BROWNIAN MOTION WITH GENERAL DRIFT

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Abstract. We construct and study the weak solution to stochastic differential equation $dX(t) = -b(X(t))dt + \sqrt{2}dW(t)$, $X_0 = x$, for every $x \in \mathbb{R}^d$, $d \geq 3$, with b in the class of weakly form-bounded vector fields, containing, as proper subclasses, a sub-critical class $[L^d + L^\infty]^d$, as well as critical classes such as weak L^d class, Kato class, Campanato-Morrey class, Chang-Wilson-T. Wolff class.

Let $\mathcal{L}^d$ be the Lebesgue measure on $\mathbb{R}^d$, $L^p = L^p(\mathbb{R}^d, \mathcal{L}^d)$ the standard (real) Lebesgue spaces, $C_b = C_b(\mathbb{R}^d)$ the space of bounded continuous functions endowed with the sup-norm, $C_\infty \subset C_b$ the closed subspace of functions vanishing at infinity. We denote by $\mathcal{B}(X,Y)$ the space of bounded linear operators between Banach spaces $X \to Y$, endowed with the operator norm $\|\cdot\|_{X\to Y}$; $\mathcal{B}(X) := \mathcal{B}(X,X)$. Put $\|\cdot\|_{p\to q} := \|\cdot\|_{L^p \to L^q}$.

1. Let $d \geq 3$, $b : \mathbb{R}^d \to \mathbb{R}^d$. The problem of existence and uniqueness of a weak solution to the stochastic differential equation (SDE)

$$X(t) = x - \int_0^t b(X(s))ds + \sqrt{2}W(t), \quad t \geq 0, \quad x \in \mathbb{R}^d, \tag{1}$$

with a locally unbounded vector field b, has been investigated by many authors. The first principal result is due to [Po]: if $b \in [L^p + L^\infty]^d$, $p > d$, then there exists a unique in law weak solution to (1). By the results in [CW], a unique in law weak solution to (1) exists for b from the Kato class $\mathbf{K}_0^{d+1}$. (Recall that a $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to the Kato class $\mathbf{K}_\delta^{d+1}$, $0 < \delta < 1$, if $|b| \in L^1_{\text{loc}}$ and there exists $\lambda = \lambda_\delta > 0$ such that

$$\||b|(\lambda - \Delta)^{-\frac{1}{2}}\|_{1\to 1} \leqslant \delta;$$

and $\mathbf{K}_0^{d+1} := \cap_{\delta > 0} \mathbf{K}_\delta^{d+1}$ ($\supsetneq [L^p + L^\infty]^d, p > d$).)

Definition. Fix $\delta \in]0,1[$. A $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $\mathbf{F}_\delta^{1/2}$, the class of *weakly* form-bounded vector fields, if $|b| \in L^1_{\text{loc}}$ and there exists $\lambda = \lambda_\delta > 0$ such that

$$\||b|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}}\|_{2\to 2} \leq \sqrt{\delta}.$$

The class $\mathbf{F}_\delta^{1/2}$ contains, as proper subclasses, a sub-critical class $[L^d + L^\infty]^d$ ($\subsetneq \mathbf{F}_0^{1/2} := \cap_{\delta>0} \mathbf{F}_\delta^{1/2}$), as well as critical classes such as the Kato class $\mathbf{K}_\delta^{d+1}$, the weak L^d class, the Campanato-Morrey class, the Chang-Wilson-T. Wolff class, see [KiS, sect. 4].

Set $m_d := \pi^{\frac{1}{2}}(2e)^{-\frac{1}{2}} d^{\frac{d}{2}} (d-1)^{\frac{1-d}{2}}$. Let $b \in \mathbf{F}_\delta^{1/2}$ for some δ such that $m_d \delta < \frac{4(d-2)}{(d-1)^2}$.

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Assume that $\{b_n\} \subset [L^\infty \cap C^1]^d \cap \mathbf{F}_{\delta_1}^{1/2}$, $m_d\delta_1 < \frac{4(d-2)}{(d-1)^2}$, $b_n \to b$ strongly in $[L^1_{\rm loc}]^d$. Then [Ki, Theorem 2], [KiS, Theorem 4.4]

$$s\text{-}C_\infty\text{-}\lim_n e^{-t\Lambda_{C_\infty}(b_n)}$$

exists uniformly in $t \in [0,1]$, and hence determines a positivity preserving L^∞ contraction C_0 semigroup $e^{-t\Lambda_{C_\infty}(b)}$ (Feller semigroup).

Here $\Lambda_{C_\infty}(b_n) := -\Delta + b_n \cdot \nabla$ of domain $(1-\Delta)^{-1}C_\infty(\mathbb{R}^d)$.

For instance, one can take

$$b_n := \gamma_{\varepsilon_n} * \mathbf{1}_n b, \qquad n = 1, 2, \dots, \tag{2}$$

where $\mathbf{1}_n$ is the indicator of $\{x \in \mathbb{R}^d : |x| \leq n, |b(x)| \leq n\}$ and $\gamma_\varepsilon(x) := \frac{1}{\varepsilon^d}\gamma\left(\frac{x}{\varepsilon}\right)$ is the K. Friedrichs mollifier, i.e. $\gamma(x) := c\exp\left(\frac{1}{|x|^2-1}\right)\mathbf{1}_{|x|<1}$ with the constant c adjusted to $\int_{\mathbb{R}^d}\gamma(x)dx = 1$, for appropriate $\varepsilon_n \downarrow 0$.

2. The space $D([0,\infty[,\bar{\mathbb{R}}^d)$ is defined to be the set of all right-continuous functions $X : [0,\infty[\to \bar{\mathbb{R}}^d$ (here and elsewhere, $\bar{\mathbb{R}}^d := \mathbb{R}^d \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}^d$) having the left limits, such that $X(t) = \infty$, $t > s$, whenever $X(s) = \infty$ or $X(s-) = \infty$.

