \cdot w)|w|^{q-2} \big\rangle + c(q-2)\big\langle (\nabla_r \mathsf{f}) \cdot (\nabla |w|)(\mathsf{f} \cdot w) w_r |w|^{q-3} \big\rangle,$$

$$S_2 = c\big\langle (\nabla_r \mathsf{f}) \cdot w, (\mathsf{f} \cdot \nabla w_r)|w|^{q-2} \big\rangle + c(q-2)\big\langle (\nabla_r \mathsf{f}) \cdot w, w_r |w|^{q-3} \mathsf{f} \cdot \nabla |w| \big\rangle.$$

By the quadratic estimates and the condition $(\mathbf{C}_{\delta_\mathsf{f}})$,

$$S_1 \leq |c| \bigg[\alpha \bigg(\delta_\mathsf{f} \frac{q^2}{4} J_q + \lambda \delta_\mathsf{f} \|w\|_q^q \bigg) + \frac{1}{4\alpha} I_q \bigg] + |c|(q-2) \bigg[\alpha_1 \bigg(\delta_\mathsf{f} \frac{q^2}{4} J_q + \lambda \delta_\mathsf{f} \|w\|_q^q \bigg) + \frac{1}{4\alpha_1} J_q \bigg], \quad \alpha, \alpha_1 > 0$$

$$S_2 \leq |c| \bigg[\gamma \bigg(\delta_\mathsf{f} \frac{q^2}{4} J_q + \lambda \delta_\mathsf{f} \|w\|_q^q \bigg) + \frac{1}{4\gamma} \bar{I}_q \bigg] + |c|(q-2) \bigg[\gamma_1 \bigg(\delta_\mathsf{f} \frac{q^2}{4} J_q + \lambda \delta_\mathsf{f} \|w\|_q^q \bigg) + \frac{1}{4\gamma_1} \bar{J}_q \bigg], \quad \gamma, \gamma_1 > 0.$$

Thus, selecting $\alpha = \alpha_1 = \frac{1}{q\sqrt{\delta_\mathsf{f}}}$, we obtain the inequality

$$\begin{aligned} &\mu \|w\|_q^q + I_q + c\bar{I}_q + (q-2)(J_q + c\bar{J}_q) \\ &\leq |c| \bigg[q \frac{\sqrt{\delta_\mathsf{f}}}{4} J_q + \frac{q\sqrt{\delta_\mathsf{f}}}{4} I_q \bigg] + |c|(q-2) \frac{q\sqrt{\delta_\mathsf{f}}}{2} J_q \\ &+ |c| \bigg[\gamma \delta_\mathsf{f} \frac{q^2}{4} J_q + \frac{1}{4\gamma} \bar{I}_q \bigg] + |c|(q-2) \bigg[\gamma_1 \delta_\mathsf{f} \frac{q^2}{4} J_q + \frac{1}{4\gamma_1} \bar{J}_q \bigg] \\ &+ \mu_0 \|w\|_q^q + \langle f, \phi \rangle \end{aligned} \tag{6}$$

where $\mu_0 := |c|\lambda\sqrt{\delta_\mathsf{f}}\big(q^{-1} + \gamma\sqrt{\delta_\mathsf{f}}\big) + |c|(q-2)\lambda\sqrt{\delta_\mathsf{f}}\big(q^{-1} + \gamma_1\sqrt{\delta_\mathsf{f}}\big)$.

2. Let us prove that there exists constant $\eta > 0$ such that

$$(\mu - \mu_0)\|w\|_q^q + \eta J_q \leq \langle f, \phi \rangle. \tag{$*$}$$

Case $c > 0$. First, suppose that $1 - \frac{q\sqrt{\delta_{\mathsf f}}}{4} \geq 0$. We select $\gamma = \gamma_1 := \frac{1}{4}$, so the terms $\bar I_q$, $\bar J_q$ are no longer present in (6). By the assumption of the theorem $1 - c\frac{q\sqrt{\delta_{\mathsf f}}}{4} \geq 0$, so using $J_q \leq I_q$ we obtain

$$(\mu - \mu_0)\|w\|_q^q + (q-1)\left[1 - c\frac{q\sqrt{\delta_{\mathsf f}}}{2} - c\frac{q^2\delta_{\mathsf f}}{16}\right]J_q \leq \langle f, \phi\rangle,$$

where $\mu_0 = c\lambda\sqrt{\delta_{\mathsf f}}(q-1)\big(\frac{1}{q} + \frac{\sqrt{\delta_{\mathsf f}}}{4}\big)$ and the coefficient $[\dots]$ is strictly positive by the assumptions of the theorem.