By $\mathcal{F}_t \equiv \sigma\{X(s) \mid 0 \leq s \leq t, X \in D([0,\infty[,\bar{\mathbb{R}}^d)\}$ denote the minimal σ-algebra containing all cylindrical sets $\{X \in D([0,\infty[,\bar{\mathbb{R}}^d) \mid \big(X(s_1),\dots,X(s_n)\big) \in A, A \subset (\bar{\mathbb{R}}^d)^n \text{ is open}\}_{0\leq s_1\leq\dots\leq s_n\leq t}$;

By a classical result, for a given Feller semigroup T^t on $C_\infty(\mathbb{R}^d)$, there exist probability measures $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ on $\mathcal{F}_\infty \equiv \sigma\{X(s) \mid 0 \leq s < \infty, X \in D([0,\infty[,\bar{\mathbb{R}}^d)\}$ such that $(D([0,\infty[,\bar{\mathbb{R}}^d),\mathcal{F}_t,\mathcal{F}_\infty,\mathbb{P}_x)$ is a Markov process (strong Markov after completing the filtration) and

$$\mathbb{E}_{\mathbb{P}_x}[f(X(t))] = T^t f(x), \quad X \in D([0,\infty[,\bar{\mathbb{R}}^d), \quad f \in C_\infty, \quad x \in \mathbb{R}^d.$$

The space $C([0,\infty[,\mathbb{R}^d)$ is defined to be the set of all continuous functions $X : [0,\infty[\to \mathbb{R}^d$.

Set $\mathcal{G}_t := \sigma\{X(s) \mid 0 \leq s \leq t, X \in C([0,\infty[,\mathbb{R}^d)\}$, $\mathcal{G}_\infty := \sigma\{X(s) \mid 0 \leq s < \infty, X \in C([0,\infty[,\mathbb{R}^d)\}$.

Theorem 1 (Main result)**.** *Let $d \geq 3$, $b \in \mathbf{F}_\delta^{1/2}$, $m_d\delta < 4\frac{d-2}{(d-1)^2}$. Let $(D([0,\infty[,\bar{\mathbb{R}}^d),\mathcal{F}_t,\mathcal{F}_\infty,\mathbb{P}_x)$ be the Markov process determined by $T^t = e^{-t\Lambda_{C_\infty}(b)}$. The following is true for every $x \in \mathbb{R}^d$:*

(i) The trajectories of the process are $\mathbb{P}_x$ a.s. finite and continuous on $0 \leq t < \infty$.

We denote $\mathbb{P}_x \restriction (C([0,\infty[,\mathbb{R}^d),\mathcal{G}_\infty)$ again by $\mathbb{P}_x$.

(ii) $\mathbb{E}_{\mathbb{P}_x}\int_0^t |b(X(s))|ds < \infty$, $X \in C([0,\infty[,\mathbb{R}^d)$.

(iii) There exists a d-dimensional Brownian motion $W(t)$ on $(C([0,\infty[,\mathbb{R}^d),\mathcal{G}_t,\mathbb{P}_x)$ such that $\mathbb{P}_x$ a.s.

$$X(t) = x - \int_0^t b(X(s))ds + \sqrt{2}W(t), \quad t \geq 0, \tag{3}$$

i.e. $\big((X(t),W(t)),(C([0,\infty[,\mathbb{R}^d),\mathcal{G}_t,\mathcal{G}_\infty,\mathbb{P}_x)\big)$ is a weak solution to the SDE (3).

Remark 1. One can show, using the methods of this paper, that if $\{\mathbb{Q}_x\}_{x\in\mathbb{R}^d}$ is another weak solution to (3) such that

$$\mathbb{Q}_x = w\text{-}\lim_n \mathbb{P}_x(\tilde{b}_n) \quad \text{for every } x \in \mathbb{R}^d,$$

where $\{\tilde{b}_n\} \subset \mathbf{F}_{\delta_1}^{1/2}$, $m_d\delta_1 < 4\frac{d-2}{(d-1)^2}$, then $\{\mathbb{Q}_x\}_{x\in\mathbb{R}^d} = \{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$.

Theorem 1 covers critical-order singularities of b, as the following example shows.

Example 1. Consider the vector field ($d \geq 3$)

$$b(x) := c|x|^{-2}x, \qquad c > 0,$$

then $b \in \mathbf{F}_\delta^{1/2}$, $c = \frac{d-2}{2}\sqrt{\delta}$.

1) If $c < 2m_d^{-1}(d-2)^2(d-1)^{-2}$, then by Theorem 1, the SDE

$$X(t) = -\int_0^t b(X(s))ds + \sqrt{2}W(t), \quad t \geq 0.$$

has a weak solution. (For this particular vector field the result is, in fact, stronger, see Remark 2 below.)

2) If $c \geq d$, then the SDE doesn't have a weak solution.

Indeed, following [CE, Example 1.17], suppose by contradiction that there is a weak solution to the SDE if $c \geq d$, i.e. there are a continuous process $X(t)$ and a Brownian motion $W(t)$ on a probability space $(\Upsilon, \mathcal{F}_t, \mathbb{Q})$ such that $\int_0^t |b(X(s))|ds < \infty$ and the SDE holds $\mathbb{Q}$ a.s. Then $X(t) = (X_1(t), \dots, X_d(t))$ is a continuous semimartingale with cross-variation $[X_i, X_k]_t = 2\delta_{ik}t$. By Itô's formula,

$$|X(t)|^2 = -2\int_0^t X(s) \cdot b(X(s))ds + 2\sqrt{2}\int_0^t X(s)dW(s) + 2\int_0^t d[W,W]_s,$$

i.e.

$$|X(t)|^2 = -2c\int_0^t \mathbf{1}_{X(s)\neq 0}ds + 2\sqrt{2}\int_0^t X(s)dW(s) + 2td.$$

If we accept that $\int_0^t \mathbf{1}_{X(s)=0}ds = 0$ a.s., then, clearly,

$$|X(t)|^2 = 2(d-c)\int_0^t \mathbf{1}_{X(s)\neq 0}ds + 2\sqrt{2}\int_0^t X(s)dW(s) \quad \text{a.s.}$$

Therefore, $|X(t)|^2 \geq 0$ is a local supermartingale if $c > d$ and is a local martingale if $c = d$. Then a.s. $|X(0)| = 0 \Rightarrow X(t) = 0$, which contradicts to $[X_1, X_1]_t = 2t$.

It remains to prove that $\int_0^t \mathbf{1}_{X(s)=0}ds = 0$ a.s. It suffices to show that $\int_0^t \mathbf{1}_{X_1(s)=0}ds = 0$ a.s. Since $X_1(t)$ is a continuous semimartingale, $[X_1, X_1]_t = 2t$, by the occupation times formula $\int_0^t \mathbf{1}_{X_1(s)=0}d[X_1,X_1]_s = \int_{-\infty}^{\infty} \mathbf{1}_{a=0}L^a_{X_1}(t)da = 0$ a.s., where $L^a_{X_1}(t)$ is the local time of X_1 at a on $[0,t]$. □

Remark 2. Recall the following

Definition. A $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $\mathbf{F}_\delta$, the class of form-bounded vector fields, if $|b| \in L^2_{\text{loc}}$ and there exists $\lambda = \lambda_\delta > 0$ such that

$$\||b|(\lambda - \Delta)^{-\frac{1}{2}}\|_{2\to 2} \leq \sqrt{\delta}.$$

Note that $\mathbf{F}_{\delta_1} \subsetneq \mathbf{F}_\delta^{1/2}$ for $\delta = \sqrt{\delta_1}$.

For $b \in \mathbf{F}_{\delta_1}$, the constraint $m_d\sqrt{\delta_1} < 4\frac{d-2}{(d-1)^2}$ in Theorem 1 can be relaxed to $\delta_1 < 1 \wedge \left(\frac{2}{d-2}\right)^2$. The proof of Theorem 1 extends to such b after replacing Lemma A below by evident modifications of [KS, Lemma 5], [KiS, Theorem 3.7].

For $b(x) := c|x|^{-2}x \in \mathbf{F}_{\delta_1}$, $\delta_1 := c^2\frac{4}{(d-2)^2}$, the result is even stronger: $-1 < c < \frac{1}{2}$ if $d = 3$, $-\infty < c < 1$ if $d = 4$, $-\infty < c < (d-3)/2$ if $d \geq 5$ (after replacing Lemma A by evident modifications of Theorems 3.8, 3.9 in [KiS]).

We refer to [KiS] for a more detailed discussion on classes $\mathbf{F}_{\delta_1}$, $\mathbf{F}_\delta^{1/2}$.

1. Preliminaries

Denote by $C^{0,\alpha} = C^{0,\alpha}(\mathbb{R}^d)$ the space of Hölder continuous functions $(0 < \alpha < 1)$, $\mathcal{S}$ the L. Schwartz space of test functions, $W^{k,p} = W^{k,p}(\mathbb{R}^d, \mathcal{L}^d)$ $(k = 1,2)$ the standard Sobolev spaces, $\mathcal{W}^{\alpha,p}$, $\alpha > 0$, the Bessel potential space endowed with norm $\|u\|_{p,\alpha} := \|g\|_p$, $u = (1-\Delta)^{-\frac{\alpha}{2}}g$, $g \in L^p$, and $\mathcal{W}^{-\alpha,p'}$, $p' = p/(p-1)$, the anti-dual of $\mathcal{W}^{\alpha,p}$.

The proof of Theorem 1 is based on the following analytic results [Ki, Theorems 1, 2], [KiS, Theorems 4.3, 4.4].

Set

$$I_s := \left]\frac{2}{1+\sqrt{1-m_d\delta}}, \frac{2}{1-\sqrt{1-m_d\delta}}\right[.$$

For every $p \in I_s$, there exists a holomorphic semigroup $e^{-t\Lambda_p(b)}$ on L^p such that the resolvent set of $-\Lambda_p(b)$ contains the half-plane $\mathcal{O} := \{\zeta \in \mathbb{C} : \mathrm{Re}\zeta \geq \kappa_d\lambda\}$,

$$(\zeta+\Lambda_p(b))^{-1} = (\zeta-\Delta)^{-1} - (\zeta-\Delta)^{-\frac{1}{2}-\frac{1}{2q}} Q_p(q)(1+T_p)^{-1} G_p(r)(\zeta-\Delta)^{-\frac{1}{2r'}}, \quad \zeta \in \mathcal{O}, \tag{4}$$

where $1 \leq r < p < q$, $\kappa_d := \frac{d}{d-1}$, $Q_p(q), G_p(r), T_p \in \mathcal{B}(L^p)$,

$$G_p(r) := b^{\frac{1}{p}} \cdot \nabla(\zeta-\Delta)^{-\frac{1}{2}-\frac{1}{2r}}, \quad b^{\frac{1}{p}} := |b|^{\frac{1}{p}-1} b,$$

$Q_p(q)$, $T(p)$ are the extensions by continuity of densely defined on $\mathcal{E} := \bigcup_{\epsilon>0} e^{-\epsilon|b|}L^p$ operators

$$Q_p(q) \upharpoonright \mathcal{E} := (\zeta-\Delta)^{-\frac{1}{2q'}} |b|^{\frac{1}{p'}}, \quad T_p \upharpoonright \mathcal{E} := b^{\frac{1}{p}} \cdot \nabla(\zeta-\Delta)^{-1}|b|^{\frac{1}{p'}},$$

$$\|T_p\|_{p\to p} \leq m_d c_p \delta, \quad c_p := \frac{pp'}{4}, \quad m_d c_p\delta < 1 \ (\Leftrightarrow p \in I_s).$$

$$e^{-t\Lambda_p(b)} = s\text{-}L^p\text{-}\lim_n e^{-t\Lambda_p(b_n)} \quad \text{(uniformly on every compact interval of } t \geq 0\text{)} ,$$

where b_n's are given by (2), $\Lambda_p(b_n) := -\Delta + b_n \cdot \nabla$, $D(\Lambda_p(b_n)) = W^{2,p}$.

By (4),

$$(\zeta+\Lambda_p(b))^{-1} \in \mathcal{B}(\mathcal{W}^{-\frac{1}{r'},p}, \mathcal{W}^{1+\frac{1}{q},p}).$$

Fix numbers $p \in I_s$, $p > d-1$ [1] and q sufficiently close to p. By (4) and the Sobolev Embedding Theorem, $(\zeta+\Lambda_p(b))^{-1}[L^p] \subset C^{0,\alpha}$, $\alpha < 1 - \frac{d-1}{p}$. Define $\Lambda_{C_\infty}(b)$ by

$$(\mu+\Lambda_{C_\infty}(b))^{-1} := \big((\mu+\Lambda_p(b))^{-1} \upharpoonright L^p \cap C_\infty\big)^{\mathrm{clos}}_{C_\infty\to C_\infty}, \quad \mu \geq \kappa_d\lambda.$$

Then

$$\big(e^{-t\Lambda_{C_\infty}(b)} \upharpoonright L^p \cap C_\infty\big)^{\mathrm{clos}}_{L^p\to C_\infty} \in \mathcal{B}(L^p, C_\infty), \qquad p \in \left]d-1, \frac{2}{1-\sqrt{1-m_d\delta}}\right[,\ t>0. \tag{5}$$

$$e^{-t\Lambda_{C_\infty}(b)} = s\text{-}C_\infty\text{-}\lim_n e^{-t\Lambda_{C_\infty}(b_n)} \quad \text{(uniformly on every compact interval of } t \geq 0\text{)}, \tag{6}$$

where $D(\Lambda_{C_\infty}(b_n)) = (1-\Delta)^{-1}C_\infty$.