Now, suppose that $1 - \frac{q\sqrt{\delta_{\mathsf f}}}{4} < 0$. We select $\gamma = \gamma_1 := \frac{1}{q\sqrt{\delta_{\mathsf f}}}$ and replace $\bar J_q$, $\bar I_q$ by J_q, I_q. Then, since $1 - c\big(\frac{q\sqrt{\delta_{\mathsf f}}}{2} - 1\big) \geq 0$ by the assumptions of the theorem, we apply $J_q \leq I_q$ to obtain

$$(\mu - \mu_0)\|w\|_q^q + (q-1)\left[1 - c\big(q\sqrt{\delta_{\mathsf f}} - 1\big)\right]J_q \leq \langle f, \phi\rangle,$$

where $\mu_0 = c\lambda\sqrt{\delta_{\mathsf f}}(q-1)\big(\frac{1}{q} + \frac{1}{q}\big)$ and the coefficient $[\dots]$ is strictly positive by the assumption of the theorem. We have proved $(*)$ with $\mu_0 = c\lambda\sqrt{\delta_{\mathsf f}}(q-1)\big(\frac{1}{q} + \frac{\sqrt{\delta_{\mathsf f}}}{4} \vee \frac{1}{q}\big)$.

Remark. Elementary considerations show that the above choice of α, α_1, γ, γ_1 is the best possible.

Case $c < 0$. We select $\gamma = \gamma_1 := \frac{1}{q\sqrt{\delta_{\mathsf f}}}$, so that

$$\begin{aligned}
&\mu\|w\|_q^q + \left(1 - |c|\frac{q\sqrt{\delta_{\mathsf f}}}{4}\right)I_q + \left[q - 2 - |c|(q-1)\frac{q\sqrt{\delta_{\mathsf f}}}{2} - |c|(q-2)\frac{q\sqrt{\delta_{\mathsf f}}}{4}\right]J_q \\
&\leq |c|\left(1 + \frac{q\sqrt{\delta_{\mathsf f}}}{4}\right)\bar I_q + |c|(q-2)\left(1 + \frac{q\sqrt{\delta_{\mathsf f}}}{4}\right)\bar J_q + \mu_0\|w\|_q^q + \langle f, \phi\rangle,
\end{aligned}$$

where $\mu_0 = 2c\lambda\sqrt{\delta_{\mathsf f}}\frac{q-1}{q}$. Next, using $\bar I_q \leq I_q$, $\bar J_q \leq J_q$, we obtain

$$\begin{aligned}
&(\mu - \mu_0)\|w\|_q^q + \left(1 - |c| - |c|\frac{q\sqrt{\delta_{\mathsf f}}}{2}\right)I_q \\
&+ \left[q - 2 - |c|(q-1)\frac{q\sqrt{\delta_{\mathsf f}}}{2} - |c|(q-2)\frac{q\sqrt{\delta_{\mathsf f}}}{4} - |c|(q-2)\left(1 + \frac{q\sqrt{\delta_{\mathsf f}}}{4}\right)\right]J_q \leq \langle f, \phi\rangle.
\end{aligned}$$

By the assumptions of the theorem, $1 - |c| - |c|\frac{q\sqrt{\delta_{\mathsf f}}}{2} \geq 0$. Therefore, by $I_q \geq J_q$,

$$(\mu - \mu_0)\|w\|_q^q + \left[q - 1 - |c|(q-1) - |c|q^2\sqrt{\delta_{\mathsf f}}\right]J_q \leq \langle f, \phi\rangle,$$

and hence the coefficient $[\dots]$ is strictly positive. We have proved $(*)$.

3. We estimate the term $\langle f, \phi\rangle$ as follows.

Lemma 1. *For each $\varepsilon_0 > 0$ there exists a constant $C = C(\varepsilon_0) < \infty$ such that*

$$\langle f, \phi\rangle \leq \varepsilon_0 I_q + C\|w\|_q^{q-2}\|f\|_q^2.$$

Proof of Lemma 1. We have:

$$\langle f, \phi\rangle = \langle -\Delta u, |w|^{q-2}f\rangle + (q-2)\langle |w|^{q-3}w \cdot \nabla|w|, f\rangle =: F_1 + F_2.$$

Due to $|\Delta u|^2 \leq d|\nabla_r w|^2$ and $\langle |w|^{q-2} f\rangle \leq \|w\|_q^{q-2}\|f\|_q^2$,

$$F_1 \leq \sqrt{d} I_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|f\|_q, \qquad F_2 \leq (q-2) J_q^{\frac{1}{2}} \|w\|_q^{\frac{q-2}{2}} \|f\|_q,$$

Now the standard quadratic estimates yield the lemma. □

We choose $\varepsilon_0 > 0$ in Lemma 1 so small that in the estimates below we can ignore $\varepsilon_0 I_q$.

4. Clearly, $(*)$ yields the inequalities

$$\|\nabla u_n\|_q \leq K_1(\mu-\mu_0)^{-\frac{1}{2}}\|f\|_q, \quad K_1 := C^{\frac{1}{2}},$$

$$\|\nabla u_n\|_{qj} \leq K_2(\mu-\mu_0)^{-\frac{1}{2}+\frac{1}{q}}\|f\|_q, \quad K_2 := C_S \eta^{-\frac{1}{q}}(q^2/4)^{\frac{1}{q}} C^{\frac{1}{2}-\frac{1}{q}},$$

where C_S is the constant in the Sobolev Embedding Theorem. So, [KiS, Theorem 3.5] $((\mu + A_q)^{-1} = s\text{-}L^q\text{-}\lim_n(\mu + A_q^n)^{-1})$ yields $(\star\star)$. The proof of Theorem 1 is completed.