The following estimates are direct consequences of (4): There exist constants $C_i = C_i(\delta,p)$, $i=1,2$, such that, for all $h \in C_c$ and $\mu \geq \kappa_d\lambda_\delta$,

$$\big\|(\mu+\Lambda_{C_\infty}(b))^{-1}|b_m|h\big\|_\infty \leq C_1 \||b_m|^{\frac{1}{p}}h\|_p, \tag{7}$$

$$\|(\mu+\Lambda_{C_\infty}(b))^{-1}|b_m-b_n|h\|_\infty \leq C_2 \big\||b_m-b_n|^{\frac{1}{p}}h\big\|_p. \tag{8}$$

Our proof of Theorem 1 employs also the following weighted estimates. Set

$$\rho(y) \equiv \rho_l(y) := (1+l|y|^2)^{-\nu}, \quad \nu > \frac{d}{2p}+1, \ \ l>0, \ \ y \in \mathbb{R}^d.$$

Lemma A. *Fix $p \in I_s$, $p > d-1$. There exist constants $K_i = K_i(\delta,p)$, $i=1,2$ such that, for all $h \in C_c(\mathbb{R}^d)$, $\mu \geq \kappa_d\lambda_\delta$ and all sufficiently small $l = l(\delta,p) > 0$,*

$$\big\|\rho(\mu+\Lambda_{C_\infty}(b_n))^{-1}h\big\|_\infty \leq K_1\|\rho h\|_p, \tag{E_1}$$

$$\big\|\rho(\mu+\Lambda_{C_\infty}(b_n))^{-1}|b_m|h\big\|_\infty \leq K_2\||b_m|^{\frac{1}{p}}\rho h\|_p. \tag{E_2}$$

[1] Since $m_d\delta < 4\frac{d-2}{(d-1)^2}$, such p exists.

This technical lemma is proven in the appendix.

2. Proof of Theorem 1

Lemma 1. *For every $x \in \mathbb{R}^d$ and $t > 0$, $b_n(X(t)) \to b(X(t))$ $\mathbb{P}_x$ a.s. as $n \uparrow \infty$.*

Proof. By (5) and the Dominated Convergence Theorem, for any $\mathcal{L}^d$-measure zero set $G \subset \mathbb{R}^d$ and every $t > 0$, $\mathbb{P}_x[X(t) \in G] = 0$. Since $b_n \to b$ pointwise in $\mathbb{R}^d$ outside of an $\mathcal{L}^d$-measure zero set, we have the required. □

Let $\mathbb{P}_x^n$ be the probability measures associated with $e^{-t\Lambda_{C_\infty}(b_n)}$, $n = 1, 2, \dots$ Set $\mathbb{E}_x := \mathbb{E}_{\mathbb{P}_x}$, and $\mathbb{E}_x^n := \mathbb{E}_{\mathbb{P}_x^n}$.

Fix a $v \in C^\infty([0,\infty[)$, $v(s) = 1$ if $0 \leq s \leq 1$, $v(s) = 0$ if $s \geq 2$. Set

$$\xi_k(y) := \begin{cases} v(|y| + 1 - k) & |y| \geq k, \\ 1 & |y| < k. \end{cases} \tag{9}$$

Lemma 2. *For every $x \in \mathbb{R}^d$ and $t > 0$, $\mathbb{P}_x[X(t) = \infty] = 0$.*

Proof. First, let us show that for every $\mu \geq \kappa_d \lambda_\delta$,

$$\int_0^\infty e^{-\mu t} \mathbb{E}_x^n[\xi_k(X(t))]dt \to \frac{1}{\mu} \quad \text{as } k \uparrow \infty \text{ \textit{uniformly in} } n. \tag{10}$$

Since $\int_0^\infty e^{-\mu t} \mathbb{E}_x^n[\mathbf{1}_{\mathbb{R}^d}(X(t))]dt = \frac{1}{\mu}$, (10) is equivalent to $\int_0^\infty e^{-\mu t} \mathbb{E}_x^n[(\mathbf{1}_{\mathbb{R}^d} - \xi_k)(X(t))]dt \to 0$ as $k \uparrow \infty$ uniformly in n. We have

$$\begin{aligned} &\int_0^\infty e^{-\mu t} \mathbb{E}_x^n[(\mathbf{1}_{\mathbb{R}^d} - \xi_k)(X(t))]dt \\ &\text{(we use the Dominated Convergence Theorem)} \\ &= \lim_{r \uparrow \infty} \int_0^\infty e^{-\mu t} \mathbb{E}_x^n[\xi_r(1 - \xi_k)(X(t))]dt \\ &= \lim_{r \uparrow \infty} (\mu + \Lambda_{C_\infty}(b_n))^{-1}[\xi_r(1 - \xi_k)](x) \\ &\text{(we apply crucially } (E_1)) \\ &\leq \rho(x)^{-1} K_1 \lim_{r \uparrow \infty} \|\rho \xi_r (1 - \xi_k)\|_p \leq \rho(x)^{-1} K_1 \|\rho(1 - \xi_k)\|_p \to 0 \quad \text{as } k \uparrow \infty, \end{aligned}$$

which yields (10).

Now, since $\mathbb{E}_x[\xi_k(X(t))] = \lim_n \mathbb{E}_x^n[\xi_k(X(t))]$ uniformly on every compact interval of $t \geq 0$, see (6), it follows from (10) that

$$\int_0^\infty e^{-\mu t} \mathbb{E}_x[\xi_k(X(t))]dt \to \frac{1}{\mu} \quad \text{as } k \uparrow \infty.$$

Finally, suppose that $\mathbb{P}_x[X(t) = \infty]$ is strictly positive for some $t > 0$. By the construction of $\mathbb{P}_x$, $t \mapsto \mathbb{P}_x[X(t) = \infty]$ is non-decreasing, and so $\varkappa := \int_0^\infty e^{-\mu t} \mathbb{E}_x[\mathbf{1}_{X(t)=\infty}]dt > 0$. Now,

$$\frac{1}{\mu} = \int_0^\infty e^{-\mu t} \mathbb{E}_x[\mathbf{1}_{\bar{\mathbb{R}}^d}(X(t))]dt \geq \varkappa + \int_0^\infty e^{-\mu t} \mathbb{E}_x[\xi_k(X(t))]dt.$$

Selecting k sufficiently large, we arrive at contradiction. □

The space $D([0,\infty[, \mathbb{R}^d)$ is defined to be the subspace of $D([0,\infty[, \bar{\mathbb{R}}^d)$ consisting of the trajectories $X(t) \neq \infty$, $0 \leq t < \infty$. Let $\mathcal{F}'_t := \sigma(X(s) \mid 0 \leq s \leq t, X \in D([0,\infty[, \mathbb{R}^d))$, $\mathcal{F}'_\infty := \sigma(X(s) \mid 0 \leq s < \infty, X \in D([0,\infty[, \mathbb{R}^d))$.

By Lemma 2, $(D([0,\infty[, \mathbb{R}^d), \mathcal{F}'_\infty)$ has full $\mathbb{P}_x$-measure in $(D([0,\infty[, \bar{\mathbb{R}}^d), \mathcal{F}_\infty)$. We denote the restriction of $\mathbb{P}_x$ from $(D([0,\infty[, \bar{\mathbb{R}}^d), \mathcal{F}_\infty)$ to $(D([0,\infty[, \mathbb{R}^d), \mathcal{F}'_\infty)$ again by $\mathbb{P}_x$.

Lemma 3. *For every $x \in \mathbb{R}^d$ and $g \in C_c^\infty(\mathbb{R}^d)$,*

$$g(X(t)) - g(x) + \int_0^t (-\Delta g + b \cdot \nabla g)(X(s))ds,$$

is a martingale relative to $(D([0,\infty[,\mathbb{R}^d), \mathcal{F}'_t, \mathbb{P}_x)$.

Proof. Fix $\mu \geq \kappa_d \lambda_\delta$. In what follows, $0 < t \leq T < \infty$.

(**a**) $\mathbb{E}_x \int_0^t \big|b \cdot \nabla g\big|(X(s))ds < \infty$. Indeed,

$$\begin{aligned}
&\mathbb{E}_x \int_0^t \big|b \cdot \nabla g\big|(X(s))ds \\
&\text{(we apply Fatou's Lemma, cf. Lemma 1)} \\
&\leq \liminf_n \mathbb{E}_x \int_0^t \big|b_n \cdot \nabla g\big|(X(s))ds = \liminf_n \int_0^t e^{-s\Lambda_{C_\infty}(b)} \big|b_n \cdot \nabla g\big|(x)ds \\
&= \liminf_n \int_0^t e^{\mu s} e^{-\mu s} e^{-s\Lambda_{C_\infty}(b)} \big|b_n \cdot \nabla g\big|(x)ds \\
&\leq e^{\mu T} \liminf_n (\mu + \Lambda_{C_\infty}(b))^{-1} |b_n||\nabla g|(x) \\
&\text{(we apply (7) with } h = |\nabla g|) \\
&\leq C_1 e^{\mu T} \liminf_n \langle |b_n||\nabla g|^p \rangle^{\frac{1}{p}} \leq C_1 e^{\mu T} 2^{\frac{1}{p}} \big(\langle |b||\nabla g|^p \rangle^{\frac{1}{p}} + \lim_n \langle |b - b_n||\nabla g|^p \rangle^{\frac{1}{p}} \big) \\
&= C_1 e^{\mu T} 2^{\frac{1}{p}} \langle |b||\nabla g|^p \rangle^{\frac{1}{p}} < \infty.
\end{aligned}$$

(**b**) $\mathbb{E}_x^n[g(X(t))] \to \mathbb{E}_x[g(X(t))]$, $\mathbb{E}_x^n \int_0^t (b_n \cdot \nabla g)(X(s))ds \to \mathbb{E}_x \int_0^t (b \cdot \nabla g)(X(s))ds$, and also, for $h \in C_c^\infty$, $\mathbb{E}_x^n \int_0^t (|b_n|h)(X(s))ds \to \mathbb{E}_x \int_0^t (|b|h)(X(s))ds$ as $n \uparrow \infty$. Indeed, the first convergence follows from (6), the second one follows from (**a**), and the third one from $\mathbb{E}_x \int_0^t (|b||h|)(X(s))ds < \infty$, a straightforward modification of (**a**).

(**c**) $\mathbb{E}_x \int_0^t (b_n \cdot \nabla g)(X(s))ds - \mathbb{E}_x^n \int_0^t (b_n \cdot \nabla g)(X(s))ds \to 0$. We have:

$$\begin{aligned}
&\mathbb{E}_x \int_0^t (b_n \cdot \nabla g)(X(s))ds - \mathbb{E}_x^n \int_0^t (b_n \cdot \nabla g)(X(s))ds \\
&= \int_0^t \left(e^{-s\Lambda_{C_\infty}(b)} - e^{-s\Lambda_{C_\infty}(b_n)} \right) (b_n \cdot \nabla g)(x)ds \\
&= \int_0^t \left(e^{-s\Lambda_{C_\infty}(b)} - e^{-s\Lambda_{C_\infty}(b_n)} \right) ((b_n - b_m) \cdot \nabla g)(x)ds \\
&+ \int_0^t \left(e^{-s\Lambda_{C_\infty}(b)} - e^{-s\Lambda_{C_\infty}(b_n)} \right) (b_m \cdot \nabla g)(x)ds =: S_1 + S_2,
\end{aligned}$$

where m is to be chosen. Arguing as in the proof of (**a**), we obtain:

$$S_1(x) \leq e^{\mu T}(\mu + \Lambda_{C_\infty}(b))^{-1} |(b_n - b_m) \cdot \nabla g|(x) + e^{\mu T}(\mu + \Lambda_{C_\infty}(b_n))^{-1} |(b_n - b_m) \cdot \nabla g|(x).$$

Since $b_n - b_m \to 0$ in L^1_{loc} as $n, m \uparrow \infty$, (8) yields $S_1 \to 0$ as $n, m \uparrow \infty$. Now, fix a sufficiently large m. Since $e^{-s\Lambda_{C_\infty}(b)} = s\text{-}C_\infty\text{-}\lim_n e^{-s\Lambda_{C_\infty}(b_n)}$ uniformly in $0 \leq s \leq T$, cf. (6), we have $S_2 \to 0$ as $n \uparrow \infty$. The proof of (**c**) is completed.