2. Proof of Theorem 2

Proof of (i). Recall that a vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $\mathbf{F}_{\delta_1}(A)$, $\delta_1 > 0$, the class of form-bounded vector fields (with respect to $A \equiv A_2 := [-\nabla \cdot a \cdot \nabla \upharpoonright C_c^\infty]_{2\to 2}^{\text{clos}})$, if $b_a^2 := b \cdot a^{-1} \cdot b \in L^1_{\text{loc}}$ and there exists a constant $\lambda = \lambda_{\delta_1} > 0$ such that

$$\|b_a(\lambda + A)^{-\frac{1}{2}}\|_{2\to 2} \leq \sqrt{\delta_1}.$$

It is easily seen that if $b \in \mathbf{F}_\delta$, then $b \in \mathbf{F}_{\delta_1}(A)$, with $\delta_1 := [1 \vee (1+c)^{-2}]\,\delta$. By the assumptions of the theorem, $\delta_1 < 4$. Therefore, by [KiS, Theorem 3.2], $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ has an operator realization $\Lambda_q(a, b)$ in L^q, $q \in \big[\frac{2}{2-\sqrt{\delta_1}}, \infty\big[$, as the (minus) generator of a positivity preserving L^∞ contraction quasi contraction C_0 semigroup. Moreover, $(\mu + \Lambda_q(a, b))^{-1}$ is well defined on L^q for all $\mu > \frac{\lambda\delta}{2(q-1)}$. This completes the proof of (i).

Proof of (ii). First, we prove an a priori variant of $(\star\star)$. Set $a_n := I + c\mathsf{f}_n \otimes \mathsf{f}_n$, where f_n have been defined in the beginning of the paper. Since our assumptions on δ_{f}, δ_a and δ involve only strict inequalities, we may assume that $(\mathbf{C}_{\delta_{\mathsf{f}}})$ holds for f_n, $\nabla a_n \in \mathbf{F}_{\delta_a}$, $b_n \in \mathbf{F}_\delta$ with $\lambda \neq \lambda(n)$ for appropriate $\epsilon_n \downarrow 0$. We also note that $\|\mathsf{f}_n\|_\infty = 1$.

Denote $A_q^n := -\nabla \cdot a_n \cdot \nabla$, $D(A_q^n) = W^{2,q}$. Set $u \equiv u_n := (\mu + \Lambda_q(a_n, b_n))^{-1} f$, $0 \leq f \in C_c^1$, $n \geq 1$, where $\Lambda_q(a_n, b_n) = A_q^n + b_n \cdot \nabla$, $D(\Lambda_q(a_n, b_n)) = D(A_q^n)$. Clearly, $0 \leq u_n \in W^{3,q}$. It is easily seen that $b_n \in \mathbf{F}_{\delta_1}(A^n)$ with $\lambda \neq \lambda(n)$, so $(\mu + \Lambda_q(a_n, b_n))^{-1}$ are well defined on L^q for all $n \geq 1$, $\mu > \frac{\lambda\delta}{2(q-1)}$.

1. Denote $w \equiv w_n := \nabla u_n$. Below we omit the index n: $\mathsf{f} \equiv \mathsf{f}_n$, $a \equiv a_n$, $b \equiv b_n$, $A_q \equiv A_q^n$. Set

$$I_q := \langle (\nabla_r w)^2 |w|^{q-2}\rangle, \quad J_q := \langle (\nabla |w|)^2 |w|^{q-2}\rangle,$$

$$\bar{I}_q := \langle \big(\mathsf{f} \cdot \nabla w\big)^2 |w|^{q-2}\rangle, \quad \bar{J}_q := \langle (\mathsf{f} \cdot \nabla |w|)^2 |w|^{q-2}\rangle.$$

Arguing as in the proof of Theorem 1, we arrive at

$$\begin{aligned}
&\mu\langle|w|^q\rangle + I_q + c\bar{I}_q + (q-2)(J_q + c\bar{J}_q) \\
&\leq |c|\left[\alpha\delta_{\mathsf{f}}\frac{q^2}{4}J_q + \frac{1}{4\alpha}I_q\right] + |c|(q-2)\left[\alpha_1\delta_{\mathsf{f}}\frac{q^2}{4}J_q + \frac{1}{4\alpha_1}J_q\right] \\
&+ |c|\left[\gamma\delta_{\mathsf{f}}\frac{q^2}{4}J_q + \frac{1}{4\gamma}\bar{I}_q\right] + |c|(q-2)\left[\gamma_1\delta_{\mathsf{f}}\frac{q^2}{4}J_q + \frac{1}{4\gamma_1}\bar{J}_q\right] \\
&+ \mu_{00}\|w\|_q^q + \langle -b\cdot w, \phi\rangle + \langle f, \varphi\rangle, \qquad \text{with } \alpha = \alpha_1 := \frac{1}{q\sqrt{\delta_{\mathsf{f}}}},
\end{aligned} \tag{7}$$

where $\mu_{00} := |c|\lambda\sqrt{\delta_{\mathsf{f}}}\big(q^{-1} + \gamma\sqrt{\delta_{\mathsf{f}}}\big) + |c|(q-2)\lambda\sqrt{\delta_{\mathsf{f}}}\big(q^{-1} + \gamma_1\sqrt{\delta_{\mathsf{f}}}\big)$, and $\gamma, \gamma_1 > 0$ are to be chosen.

2. We estimate the term $\langle -b\cdot w, \phi\rangle$ as follows.

Lemma 2. *There exist constants* C_i $(i = 0, 1)$ *such that*

$$\langle -b\cdot w, \phi\rangle \leq \left[\big(\sqrt{\delta}\sqrt{\delta_a} + \delta\big)\frac{q^2}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right]J_q + |c|\frac{q\sqrt{\delta}}{2}J_q^{\frac{1}{2}}\bar{I}_q^{\frac{1}{2}} + C_0\|w\|_q^q + C_1\|w\|_q^{q-2}\|f\|_q^2.$$

Proof. We have:

$$\begin{aligned}
\langle -b\cdot w, \phi\rangle &= \langle -\Delta u, |w|^{q-2}(-b\cdot w)\rangle + (q-2)\langle |w|^{q-3}w\cdot\nabla|w|, -b\cdot w\rangle \\
&=: F_1 + F_2.
\end{aligned}$$

Set $B_q := \langle |b\cdot w|^2|w|^{q-2}\rangle$. We have

$$F_2 \leq (q-2)B_q^{\frac{1}{2}}J_q^{\frac{1}{2}}.$$

Next, we bound F_1. Recall that $\nabla a = c\big[(\mathrm{div}\mathsf{f})\mathsf{f} + \mathsf{f}\cdot\nabla\mathsf{f}\big]$. We represent $-\Delta u = \nabla\cdot(a-1)\cdot w - \mu u - b\cdot w + f$, and evaluate: $\nabla\cdot(a-1)\cdot w = \nabla a\cdot w + c\mathsf{f}\cdot(\mathsf{f}\cdot\nabla w)$, so

$$\begin{aligned}
F_1 &= \langle\nabla\cdot(a-1)\cdot w, |w|^{q-2}(-b\cdot w)\rangle + \langle(-\mu u - b\cdot w + f), |w|^{q-2}(-b\cdot w)\rangle \\
&= \langle\nabla a\cdot w, |w|^{q-2}(-b\cdot w)\rangle \\
&+ c\langle\mathsf{f}\cdot(\mathsf{f}\cdot\nabla w), |w|^{q-2}(-b\cdot w)\rangle \\
&+ \langle(-\mu u - b\cdot w + f), |w|^{q-2}(-b\cdot w)\rangle.
\end{aligned}$$

Set $P_q := \langle|\nabla a\cdot w|^2|w|^{q-2}\rangle$. We bound F_1 from above by applying consecutively the following estimates:

1°) $\langle\nabla a\cdot w, |w|^{q-2}(-b\cdot w)\rangle \leq P_q^{\frac{1}{2}}B_q^{\frac{1}{2}}$.

2°) $\langle\mathsf{f}\cdot(\mathsf{f}\cdot\nabla w), |w|^{q-2}(-b\cdot w)\rangle \leq \bar{I}_q^{\frac{1}{2}}B_q^{\frac{1}{2}}$.

3°) $\langle\mu u, |w|^{q-2}(-b\cdot w)\rangle \leq \frac{\mu}{\mu-\omega_q}B_q^{\frac{1}{2}}\|w\|_q^{\frac{q-2}{2}}\|f\|_q$ $\big($here $\frac{2}{2-\sqrt{\delta}} < q \Rightarrow \|u\|_q \leq (\mu-\omega_q)^{-1}\|f\|_q\big)$.

4°) $\langle b\cdot w, |w|^{q-2}b\cdot w\rangle = B_q$.

5°) $\langle f, |w|^{q-2}(-b\cdot w)\rangle| \leq B_q^{\frac{1}{2}}\|w\|_q^{\frac{q-2}{2}}\|f\|_q$.

In 3°) and 5°) we estimate $B_q^{\frac{1}{2}}\|w\|_q^{\frac{q-2}{2}}\|f\|_q \leq \varepsilon_0 B_q + \frac{1}{4\varepsilon_0}\|w\|_q^{q-2}\|f\|_q^2$ $(\varepsilon_0 > 0)$.

Therefore,

$$\langle -b\cdot w, \phi\rangle \leq P_q^{\frac{1}{2}}B_q^{\frac{1}{2}} + |c|\bar{I}_q^{\frac{1}{2}}B_q^{\frac{1}{2}} + B_q + (q-2)B_q^{\frac{1}{2}}J_q^{\frac{1}{2}} + \varepsilon_0 B_q + C_1(\varepsilon_0)\|w\|_q^{q-2}\|f\|_q^2.$$

It is easily seen that $b \in \mathbf{F}_\delta$ is equivalent to the inequality

$$\langle b^2|\varphi|^2\rangle \leq \delta\langle|\nabla\varphi|^2\rangle + \lambda\delta\langle|\varphi|^2\rangle, \quad \varphi \in W^{1,2}.$$

Thus,

$$B_q \leq \|b|w|^{\frac{q}{2}}\|_2^2 \leq \delta\|\nabla|w|^{\frac{q}{2}}\|_2^2 + \lambda\delta\|w\|_q^q = \frac{q^2\delta}{4}J_q + \lambda\delta\|w\|_q^q.$$