Now we are in position to complete the proof of Lemma 3. Since $b_n \in [C_c^\infty(\mathbb{R}^d)]^d$,

$$g(X(t)) - g(x) + \int_0^t (-\Delta g + b_n \cdot \nabla g)(X(s))ds \text{ is a martingale under } \mathbb{P}_x^n,$$

so the function

$$x \mapsto \mathbb{E}_x^n[g(X(t))] - g(x) + \mathbb{E}_x^n \int_0^t (-\Delta g + b_n \cdot \nabla g)(X(s))ds \quad \text{is identically zero in } \mathbb{R}^d.$$

Thus by (**b**), the function

$$x \mapsto \mathbb{E}_x[g(X(t))] - g(x) + \mathbb{E}_x \int_0^t (-\Delta g + b \cdot \nabla g)(X(s))ds \quad \text{is identically zero in } \mathbb{R}^d,$$

i.e. $g(X(t)) - g(x) + \int_0^t (-\Delta g + b \cdot \nabla g)(X(s))ds$ is a martingale under $\mathbb{P}_x$. □

Lemma 4. *For $x \in \mathbb{R}^d$, $C([0,\infty[,\mathbb{R}^d)$ has full $\mathbb{P}_x$-measure in $D([0,\infty[,\mathbb{R}^d)$.*

Proof. Let A, B be arbitrarily bounded closed sets in $\mathbb{R}^d$, $\operatorname{dist}(A,B) > 0$. Fix $g \in C_c^\infty(\mathbb{R}^d)$ such that $g = 0$ on A, $g = 1$ on B. Set ($X \in D([0,\infty[,\mathbb{R}^d)$)

$$M^g(t) := g(X(t)) - g(x) + \int_0^t (-\Delta g + b \cdot \nabla g)(X(s))ds, \quad K^g(t) := \int_0^t \mathbf{1}_A(X(s-))dM^g(s),$$

then

$$\begin{aligned} K^g(t) &= \sum_{s \le t} \mathbf{1}_A\left(X(s-)\right) g(X(s)) + \int_0^t \mathbf{1}_A(X(s-))\big(-\Delta g + b \cdot \nabla g\big)(X(s))ds \\ &= \sum_{s \le t} \mathbf{1}_A\left(X(s-)\right) g(X(s)). \end{aligned}$$

By Lemma 3, $M^g(t)$ is a martingale, and hence so is $K^g(t)$. Thus, $\mathbb{E}_x\big[\sum_{s\le t} \mathbf{1}_A(X(s-))g(X(s))\big] = 0$. Using the Dominated Convergence Theorem, we obtain $\mathbb{E}_x\big[\sum_{s\le t} \mathbf{1}_A(X(s-))\mathbf{1}_B(X(s))\big] = 0$. The proof of Lemma 4 is completed. □

We denote the restriction of $\mathbb{P}_x$ from $(D([0,\infty[,\mathbb{R}^d), \mathcal{F}'_\infty)$ to $(C([0,\infty[,\mathbb{R}^d), \mathcal{G}_\infty)$ again by $\mathbb{P}_x$. Lemma 3 and Lemma 4 combined yield

Lemma 5. *For every $x \in \mathbb{R}^d$ and $g \in C_c^\infty(\mathbb{R}^d)$,*

$$g(X(t)) - g(x) + \int_0^t (-\Delta g + b \cdot \nabla g)(X(s))ds, \quad X \in C([0,\infty), \mathbb{R}^d),$$

is a continuous martingale relative to $(C([0,\infty[,\mathbb{R}^d), \mathcal{G}_t, \mathbb{P}_x)$.

Lemma 6. *For every $x \in \mathbb{R}^d$ and $t > 0$, $\mathbb{E}_x \int_0^t |b(X(s))|ds < \infty$, and, for $f(y) = y_i$ or $f(y) = y_i y_j$, $1 \le i,j \le d$,*

$$f(X(t)) - f(x) + \int_0^t (-\Delta f + b \cdot \nabla f)(X(s))ds, \quad X \in C([0,\infty[,\mathbb{R}^d),$$

is a continuous martingale relative to $(C([0,\infty[,\mathbb{R}^d), \mathcal{G}_t, \mathbb{P}_x)$.

Proof. Define $f_k := \xi_k f \in C_c^\infty(\mathbb{R}^d)$ (see (9) for the definition of ξ_k). Set $\alpha := \|\nabla \xi_k\|_\infty$, $\beta := \|\Delta \xi_k\|_\infty$ (α, β don't depend on k). Fix $0 < T < \infty$. In what follows, $0 < t \le T$.

(**a**) $\mathbb{E}_x \int_0^t (|b|(|\nabla f| + \alpha|f|))(X(s))ds < \infty$.

Indeed, set $\varphi := |\nabla f| + \alpha|f| \in C \cap W^{1,2}_{\mathrm{loc}}$, $\varphi_k := \xi_{k+1}\varphi \in C_c \cap W^{1,2}$. First, let us prove that

$$\mathbb{E}_x^n \int_0^t (|b_n|\varphi_k)(X(s))ds \le \text{const independent of } n, k.$$

Fix $p \in \left]d-1, \frac{2}{1-\sqrt{1-m_d\delta}}\right[$. Then $\sqrt{(\rho\varphi)^p} \in W^{1,2}$ (recall that $\rho(x) := (1+l|x|^2)^{-\nu}$, $\nu > \frac{d}{2p}+1$.) We have

$$\begin{aligned}
&\mathbb{E}_x^n \int_0^t (|b_n|\varphi_k)(X(s))ds = \int_0^t e^{-s\Lambda_{C_\infty}(b_n)}|b_n|\varphi_k(x)ds \\
&\leq e^{\mu T}(\mu + \Lambda_{C_\infty}(b_n))^{-1}|b_n|\varphi_k(x) \\
&\text{(we apply } (E_2)) \\
&\leq e^{\mu T}\rho(x)^{-1}K_2\langle |b_n|(\rho\varphi_k)^p\rangle^{\frac{1}{p}} \leq e^{\mu T}\rho(x)^{-1}K_2\langle |b_n|(\rho\varphi)^p\rangle^{\frac{1}{p}} \\
&\left(\text{we use } b_n \in \mathbf{F}_{\delta_1}^{1/2},\ m_d\delta_1 < 4\frac{d-1}{(d-2)^2}\right) \\
&\leq e^{\mu T}\rho(x)^{-1}K_2\delta_1^{\frac{1}{p}}\|(\lambda-\Delta)^{\frac{1}{4}}\sqrt{(\rho\varphi)^p}\|_2^{\frac{2}{p}} < \infty.
\end{aligned}$$

By step (**b**) in the proof of Lemma 3, $\mathbb{E}_x^n\int_0^t(|b_n|\varphi_k)(X(s))ds \to \mathbb{E}_x\int_0^t(|b|\varphi_k)(X(s))ds$ as $n \uparrow \infty$. Therefore, $\mathbb{E}_x^n\int_0^t(|b_n|\varphi_k)(X(s))ds \leq C$ implies $\mathbb{E}_x\int_0^t(|b|\varphi_k)(X(s))ds \leq C$ ($C \neq C(k)$). Now, Fatou's Lemma yields the required.

(**b**) For every $t > 0$, $\mathbb{E}_x\int_0^t(|\Delta f| + 2\alpha|\nabla f| + \beta|f|)(X(t))ds < \infty$.

The proof is similar to the proof of (**a**) (use (E_1) instead of (E_2)).

(**c**) For every $t > 0$, $\mathbb{E}_x[|f|(X(t))] < \infty$.

Indeed, set $g(y) := 1 + |y|^2$, $y \in \mathbb{R}^d$. Since $|f| \leq g$, it suffices to show that $\mathbb{E}_x[g(X(t))] < \infty$. Set $g_k(y) := \xi_k(y)g(y)$. By Lemma 5,

$$\mathbb{E}_x[g_k(X(t))] = g_k(x) - \mathbb{E}_x\int_0^t(-\Delta g_k)(X(s))ds - \mathbb{E}_x\int_0^t(b\cdot\nabla g_k)(X(s))ds.$$

Note that

$$\sup_k \mathbb{E}_x\int_0^t(|b||g_k|)(X(s))ds < \infty, \quad \sup_k \mathbb{E}_x\int_0^t|\Delta g_k|(X(s))ds < \infty$$

for, arguing as in the proofs of (**a**) and (**b**), we have:

$$\mathbb{E}_x\int_0^t(|b|(|\nabla g| + \alpha|g|))(X(s))ds < \infty, \quad \mathbb{E}_x\int_0^t(|\Delta g| + 2\alpha|\nabla g| + \beta|g|)(X(t))ds < \infty.$$

Therefore, $\sup_k \mathbb{E}_x[g_k(X(t))] < \infty$, and so, by the Monotone Convergence Theorem, $\mathbb{E}_x[g(X(t))] < \infty$. This completes the proof of (**c**).

Let us complete the proof of Lemma 6. By (**a**), $\mathbb{E}_x\int_0^t|b(X(s))|ds < \infty$. By (**a**)-(**c**),

$$M^f(t) := f(X(t)) - f(x) + \int_0^t(-\Delta f + b\cdot\nabla f)(X(s))ds, \quad t > 0,$$

satisfies $\mathbb{E}_x[|M^f(t)|] < \infty$ for all $t > 0$. By Lemma 5, for every k, $M^{f_k}(t)$ is a martingale relative to $(C([0,\infty[,\mathbb{R}^d), \mathcal{G}_t, \mathbb{P}_x)$. By (**a**) and the Dominated Convergence Theorem, since $|\nabla f_k| \leq |\nabla f| + \alpha|f|$ for all k, we have $\mathbb{E}_x\int_0^t(b\cdot\nabla f_k)(X(s))ds \to \mathbb{E}_x\int_0^t(b\cdot\nabla f)(X(s))ds$. By (**b**), $\mathbb{E}_x\int_0^t(-\Delta f_k)(X(s))ds \to \mathbb{E}_x\int_0^t(-\Delta f)(X(s))ds$. By (**c**), $\mathbb{E}_x[f_k(X(t))] \to \mathbb{E}_x[f(X(t))]$. So, $M^f(t)$ is also a martingale on $(C([0,\infty[,\mathbb{R}^d), \mathcal{G}_t, \mathbb{P}_x)$. The proof of Lemma 6 is completed. □

We are in position to complete the proof of Theorem 1. Lemma 4 yields (i). Lemma 6 yields (ii). By classical results, Lemma 6 yields existence of a d-dimensional Brownian motion $W(t)$ on $(C([0,\infty[,\mathbb{R}^d), \mathcal{G}_t, \mathbb{P}_x)$ such that $X(t) = x - \int_0^t b(X(s))ds + \sqrt{2}W(t)$, $0 \leq t < \infty$, $\mathbb{P}_x$ a.s. $\Rightarrow$ (iii). The proof of Theorem 1 is completed.

Appendix: Proof of Lemma A

The proofs of (E_1) and (E_2) are similar. For instance, let us prove (E_1).

We will use the bounds:

$$\|(\mu-\Delta)^{-\frac{1}{2}}|b|^{\frac{1}{p'}}\|_{p\to p}\leq C_{p,\delta}<\infty,\quad \||b|^{\frac{1}{p}}(\mu-\Delta)^{-\frac{1}{2}}\|_{p\to p}\leq C_{p',\delta}<\infty\quad\text{(by duality)}\tag{11}$$

(for $\|Q_p(q)\|_{p\to p}\leq C_{p,q,\delta}<\infty$, see section 1).

By the definition of ρ,

$$|\nabla\rho|\leq\nu\sqrt{l}\rho\equiv C_1\sqrt{l}\rho,\quad |\Delta\rho|\leq 2\nu(2\nu+d+2)l\rho\equiv C_2l\rho.\tag{$\star$}$$

Set $u=(\mu-\Delta)^{-1}f,\ f\in C_c(\mathbb{R}^d)$. We have $(\mu-\Delta)\rho u=-(\Delta\rho)u-2\nabla\rho\cdot\nabla u+\rho(\mu-\Delta)u$, and so

$$\rho u=-(\mu-\Delta)^{-1}(\Delta\rho)u-2(\mu-\Delta)^{-1}\nabla\rho\cdot\nabla u+(\mu-\Delta)^{-1}\rho(\mu-\Delta)u.$$

Thus,

$$\begin{aligned}\rho(\mu-\Delta)^{-1}f=&-(\mu-\Delta)^{-1}(\Delta\rho)(\mu-\Delta)^{-1}f\\&-2(\mu-\Delta)^{-1}\nabla\rho\cdot\nabla(\mu-\Delta)^{-1}f\\&+(\mu-\Delta)^{-1}\rho f.\end{aligned}\tag{$\star\star$}$$

We obtain from $(\star\star)$:

$$\begin{aligned}\rho\nabla(\mu-\Delta)^{-1}f=&-(\nabla\rho)(\mu-\Delta)^{-1}f\\&-\nabla(\mu-\Delta)^{-1}(\Delta\rho)(\mu-\Delta)^{-1}f\\&-2\nabla(\mu-\Delta)^{-1}\nabla\rho\cdot\nabla(\mu-\Delta)^{-1}f\\&+\nabla(\mu-\Delta)^{-1}\rho f.\end{aligned}$$

Then

$$\begin{aligned}I_0:=&\|\rho(|b_n|^{\frac{1}{p}}+1)\nabla(\mu-\Delta)^{-1}f\|_p\\\leq&\, C_1\sqrt{l}\|(|b_n|^{\frac{1}{p}}+1)\rho(\mu-\Delta)^{-1}f\|_p\\&+C_2lm_d\|(|b_n|^{\frac{1}{p}}+1)(\kappa_d^{-1}\mu-\Delta)^{-\frac{1}{2}}\rho|(\mu-\Delta)^{-1}f|\|_p\\&+2C_1\sqrt{l}m_d\|(|b_n|^{\frac{1}{p}}+1)(\kappa_d^{-1}\mu-\Delta)^{-\frac{1}{2}}\rho|\nabla(\mu-\Delta)^{-1}f|\|_p\\&+\|(|b_n|^{\frac{1}{p}}+1)\nabla(\mu-\Delta)^{-1}\rho f\|_p\\=:&\, C_1\sqrt{l}I_1+C_2lm_dI_2+2C_1\sqrt{l}m_dI_3+\|(|b_n|^{\frac{1}{p}}+1)\nabla(\mu-\Delta)^{-1}\rho f\|_p.\end{aligned}$$