Similarly, using that $\nabla a \in \mathbf{F}_{\delta_a}$, we obtain

$$P_q \leq \|(\nabla a)|w|^{\frac{q}{2}}\|_2^2 \leq \delta_a\|\nabla|w|^{\frac{q}{2}}\|_2^2 + \lambda\delta_a\|w\|_q^q = \frac{q^2\delta_a}{4}J_q + \lambda\delta_a\|w\|_q^q.$$

Then selecting $\varepsilon_0 > 0$ sufficiently small, and noticing that the assumption on δ, δ_a in the theorem are strict inequalities, we can and will ignore below the terms multiplied by ε_0. The proof of Lemma 2 is completed. □

In (7), we apply Lemma 2 where the inequality $\frac{q\sqrt{\delta}}{2}J_q^{\frac{1}{2}}\bar{I}_q^{\frac{1}{2}} \leq \gamma_2\frac{q^2\delta}{4}J_q + \frac{1}{4\gamma_2}\bar{I}_q$, $\gamma_2 > 0$, is used. Thus, we have

$$\begin{aligned}
&\mu\|w\|_q^q + I_q + c\bar{I}_q + (q-2)(J_q + c\bar{J}_q) \\
&\leq |c|\left[\frac{q\sqrt{\delta_{\mathsf{f}}}}{4}J_q + \frac{q\sqrt{\delta_{\mathsf{f}}}}{4}I_q\right] + |c|(q-2)\frac{q\sqrt{\delta_{\mathsf{f}}}}{2}J_q \\
&+ |c|\left[(\gamma\delta_{\mathsf{f}} + \gamma_2\delta)\frac{q^2}{4}J_q + \left(\frac{1}{4\gamma} + \frac{1}{4\gamma_2}\right)\bar{I}_q\right] + |c|(q-2)\left[\gamma_1\delta_{\mathsf{f}}\frac{q^2}{4}J_q + \frac{1}{4\gamma_1}\bar{J}_q\right] \\
&+ \left[\left(\sqrt{\delta}\sqrt{\delta_a} + \delta\right)\frac{q^2}{4} + (q-2)\frac{q\sqrt{\delta}}{2}\right]J_q + \mu_0\|w\|_q^q + C_1\|w\|_q^{q-2}\|f\|_q^2 + \langle f, \phi\rangle,
\end{aligned} \tag{8}$$

where $\mu_0 := |c|\lambda\sqrt{\delta_{\mathsf{f}}}\big(q^{-1} + \gamma\sqrt{\delta_{\mathsf{f}}}\big) + |c|(q-2)\lambda\sqrt{\delta_{\mathsf{f}}}\big(q^{-1} + \gamma_1\sqrt{\delta_{\mathsf{f}}}\big) + C_0$.

3. Let us prove that there exists constant $\eta > 0$ such that

$$(\mu - \mu_0)\|w\|_q^q + \eta J_q \leq C_1\|w\|_q^{q-2}\|f\|_q^2 + \langle f, \phi\rangle. \tag{$*$}$$

Set $Q := \big(\sqrt{\delta}\sqrt{\delta_a} + \delta\big)\frac{q^2}{4} + (q-2)\frac{q\sqrt{\delta}}{2}$.

Case $c > 0$. First, suppose that $1 - \frac{q\sqrt{\delta_{\mathsf{f}}}}{4} - \frac{q\sqrt{\delta}}{4} \geq 0$. We select γ, $\gamma_2 > 0$ such that $\frac{1}{4\gamma} + \frac{1}{4\gamma_2} = 1$ while $\gamma\delta_{\mathsf{f}} + \gamma_2\delta$ attains its minimal value. It is easily seen that $\gamma = \frac{1}{4}\big(1 + \sqrt{\frac{\delta}{\delta_{\mathsf{f}}}}\big)$, $\gamma_2 = \frac{1}{4}\big(1 + \sqrt{\frac{\delta_{\mathsf{f}}}{\delta}}\big)$. We have $1 - \frac{q\sqrt{\delta_{\mathsf{f}}}}{4} \geq 0$, and select $\gamma_1 = \frac{1}{4}$. Thus, the terms $\bar{I}_q$, $\bar{J}_q$ are no longer present in (8):

$$\begin{aligned}
&\mu\|w\|_q^q + \left(1 - c\frac{q\sqrt{\delta_{\mathsf{f}}}}{4}\right)I_q \\
&+ \left[q - 2 - c\frac{q\sqrt{\delta_{\mathsf{f}}}}{4} - c(q-2)\frac{q\sqrt{\delta_{\mathsf{f}}}}{2} - c(\delta_{\mathsf{f}} + 2\sqrt{\delta_{\mathsf{f}}\delta} + \delta)\frac{q^2}{16} - c(q-2)\frac{q^2\delta_{\mathsf{f}}}{16} - Q\right]J_q \\
&\leq \mu_0\|w\|_q^q + C_1\|w\|_q^{q-2}\|f\|_q^2 + \langle f, \phi\rangle.
\end{aligned}$$

By the assumptions of the theorem, $1-c\frac{q\sqrt{\delta_{\mathsf f}}}{4}\geq 0$, so by $J_q\leq I_q$ we obtain

$$\begin{aligned}&\mu\|w\|_q^q+\Big[q-1-c(q-1)\frac{q\sqrt{\delta_{\mathsf f}}}{2}-c(\delta_{\mathsf f}+2\sqrt{\delta_{\mathsf f}\delta}+\delta)\frac{q^2}{16}-c(q-2)\frac{q^2\delta_{\mathsf f}}{16}-Q\Big]J_q\\&\leq\mu_0\|w\|_q^q+C_1\|w\|_q^{q-2}\|f\|_q^2+\langle f,\phi\rangle.\end{aligned}$$

Next, suppose that $1-\frac{q\sqrt{\delta_{\mathsf f}}}{4}-\frac{q\sqrt{\delta}}{4}<0$, but $1-\frac{q\sqrt{\delta_{\mathsf f}}}{4}\geq 0$. We select $\gamma=\frac{1}{q\sqrt{\delta_{\mathsf f}}}$, $\gamma_2=\frac{1}{q\sqrt{\delta}}$, and $\gamma_1=\frac14$. Then the term $\bar J_q$ is no longer present, so using $\bar I_q\leq I_q$ we obtain

$$\begin{aligned}&\mu\|w\|_q^q+\Big[1+c\Big(1-\frac{q\sqrt{\delta_{\mathsf f}}}{2}-\frac{q\sqrt{\delta}}{4}\Big)\Big]I_q\\&+\Big[q-2-c\frac{q\sqrt{\delta_{\mathsf f}}}{4}-c(q-2)\frac{q\sqrt{\delta_{\mathsf f}}}{2}-c\frac{q\sqrt{\delta_{\mathsf f}}+q\sqrt{\delta}}{4}-c(q-2)\frac{q^2\delta_{\mathsf f}}{16}-Q\Big]J_q\\&\leq\mu_0\|w\|_q^q+C_1\|w\|_q^{q-2}\|f\|_q^2+\langle f,\phi\rangle.\end{aligned}$$

Thus, since $1+c\big(1-\frac{q\sqrt{\delta_{\mathsf f}}}{2}-\frac{q\sqrt{\delta}}{4}\big)\geq 0$ by the assumptions of the theorem, we have using $J_q\leq I_q$

$$\begin{aligned}&\mu\langle|w|^q\rangle+\Big[q-1+c-c\frac{q\sqrt{\delta}}{2}-c\frac{q^2\sqrt{\delta_{\mathsf f}}}{2}-c(q-2)\frac{q^2\delta_{\mathsf f}}{16}-Q\Big]J_q\\&\leq\mu_0\|w\|_q^q+C_1\|w\|_q^{q-2}\|f\|_q^2+\langle f,\phi\rangle,\end{aligned}$$

Finally, suppose that $1-\frac{q\sqrt{\delta_{\mathsf f}}}{4}<0$. We select $\gamma=\gamma_1=\frac{1}{q\sqrt{\delta_{\mathsf f}}}$, $\gamma_2=\frac{1}{q\sqrt{\delta}}$. Then using $\bar I_q\leq I_q$, $\bar J_q\leq J_q$ we obtain

$$\begin{aligned}&\mu\|w\|_q^q+\Big[1+c\Big(1-\frac{q\sqrt{\delta_{\mathsf f}}}{2}-\frac{q\sqrt{\delta}}{4}\Big)\Big]I_q+\Big[q-2+c(q-2)\Big(1-\frac{q\sqrt{\delta_{\mathsf f}}}{4}\Big)\\&-c\frac{q\sqrt{\delta_{\mathsf f}}}{4}-c(q-2)\frac{q\sqrt{\delta_{\mathsf f}}}{2}-c\frac{q\sqrt{\delta_{\mathsf f}}+q\sqrt{\delta}}{4}-c(q-2)\frac{q\sqrt{\delta_{\mathsf f}}}{4}-Q\Big]J_q\\&\leq\mu_0\|w\|_q^q+C_1\|w\|_q^{q-2}\|f\|_q^2+\langle f,\phi\rangle.\end{aligned}$$

Since $1+c\big(1-\frac{q\sqrt{\delta_{\mathsf f}}}{2}-\frac{q\sqrt{\delta}}{4}\big)\geq 0$ by the assumptions of the theorem, we have using $J_q\leq I_q$

$$\begin{aligned}&\mu\|w\|_q^q+\Big[q-1+c(q-1)-c\frac{q\sqrt{\delta}}{2}-c(q-1)q\sqrt{\delta_{\mathsf f}}-Q\Big]J_q\\&\leq\mu_0\|w\|_q^q+C_1\|w\|_q^{q-2}\|f\|_q^2+\langle f,\phi\rangle,\end{aligned}$$

In all three cases, the coefficient of J_q is positive. We have proved $(*)$.