We have:

$$\begin{aligned}I_3\leq&\|(|b_n|^{\frac{1}{p}}+1)(\kappa_d^{-1}\mu-\Delta)^{-\frac{1}{2}}\|_{p\to p}\|\rho\nabla(\mu-\Delta)^{-1}f\|_p\\&\text{(we use (11))}\\\leq&\, c\|\rho\nabla(\mu-\Delta)^{-1}f\|_p\leq cI_0.\end{aligned}$$

We estimate I_1 using $(\star\star)$ and $(\star)$:

$$\begin{aligned}I_1\leq&\, C_2l\|(|b_n|^{\frac{1}{p}}+1)(\mu-\Delta)^{-1}\rho(\mu-\Delta)^{-1}f\|_p\\&+2C_1\sqrt{l}\|(|b_n|^{\frac{1}{p}}+1)(\mu-\Delta)^{-1}\|_{p\to p}\|\rho\nabla(\mu-\Delta)^{-1}f\|_p\\&+\|(|b_n|^{\frac{1}{p}}+1)(\mu-\Delta)^{-1}\rho f\|_p,\end{aligned}$$

and so $I_1\leq C_2lI_1+2C_1\sqrt{l}cI_3+\|(|b_n|^{\frac{1}{p}}+1)(\mu-\Delta)^{-1}\rho f\|_p$.

We estimate I_2 again using $(\star)$ and $(\star\star)$:

$$I_2 \leq C_2 l \|(|b_n|^{\frac{1}{p}}+1)(\kappa_d^{-1}\mu-\Delta)^{-\frac{1}{2}}(\mu-\Delta)^{-1}\rho|(\mu-\Delta)^{-1}f|\|_p$$
$$+2C_1\sqrt{l}\|(|b_n|^{\frac{1}{p}}+1)(\kappa_d^{-1}\mu-\Delta)^{-\frac{1}{2}}(\mu-\Delta)^{-1}\rho|\nabla(\mu-\Delta)^{-1}f|\|_p$$
$$+\|(|b_n|^{\frac{1}{p}}+1)(\kappa_d^{-1}\mu-\Delta)^{-\frac{1}{2}}|(\mu-\Delta)^{-1}\rho f|\|_p,$$

and so $I_2 \leq C_2 c' l I_1 + 2C_1 c'\sqrt{l} I_3 + \|(|b_n|^{\frac{1}{p}}+1)(\kappa_d^{-1}\mu-\Delta)^{-\frac{1}{2}}|(\mu-\Delta)^{-1}\rho f|\|_p$.

Assembling the above estimates, we conclude that there exists a constant $C>0$ such that, for any $\varepsilon_0>0$, there exists a sufficiently small $l>0$ such that

$$(1-\varepsilon_0)I_0 \leq \|(|b_n|^{\frac{1}{p}}+1)\nabla(\mu-\Delta)^{-1}\rho f\|_p$$
$$+C\varepsilon_0\left[\left\|(|b_n|^{\frac{1}{p}}+1)(\mu-\Delta)^{-1}\rho f\right\|_p+\left\|(|b_n|^{\frac{1}{p}}+1)(\kappa_d^{-1}\mu-\Delta)^{-\frac{1}{2}}|(\mu-\Delta)^{-1}\rho f|\right\|_p\right].$$

Put $f:=|b_n|^{\frac{1}{p'}}h$, $h\in C_c$. Then, using $\|T_p(b_n)\|_{p\to p}\leq m_d c_p\delta$ (cf. section 1), and applying (11) to the terms in brackets [ ], we obtain: For any $\varepsilon>0$ there exists $l>0$ such that, uniformly in n,

$$\left\|\rho(|b_n|^{\frac{1}{p}}+1)\nabla(\mu-\Delta)^{-1}|b_n|^{\frac{1}{p'}}h\right\|_p<(1+\varepsilon)m_d c_p\delta\|\rho h\|_p, \tag{12}$$

so

$$\|\rho T_p(b_n)h\|_p\leq(1+\varepsilon)m_d c_p\delta\|\rho h\|_p. \tag{13}$$

We select $\varepsilon>0$ so that $(1+\varepsilon)m_d c_p\delta<1$. (Recall that $m_d c_p\delta<1$.)

Arguing as in the proof of (12) but taking $f:=h$ we find a constant $M_1<\infty$ such that

$$\|\rho\,|b_n|^{\frac{1}{p}}\nabla(\mu-\Delta)^{-1}h\|_p\leq M_1\|\rho h\|_p,\quad\text{uniformly in } n. \tag{14}$$

Also, we find a constant $M_2<\infty$ such that

$$\|\rho(\mu-\Delta)^{-1}|b_n|^{\frac{1}{p'}}h\|_\infty\leq M_2\|\rho h\|_p,\quad\text{uniformly in } n. \tag{15}$$

Indeed, using $(\star\star)$ with $f:=|b_n|^{\frac{1}{p'}}h$, we obtain

$$\|\rho(\mu-\Delta)^{-1}|b_n|^{\frac{1}{p'}}h\|_\infty\leq C_2 l\|(\mu-\Delta)^{-1}\|_{\infty\to\infty}\|\rho(\mu-\Delta)^{-1}|b_n|^{\frac{1}{p'}}h\|_\infty$$
$$+2C_1\sqrt{l}\|(\mu-\Delta)^{-1}\|_{p\to\infty}\|\rho\nabla(\mu-\Delta)^{-1}|b_n|^{\frac{1}{p'}}h\|_p$$
$$+\|(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}}\|_{p\to\infty}\|Q_p(q)\rho h\|_p,$$

where $\|\rho\nabla(\mu-\Delta)^{-1}|b_n|^{\frac{1}{p'}}h\|_p\leq(1+\varepsilon)m_d c_p\delta\|\rho h\|_p$ by (12), and $\|(\mu-\Delta)^{-\frac{1}{2}-\frac{1}{2q}}\|_{p\to\infty}<\infty$ because $p>d-1$ and q can be chosen arbitrarily close to p. Select $l>0$ so that $C_2 l\mu^{-1}<1$. (15) follows.

Now, (4) combined with (13)-(15) yields (E_1).

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