Case $c<0$. In (8), select $\gamma=\gamma_1=\frac{1}{q\sqrt{\delta_{\mathsf f}}}$, $\gamma_2=\frac{1}{q\sqrt{\delta}}$:

$$\begin{aligned}&\mu\|w\|_q^q+\Big(1-|c|\frac{q\sqrt{\delta_{\mathsf f}}}{4}\Big)I_q\\&+\Big[q-2-|c|(q-1)\frac{q\sqrt{\delta_{\mathsf f}}}{2}-|c|(q-2)\frac{q\sqrt{\delta_{\mathsf f}}}{4}-|c|\frac{q\sqrt{\delta}}{4}-Q\Big]J_q\\&-|c|\Big(1+\frac{q\sqrt{\delta_{\mathsf f}}}{4}+\frac{q\sqrt{\delta}}{4}\Big)\bar I_q-|c|(q-2)\Big(1+\frac{q\sqrt{\delta_{\mathsf f}}}{4}\Big)\bar J_q\leq\mu_0\|w\|_q^q+C_1\|w\|_q^{q-2}\|f\|_q^2+\langle f,\phi\rangle.\end{aligned}$$

Using $I_q \geq \bar{I}_q$, $J_q \geq \bar{J}_q$, we obtain

$$\begin{aligned}&\mu\|w\|_q^q + \biggl(1-|c|\Bigl(1+\frac{q\sqrt{\delta_{\mathsf f}}}{2}+\frac{q\sqrt{\delta}}{4}\Bigr)\biggr)I_q \\ &+\biggl[q-2-|c|(q-1)\frac{q\sqrt{\delta_{\mathsf f}}}{2}-|c|(q-2)\frac{q\sqrt{\delta_{\mathsf f}}}{4}-|c|\frac{q\sqrt{\delta}}{4}-|c|(q-2)\Bigl(1+\frac{q\sqrt{\delta_{\mathsf f}}}{4}\Bigr)-Q\biggr]J_q \\ &\leq \mu_0\|w\|_q^q + C_1\|w\|_q^{q-2}\|f\|_q^2+\langle f,\phi\rangle.\end{aligned}$$

By the assumptions of the theorem, $1-|c|\bigl(1+\frac{q\sqrt{\delta_{\mathsf f}}}{2}+\frac{q\sqrt{\delta}}{4}\bigr)\geq 0$. Therefore, by $I_q \geq J_q$,

$$\begin{aligned}&\mu\|w\|_q^q + \biggl[q-1-|c|\Bigl(1+\frac{q\sqrt{\delta_{\mathsf f}}}{2}+\frac{q\sqrt{\delta}}{4}\Bigr) \\ &-|c|(q-1)\frac{q\sqrt{\delta_{\mathsf f}}}{2}-|c|(q-2)\frac{q\sqrt{\delta_{\mathsf f}}}{4}-|c|\frac{q\sqrt{\delta}}{4}-|c|(q-2)\Bigl(1+\frac{q\sqrt{\delta_{\mathsf f}}}{4}\Bigr)-Q\biggr]J_q \\ &\leq \mu_0\|w\|_q^q + C_1\|w\|_q^{q-2}\|f\|_q^2+\langle f,\phi\rangle,\end{aligned}$$

where the coefficient of J_q is strictly positive by the assumptions of the theorem. We have proved $(*)$.

4. We estimate the term $\langle f,\phi\rangle$ by Lemma 1: For each $\varepsilon_0>0$ there exists a constant $C=C(\varepsilon_0)<\infty$ such that

$$\langle f,\phi\rangle \leq \varepsilon_0 I_q + C\|w\|_q^{q-2}\|f\|_q^2.$$

We choose $\varepsilon_0>0$ so small that in the estimates below we can ignore $\varepsilon_0 I_q$.

Then $(*)$ yields the inequalities

$$\|\nabla u_n\|_q \leq K_1(\mu-\mu_0)^{-\frac{1}{2}}\|f\|_q, \quad K_1 := (C+C_1)^{\frac{1}{2}},$$

$$\|\nabla u_n\|_{qj} \leq K_2(\mu-\mu_0)^{\frac{1}{q}-\frac{1}{2}}\|f\|_q, \quad K_2 := C_S\eta^{-\frac{1}{q}}(q^2/4)^{\frac{1}{q}}(C+C_1)^{\frac{1}{2}-\frac{1}{q}},$$

where C_S is the constant in the Sobolev Embedding Theorem.

If $c>0$ then $\delta_1=\delta<1$. If $c<0$ then elementary arguments show that, by the assumptions of the theorem, $\delta_1=(1-|c|)^{-2}\delta<1$. Therefore, [KiS, Theorem 3.5] ($(\mu+\Lambda_q(a,b))^{-1} = s\text{-}L^q\text{-}\lim_n(\mu+\Lambda_q(a_n,b_n))^{-1}$) yields $(\star\star)$. The proof of Theorem 2 is completed.

3. The iteration procedure

The following is a direct extension of the iteration procedure in [KS]. Let $a\in(H_u)$.

Recall that a vector field $b:\mathbb{R}^d\to\mathbb{R}^d$ belongs to $\mathbf{F}_{\delta_1}(A)$, $\delta_1>0$, the class of form-bounded vector fields (with respect to $A\equiv A_2 := [-\nabla\cdot a\cdot\nabla \upharpoonright C_c^\infty]_{2\to 2}^{\text{clos}}$), if $b_a^2 := b\cdot a^{-1}\cdot b\in L^1_{\text{loc}}$ and there exists a constant $\lambda=\lambda_{\delta_1}>0$ such that $\|b_a(\lambda+A)^{-\frac{1}{2}}\|_{2\to 2}\leq\sqrt{\delta_1}$.

Consider

$$\{a_n\}_{n=1}^\infty \subset [C^1]^{d\times d}\cap(H_{u,\sigma,\xi})$$

and

$$\{b_n\}_{n=1}^\infty \subset [C^1]^d \cap \bigcap_{m\geq 1}\mathbf{F}_{\delta_1}(A^m), \quad \delta_1<4, \quad \lambda\neq\lambda(n,m).$$

Here $A^m\equiv A(a_m)$.

By [KiS, Theorem 3.2], $-\Lambda_r(a_n,b_n) := \nabla \cdot a_n \cdot \nabla - b_n \cdot \nabla$, $D(\Lambda_r(a_n,b_n)) = W^{2,r}$, is the generator of a positivity preserving L^∞ contraction quasi contraction C_0 semigroup on L^r, $r \in \,]\frac{2}{2-\sqrt{\delta_1}},\infty[$, with the resolvent set of $-\Lambda_r(a_n,b_n)$ containing $\mu > \omega_r := \frac{\lambda\delta_1}{2(r-1)}$ for all $n \geq 1$.

Set $u_n := (\mu + \Lambda_r(a_n,b_n))^{-1} f$, $f \in L^1 \cap L^\infty$ and $g := u_m - u_n$.

Lemma 3. *There are positive constants $C = C(d), k = k(\delta_1)$ such that*

$$\|g\|_{rj} \leq \big(C\sigma^{-1}(\delta_1 + 2\xi\sigma^{-1})(1+2\xi)\|\nabla u_m\|_{qj}^2\big)^{\frac{1}{r}} \big(r^{2k}\big)^{\frac{1}{r}} \|g\|_{x'(r-2)}^{1-\frac{2}{r}},$$

where $q \in \,]\frac{2}{2-\sqrt{\delta_1}} \vee (d-2), \frac{2}{\sqrt{\delta_1}}[$, $2x = qj$, $j = \frac{d}{d-2}$, $x' := \frac{x}{x-1}$ and $x'(r-2) > \frac{2}{2-\sqrt{\delta_1}}$, $\mu > \lambda_{\delta_1}$.

The proof follows closely [KiS, proof of Lemma 3.12] or [KS, proof of Lemma 6].

Iterating the inequality of Lemma 3, we arrive at

Lemma 4. *In the notation of Lemma 3, assume that $\sup_m \|\nabla u_m\|_{qj}^2 < \infty$, $\mu > \mu_0$. Then for any $r_0 > \frac{2}{2-\sqrt{\delta_1}}$*

$$\|g\|_\infty \leq B\|g\|_{r_0}^\gamma, \quad \mu \geq 1 + \mu_0 \vee \lambda_{\delta_1},$$

where $\gamma = \big(1 - \frac{x'}{j}\big)\big(1 - \frac{x'}{j} + \frac{2x'}{r_0}\big)^{-1} > 0$, and $B = B(d,\delta_1) < \infty$.

The proof repeats [KiS, proof of Lemma 3.13] or [KS, proof of Lemma 7].

Remark. The assumption $\sup_m \|\nabla u_m\|_{qj}^2 < \infty$ in Lemma 4 is crucial and holds e.g. in the assumptions of Theorem 2(ii).

4. Proof of Theorem 3

By Lemma 4 and the second inequality in ($\star\star$), we have for all $r_0 > \frac{2}{2-\sqrt{\delta_1}}$

$$\|u_n - u_m\|_\infty \leq B\|u_n - u_m\|_{r_0}^\gamma, \quad \mu \geq 1 + \mu_0 \vee \lambda_{\delta_1},$$

where $\gamma > 0$, $B < \infty$, and $u_n := (\mu + \Lambda_{r_0}(a_n,b_n))^{-1} f$, $f \in L^1 \cap L^\infty$. By [KiS, Theorem 3.5],

$$(\mu + \Lambda_{r_0}(a,b))^{-1} = s\text{-}L^{r_0}\text{-}\lim_n (\mu + \Lambda_{r_0}(a_n,b_n))^{-1},$$

so $\{u_n\}$ is fundamental in C_∞.

Lemma 5. *s-C_∞-$\lim_{\mu\uparrow\infty} \mu(\mu + \Lambda_{C_\infty}(a_n,b_n))^{-1} = 1$ uniformly in n.*

The proof follows closely [KiS, proof of Lemma 3.16].

We are in position to complete the proof of Theorem 3. The assertion (i) follows from the fact that $\{u_n\}$ is fundamental in C_∞ and Lemma 5 by applying the Trotter Approximation Theorem. (ii) is Theorem 2($\star\star$). The proof of (iii) is standard. The proof of Theorem 3 is completed.

Remark. The arguments of the present paper extend more or less directly to the time-dependent case $\partial_t - \nabla \cdot a(t,x) \cdot \nabla + b(t,x) \cdot \nabla$, cf. [Ki].

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